Constructing metrics with prescribed geometry

G. Calvaruso

Abstract: Geometric properties of a (pseudo-)Riemannian manifold (M, g) are encoded by its curvature. In particular, the curvature of a three-dimensional manifold is completely determined by its Ricci tensor. Therefore, the problem of finding a three-dimensional manifold with required geometrical properties, corresponds to the problem of *finding a metric with a prescribed Ricci tensor*. In this framework, the following problems arise naturally:

(i) existence results: when an algebraic symmetric $(0, 2)$-tensor can be taken as the Ricci tensor of a metric g?

(ii) explicit examples: how to provide explicit examples of such metrics?

Problem *(i)* was extensively studied ([DeTurck, Bull. A.M.S., 1980], [DeTurck, Invent. Math., 1981], [DeTurck and Goldschmidt, Adv. Math., 1999]) and solved under very general hypotheses. On the other hand, problem *(ii)* remains quite open, and deserves to be further studied under many interesting geometric assumptions.

We shall illustrate two different methods to construct three-dimensional metrics with prescribed Ricci curvature. The first method goes back to the pioneering work of Kowalski and Prüfer [Math. Ann., 1994], and has been adapted to solve several open questions in the Lorentzian case [1]-[3]. The second method has been recently introduced and permitted to construct examples of three-dimensional *IP manifolds* (both Riemannian and Lorentzian) [4], conformally flat metrics [5], pseudo-symmetric metrics [6].

References

[1] G. Calvaruso, *Pseudo-Riemannian 3-manifolds with prescribed distinct constant Ricci eigenvalues*, Diff. Geom. Appl. **26** (2008), 419–433.

[2] G. Calvaruso, *Three-dimensional homogeneous Lorentzian metrics with prescribed Ricci tensor*, J. Math. Phys. **48** (2007), 123518, 1–17.

[3] G. Calvaruso, *Curvature homogeneous Lorentzian three-manifolds*, Ann. Global Anal. Geom., 36 (2009), 1–17.

[4] G. Calvaruso, *Three-dimensional Ivanov-Petrova manifolds*, J. Math. Phys., 50 (2009) 063509, 1–12.

[5] G. Calvaruso, *Conformally flat Lorentzian three-spaces with different properties of symmetry and homogeneity*, submitted.

[6] G. Calvaruso and B. De Leo, *Pseudo-symmetric Lorentzian three-manifolds*, Int. J. Geom. Meth. Mod. Phys., to appear.

On the regularity of the space of harmonic maps from the 2-sphere to the 4-sphere, by Luis Fernández.

Abstract: The space of harmonic maps of a given degree from the 2-sphere to the 4-sphere is not regular (Wood, 2006). However, is the space of *linearly full* maps regular? This is only known when the degree d is 3, 4, 5 or 6. We will discuss some ways to study this problem in general and explain why, in the case $d = 6$, it was expected that this space would have singularities. Finally I will sketch the proof of the regularity of the space of harmonic maps from the 2-sphere to the 4-sphere when $d = 6$ (joint work with John Bolton).

Harmonic morphisms on conformally flat 3-spheres

Sebastian Heller

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Abstract. We show that under some non-degeneracy assumption the only submersive harmonic morphism on a conformally flat $3-$sphere is the Hopf fibration. The proof involves an appropriate use the Chern-Simons functional.

TRANSFORMATIONS OF WILLMORE SURFACES

KATRIN LESCHKE

Abstract

In my talk I will discuss the Bäcklund and Darboux transformation of Willmore surfaces: since the harmonicity of the conformal Gauss map characterizes Willmore surfaces, one can use an analogue of the delbar-sequence for harmonic maps into complex projective space to classify Willmore tori with non-trivial normal bundle. Furthermore, I will discuss some properties of the Darboux transformation on Willmore surfaces.

MINIMAL SURFACES IN LIE GROUPS

FRANCESCO MERCURI

Abstract

We will discuss a Weierstrass type representation for minimal surfaces in Riemannian manifolds. We will outline some applications when the ambient space is a 3-dimensional Lie group.

ON THE STRESS-BIENERGY TENSOR AND LICHNEROWICZ LAPLACIAN

CEZAR ONICIUC

The vanishing of the stress-bienergy tensor S_2 characterizes the critical points of the bienergy functional thought of as a functional defined on the set of all admissible metrics on the domain manifold M of a smooth map $\phi : M \to (N, h)$.

In this talk we shall survey some known results for the stress-bienergy tensor and present the link between S_2 and the Lichnerowicz Laplacian Δ^L.

Singular Dressing Actions on Harmonic Maps

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Joint work with Nuno Correia

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Abstract

The well known zero-curvature formulation introduced by K. Uhlenbeck [6] yields an action of a certain loop group on the space of harmonic maps from a Riemann surface into a compact symmetric space G/K; underlying this *dressing action* is the existence of Iwasawa-type decompositions of the loop groups and loop algebras concerned. At same time, Uhlenbeck introduced the fundamental procedure of *adding a uniton*, which is another way of obtaining new harmonic maps from known ones, and proved that all harmonic two-spheres in the unitary group can be factorized into finite products of *flag factors* $\mathrm{S}^2 \to \mathrm{U}(n)$ by adding unitons. This was subsequently generalized in [2, 3] to the case of an arbitray compact semisimple Lie group G.

Bergvelt and Guest [1], again inspired in [6], exploited the singularities of the dressing action in order to enlarge the corresponding orbits of harmonic maps – this limiting process is called the *modified completion procedure* or the *singular dressing action* [1, 5]. They proved that any harmonic map from the two-sphere S^2 into the complex projective space $\mathbb{C}\mathrm{P}^{\mathrm{n}}$ may be reduced to a constant by applying twice this procedure. Subsequently, Jiao [5] proved that any harmonic map ϕ from S^2 into the unitary group $\mathrm{U}(n)$ may be reduced to a constant by applying n singular dressing actions. This reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to \mathrm{U(n)}$, and the n singular dressing actions are produced from curves $\{\gamma_a\}$ of rational loops of the form $\gamma_a(\lambda) = \pi_V^{\perp} + \zeta_a(\lambda)\pi_V$, where

$$\zeta_a(\lambda) = \frac{\lambda - a}{\overline{a}\lambda - 1}\frac{\overline{a} - 1}{1 - a}.$$

These rational loops are precisely the *simple factors* of Uhlenbeck and they generate the group of rational loops in the matrix Lie group $\mathrm{Gl}(n, \mathbb{C})$ satisfying the reality condition with respect to $\mathrm{U}(n)$ [6]. In [4], the authors gave a consistent definition of simple factor for an arbitrary complex reductive Lie group and an arbitrary representation, and proved that, in the $\mathrm{SO}(n)^{\mathbb{C}}$ and $\mathrm{G}_2^{\mathbb{C}}$ cases, with respect to the fundamental representations, the simple factors generate the group of rational loops satisfying the reality condition.

In this talk I intend to outline the proof that any harmonic map ϕ from S^2 into an arbitrary compact semisimple matrix Lie group G may be reduced to a constant by applying a finite number of singular dressing actions; this reduction induces a factorization of ϕ into flag factors $\mathrm{S}^2 \to \mathrm{G}$, and the singular dressing actions are produced from curves of simple factors for $G^{\mathbb{C}}$. A version of this result for an arbitrary inner symmetric space G/K is established. I shall also present generating theorems for the rational loops of the fundamental representations of $\mathrm{Sp}(n)^{\mathbb{C}}$ and $\mathrm{SU}(n)^{\mathbb{C}}$.

References

[1] M.J. Bergvelt and M.A. Guest, *Action of loop groups on harmonic maps.* Trans. Amer. Math. Soc. **326** (1991), 861–886

[2] F.E. Burstall and M.A. Guest, *Harmonic two-spheres in compact symmetric spaces, revisited*, Math.Ann. **309** (1997), 541–572.

[3] F.E. Burstall, J. H. Rawnsley, *Twistor Theory for Riemannian Symmetric Spaces*, Lectures Notes in Math. 1424 Berlin, Heidelberg: 1990.

[4] N. Donaldson, D. Fox, O. Goertsches, *Generators for rational loop groups and geometric applications*, arXiv:0803.0029v1 [math.DG].

[5] X. Jiao, *Factorizations of Harmonic Maps of Surfaces into Lie Groups by Singular Dressing Actions*, Differential Geometry and Integrable Systems: A Conference on Integrable Systems in Differential Geometry (Contemp. Math., Vol. 308, M. Guest et al., eds.), Amer. Math. Soc., Providence, R. I. (2002), pp. 199–204.

[6] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.

NEUTRAL MINIMAL LAGRANGIAN SURFACES IN THE TANGENT BUNDLE

PASCAL ROMON

Abstract

Given an oriented Riemannian surface (Σ, g), its tangent bundle $T\Sigma$ enjoys a natural pseudo-Kähler structure, that is the combination of a complex structure $\mathbb{J}$, a pseudo-metric $\mathbb{G}$ with neutral signature and a symplectic structure Ω. We give a local classification of those surfaces of $T\Sigma$ which are both Lagrangian with respect to Ω and minimal with respect to $\mathbb{G}$. We first show that if g is nonflat, the only such surfaces are affine normal bundles over geodesics. In the flat case there is, in contrast, a large set of Lagrangian minimal surfaces, which is described explicitly. As an application, we show that motions of surfaces in $\mathbb{R}^3$ induce Hamiltonian motions of their normal congruences, which are Lagrangian surfaces in $T\mathbb{S}^2$. Joint work with Henri Anciaux and Brendan Guilfoyle.

An existence result for affine harmonic maps

Jürgen Jost and Fatma Muazzez Şimşir

Abstract

We introduce a class of maps from an Kähler affine into a Riemannian manifold that solve an elliptic system defined by the natural second order elliptic operator of the affine structure and the nonlinear Riemannian geometry of the target. These maps are called affine harmonic. We show an existence result for affine harmonic maps in a given homotopy class when the target has nonpositive sectional curvature and some global nontriviality condition is met. An example shows that such a condition is necessary. Combined estimation techniques from geometric analysis and PDE theory with global geometric considerations is needed because of the absence of a variational structure underlying affine harmonic maps.

Quartic energies and local minima on spheres

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For a map $\varphi : (M, g) \to (N, h)$ a good measure of how much it stretches areas (of 2-dimensional submanifolds of M) is the norm of $\Lambda^2 \mathrm{d}\varphi : \Lambda^2 TM \to \Lambda^2 TN$, analogously to $\|\mathrm{d}\varphi\|^2$ (the Dirichlet energy density) that is a measure of length's stretching. As $\|\Lambda^2 \mathrm{d}\varphi\|^2$ coincides with $\sigma_2(\varphi^* h)$, the second elementary symmetric function of the eigenvalues of $\varphi^* h$ with respect to g, we call it σ_2–energy.

Developing an unpublished general work of Chris Wood, we present the first and second variation of the quartic σ_2–energy together with some classes of (stable) critical maps. The case of maps from/to spheres is specially emphasized. In contrast to the harmonic case, α–Hopf construction is now able to produce σ_2–critical representants in every class of $\pi_3(\mathbb{S}^2)$ considering only canonical metrics. Moreover, the quartic character of energy allows the stability on sphers of dimension less than 4. All these phenomena have a counterpart for the 4-energy.

As σ_2–energy summed with the Dirichlet energy form the static Hamiltonian of the Skyrme-Faddeev model, the interplay between the above discussion and the problem of topological solitons from high energy physics is also briefly outlined.

The talk resumes the main results in [1, 2, 3, 4].

References

[1] Slobodeanu, R., *A special class of holomorphic mappings and the Faddeev-Hopf model*, J. Nonlin. Math. Phys., **15**(3) (2008), 270 – 282..

[2] Slobodeanu, R., *On the geometrized Skyrme and Faddeev models*, Preprint submitted to Elsevier, arXiv:0809.4864 [math.DG].

[3] Slobodeanu, R., *A note on higher-charge configurations for the Faddeev-Hopf model*, Preprint submitted to IOP, arXiv:0812.4545 [math.DG].

[4] Loubeau, E., Slobodeanu, R., *Local minima of higher power energies on spheres*, in preparation.

A PROOF OF THE DDVV CONJECTURE AND ITS EQUALITY CASE

TANG ZIZHOU

Abstract

We give a proof of the DDVV conjecture, which is a pointwise inequality involving the scalar curvature, the normal scalar curvature and the mean curvature on a submanifold of a real space form. This is based on joint work with Ge Jianquan.