1. Introduction

We shall consider the solvability for a classical pseudodifferential operator $P$ on a $C^\infty$ manifold $X$. This means that $P$ has an expansion $p_m + p_{m-1} + \ldots$ where $p_k \in S^k_{\text{hom}}$ is homogeneous of degree $k$ and $p_m = \sigma(P)$ is the principal symbol of the operator. The operator $P$ is solvable at a compact set $K \subseteq X$ if the equation

\[(1.1) \quad Pu = v\]

has a local solution $u \in \mathcal{D}'(X)$ in a neighborhood of $K$ for any $v \in C^\infty(X)$ in a set of finite codimension. We can also define the microlocal solvability at any compactly based cone $K \subset T^*X$, see Definition 2.7.

A pseudodifferential operator is of principal type if the Hamilton vector field

\[(1.2) \quad H_p = \sum_{j=1}^n \partial_{\xi_j} p \partial_{x_j} - \partial_{x_j} p \partial_{\xi_j}\]

of the principal symbol $p$ does not have the radial direction $\langle \xi, \partial \xi \rangle$ at $p^{-1}(0)$, in particular $H_p \neq 0$ then. By homogeneity $H_p$ is well defined on the cosphere bundle $S^*X = \{ (x, \xi) \in T^*X : |\xi| = 1 \}$, defined by some choice of Riemannian metric. For pseudodifferential operators of principal type, it is known \cite{1} \cite{3} that local solvability at a point is equivalent to condition $(\Psi)$ which means that

\[(1.3) \quad \text{Im}(ap) \text{ does not change sign from } - \text{ to } +\]

along the oriented bicharacteristics of $\text{Re}(ap)$ for any $0 \neq a \in C^\infty(T^*M)$. Oriented bicharacteristics are the positive flow-outs of the Hamilton vector field $H_{\text{Re}(ap)} \neq 0$ on $\text{Re}(ap) = 0$. Bicharacteristics of $\text{Re} ap$ are also called semi-bicharacteristics of $p$.

We shall consider the case when the principal symbol is real and vanishes at least second order at an involutive manifold $\Sigma_2$, thus $P$ is not of principal type. For operators which are not of principal type, the values of the subprincipal symbol $p_{m-1}$ at $\Sigma_2$ becomes important. In the case when principal symbol $p = \xi_1\xi_2$, Mendoza and Uhlman \cite{5} proved that $P$ was not solvable if the subprincipal symbol changed sign on the $x_1$ or $x_2$ lines when $\xi_1 = \xi_2 = 0$. Mendoza \cite{6} generalized this to the case when the principal symbol vanishes on an involutive submanifold having an indefinite Hessian with rank equal to the codimension of the manifold. The Hessian then gives well-defined limit bicharacteristics.

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over the submanifold, and \( P \) is not solvable if the subprincipal symbol changes sign on any of these limit bicharacteristics. This corresponds to condition \((P)\) on the limit bicharacteristics (no sign changes) because in this case one gets both directions when taking the limit.

In this paper, we shall extend this result to more general pseudodifferential operators. As in the previous cases, the operator will have real principal symbol and we shall consider the limits of bicharacteristics at the set where the principal symbol vanishes of at least second order. The convergence shall be as smooth curves, then the limit bicharacteristic also is a smooth curve. We shall also need uniform bounds on the curvature of the characteristics at the bicharacteristics, but only along the tangent space of a Lagrangean submanifold, which we call a grazing Lagrangean space, see (2.3). This gives uniform bounds on the linearization of the normalized Hamilton flow on the tangent space of this submanifold at the bicharacteristics. Our main result is Theorem 2.8, which essentially says that under these conditions the operator is not solvable at the limit bicharacteristic if the quotient of the imaginary part of the subprincipal symbol with the norm of the Hamilton vector field switches sign from \(-\) to \(+\) on the bicharacteristics and becomes unbounded as they converge to the limit bicharacteristic.

2. Statement of results

Let the principal symbol \( p \) be real valued, \( \Sigma = p^{-1}(0) \) be the characteristics, and \( \Sigma_2 \) be the set of double characteristics, i.e., the points on \( \Sigma \) where \( dp = 0 \). Let \( \{ \Gamma_j \}_{j=1}^{\infty} \) be a set of bicharacteristics of \( p \) on \( S^*X \) so that \( \Gamma_j \in \Sigma \setminus \Sigma_2 \) are uniformly bounded in \( C^\infty \) when parametrized on a uniformly bounded interval (for example with respect to the arc length). These bounds are defined with respect to some choice of Riemannian metric on \( T^*X \), but different choices of metric will only change the bounds with fixed constants. In particular, we have a uniform bound on the arc lengths:

\[
|\Gamma_j| \leq C \quad \forall j
\]

We also have that \( \Gamma_j = \{ \gamma_j(t) : t \in I_j \} \) with \( |\gamma_j'(t)| \equiv 1 \) and \( |I_j| \leq C \), then \( |\gamma_j^{(k)}(t)| \leq C_k \) for \( t \in I_j \) and all \( j, k \geq 1 \). Let \( \tilde{p} = p/|\nabla p| \) then the normalized Hamilton vector field is equal to

\[
H_{\tilde{p}} = |H_p|^{-1}H_p \quad \text{on } p^{-1}(0)
\]

then \( \Gamma_j \) is uniformly bounded in \( C^\infty \) if

\[
|H^k_{\tilde{p}} \nabla \tilde{p}| \leq C_k \quad \text{on } \Gamma_j \quad \forall j, k
\]

where \( \nabla \tilde{p} \) is the gradient of \( \tilde{p} \). Thus the normalized Hamilton vector field \( H_{\tilde{p}} \) is uniformly bounded in \( C^\infty \) as a vector field over \( \Gamma \). Observe that the bicharacteristics have a natural orientation given by the Hamilton vector field. Now the set of bicharacteristic curves \( \{ \Gamma_j \}_{j=1}^{\infty} \) is uniformly bounded in \( C^\infty \) when parametrized with respect to the arc length, and therefore it is a precompact set. Thus there exists a subsequence \( \Gamma_{j_k}, k \to \infty \), that converge to a smooth curve \( \Gamma \) (possibly a point), called a limit bicharacteristic by the following definition.
Definition 2.1. We say that a sequence of smooth curves \( \Gamma_j \) on a smooth manifold converges to a smooth limit curve \( \Gamma \) (possibly a point) if there exist parametrizations on uniformly bounded intervals that converge in \( C^\infty \). If \( p \in C^\infty(T^*X) \), then a smooth curve \( \Gamma \subset \Sigma_2 \cap S^*X \) is a limit bicharacteristic of \( p \) if there exists bicharacteristics \( \Gamma_j \) that converge to it.

Naturally, this definition is invariant, and the set \( \{ \Gamma_j \}_{j=1}^{\infty} \) may have subsequences converging to several different limit bicharacteristics, which could be points. In fact, if \( \Gamma_j \) is parametrized with respect to the arc length on intervals \( I_j \) such that \( |I_j| \to 0 \), then we find that \( \Gamma_j \) converges to a limit curve which is a point. Observe that if \( \Gamma_j \) converge to a limit bicharacteristic \( \Gamma \), then (2.1) and (2.2) hold for \( \Gamma_j \).

Example 2.2. Let \( \Gamma_j \) be the curve parametrized by

\[
[0,1] \ni t \mapsto \gamma_j(t) = (\cos(jt), \sin(jt))/j
\]

Since \( |\gamma_j'(t)| = 1 \) the curves are parametrized with respect to arc length, and we have that \( \Gamma_j \to (0,0) \) in \( C^0 \), but not in \( C^\infty \) since \( |\gamma_j''(t)| = j \). If we parametrize \( \Gamma_j \) with \( x = jt \in [0,j] \) we find that \( \Gamma_j \) converge to \( (0,0) \) in \( C^\infty \) but not on uniformly bounded intervals.

Example 2.3. Let \( P \) have real principal symbol \( p = w^k + a(w') \) in the coordinates \( (w_1, w') \in S^*\mathbb{R}^n \), where \( w_1 = a(w') = 0 \) at \( \Sigma_2 \) and \( k \geq 2 \). This case includes the cases where the operator is microhyperbolic, then \( k = 2 \) and \( a(w') \) vanishes of exactly second order at \( \Sigma_2 \). If \( k \) is even then in order to have limit bicharacteristics, it is necessary that \( a \not\equiv 0 \) in a neighborhood of \( \Sigma_2 \), and then \( \Sigma \) is given locally by \( w_1 = \pm a(w')^{1/k} \). We find that (2.2) is satisfied if \( H_P w_1 = 0 \) and \( H_P \nabla a = \mathcal{O}(\nabla a) \) for any \( j \) when \( w_1 = a = 0 \).

But we shall also need a condition on the differential of the Hamilton vector field \( H_P \) at the bicharacteristics along a Lagrangean space, which will give bounds on the curvature of the characteristics in these directions. In the following, a section of Lagrangean spaces \( L \) over a bicharacteristic \( \Gamma \) will be a map \( \Gamma \ni w \mapsto L(w) \subset T_w(T^*X) \) such that \( L(w) \) is a Lagrangean subspace of dimension \( n \) in \( T\Sigma, \forall w \), where \( \Sigma = p^{-1}(0) \). If the section \( L \) is \( C^1 \) then it has tangent space \( TL \subset T(T_\Gamma(T^*X)) \). If we identify \( T(T_\Gamma(T^*X)) \) with \( T_\Gamma(T^*X) \), the linearization (or first order jet) of the normalized Hamilton vector field \( H_P \) at \( \Gamma \) also is in \( T(T_\Gamma(T^*X)) \), which we use in the following definition.

Definition 2.4. For a bicharacteristic \( \Gamma \) of \( p \) we say that a smooth section of Lagrangean spaces \( L \) over \( \Gamma \) is a section of grazing Lagrangean spaces of \( \Gamma \) if \( L \subset T\Sigma \) and the linearization (or first order jet) of \( H_P \in TL \).

Thus, the linearization of \( H_P \) gives a partial connection on \( L \), i.e., a connection that is only defined along \( T\Gamma \). Choosing a Lagrangean subspace of \( T\Sigma \) at \( w_0 \in \Gamma \) then determines \( L \) along \( \Gamma \), so \( L \) must be smooth. Actually, \( L \) is in the tangent space at \( \Gamma \) of a Lagrangean manifold, which is given as the zero set of the second order Taylor expansion of \( p \) at \( \Gamma \) and has the first order Taylor expansion of \( H_P \) as its Hamilton vector field.
**Example 2.5.** Let \( p = \tau - \langle A(t)x, x \rangle / 2 \) where \( A(t) \) is a real and symmetric \( n \times n \) matrix, and let \( \Gamma = \{ (t, 0, 0, \xi_0) : t \in I \} \). Then \( p^{-1}(0) = \{ \tau = \langle A(t)x, x \rangle / 2 \} \), the linearization of the Hamilton field at \( (t, 0, 0, \xi_0) \) is \( H_p = \partial_t + \langle A(t)y, \partial_x \rangle \) and a grazing Lagrangean space is given by \( L(t) = \{ (s, y, 0, A(t)y) : (s, y) \in \mathbb{R}^n \} \), where \( A'(t) = A(t) \). Thus \( L(t) \) is constant in \( t \) if and only if \( A(t) = 0 \).

Observe that we may choose symplectic coordinates \((t, x; \tau, \xi)\) so that \( \tau = p \) and the fiber of \( L(w) \) is equal to \( \{ (s, y, 0, 0) : (s, y) \in \mathbb{R}^n \} \) at \( w \in \Gamma = \{ (t, 0, 0, \xi_0) : t \in I \} \). But it is not clear that we can to that uniformly, for that we need an additional condition. Now we assume that there exists a grazing Lagrangean space \( L_j \) of \( \Gamma_j \), \( \forall j \), such that the normalized Hamilton vector field \( H_{\tilde{p}} \) satisfies

\[
(2.3) \quad \left| dH_{\tilde{p}}(w) \right|_{L_j(w)} \leq C \quad \text{for } w \in \Gamma_j \quad \forall j
\]

Since the mapping \( \Gamma_j \ni w \mapsto L_j(w) \) is determined by the linearization of \( H_{\tilde{p}} \) on \( L_j \), thus by \( dH_{\tilde{p}}(w)|_{L_j(w)} \), condition (2.3) implies that \( \Gamma_j \ni w \mapsto L_j(w) \) is uniformly in \( C^1 \), see Example 2.5. Observe that condition (2.2) gives (2.3) in the directions of \( TT_\Gamma \). Clearly condition (2.3) is invariant under changes of symplectic coordinates and multiplications with non-vanishing real factors. Also, it gives no restrictions on the variation of \( |H_p| \).

Now if \( u \in C^\infty \) has values in \( \mathbb{R}^n \) and \( \omega = u/|u| \) then

\[
\partial \omega = \partial u/|u| - \langle u, \partial u \rangle u/|u|^3
\]

This gives that (2.3) is equivalent to

\[
(2.4) \quad \left| \Pi d\nabla p \right|_{L_j} \leq C|\nabla p| \quad \text{on } \Gamma_j \quad \forall j
\]

where \( \Pi v = v - \langle v, \nabla p \rangle \nabla p/|\nabla p|^2 \) is the ON projection on the vectors orthogonal to \( \nabla p \).

Condition (2.4) gives a uniform bound on the curvature of level surface \( p^{-1}(0) \) in the directions given by \( L_j \) over \( \Gamma_j \). Observe that the invariance of condition (2.4) can be checked directly since \( d(ap) = adp \) and \( d^2(ap) = ad^2p + dadp = ad^2p \) on \( L_j \) over \( \Gamma_j \). The invariant subprincipal symbol \( p_s \) will be important for the solvability of the operator. For the usual Kohn-Nirenberg quantization of pseudodifferential operators, the subprincipal symbol is equal to

\[
(2.5) \quad p_s = p_{m-1} - \frac{1}{2i} \sum_j \partial_{\xi_j} \partial_{x_j} p
\]

and for the Weyl quantization it is \( p_{m-1} \).

Now for the principal symbol we shall denote

\[
(2.6) \quad 0 < \min_{\Gamma_j} |H_p| = \kappa_j \to 0 \quad j \to \infty
\]

and for the subprincipal symbol \( p_s \) we shall assume the following condition

\[
(2.7) \quad \min_{\partial \Gamma_j} \int \text{Im} p_s |H_p|^{-1} ds / |\log \kappa_j| \to \infty \quad j \to \infty
\]

where the integration is along the natural orientation given by \( H_p \) on \( \Gamma_j \) starting at some point \( w_j \in \Gamma_j \). Observe that if (2.7) holds then there must be a change of sign of \( \text{Im} p_s \).
from $-\infty$ to $+\infty$ on $\Gamma_j$, and
\begin{equation}
\max_{\Gamma_j}(-1)^{\pm 1}\Im p_s/|H_p| \log \kappa_j \to \infty \quad j \to \infty
\end{equation}
for both signs. Observe that condition (2.7) is invariant under symplectic changes of coordinates and multiplication with elliptic pseudodifferential operators, thus under conjugation with elliptic Fourier integral operators. In fact, multiplication only changes the subprincipal symbol with non-vanishing factors and terms proportional to $|\nabla p| = |H_p|$. Now by choosing symplectic coordinates $(t, x, \tau, \xi)$ near a given point $w_0 \in \Gamma_j$ so that $p = \alpha \tau$ near $w_0$ with $\alpha \neq 0$, we obtain that $\partial_x \partial_\xi p = 0$ at $\Gamma_j$ and $\partial_t \partial_\tau p = \partial_t \alpha = \partial_t |\nabla p|$ at $\Gamma_j$ near $w_0$. Thus, the second term in (2.5) only gives terms in condition (2.7) which are bounded by
\begin{equation}
\int \partial_t |\nabla p|/|\nabla p| \, ds / |\log (\kappa_j)| \lesssim |\log(|\nabla p|)| / |\log (\kappa_j)| \lesssim 1
\end{equation}
since $\kappa_j \leq |\nabla p|$ on $\Gamma_j$. Here $a \lesssim b$ (and $b \gtrsim a$) means that $a \leq Cb$ for some $C > 0$. Thus we obtain the following result.

**Remark 2.6.** We may replace the subprincipal symbol $p_s$ by $p_{m-1}$ in (2.7), since the difference is bounded as $j \to \infty$.

We shall study the microlocal solvability, which is given by the following definition. Recall that $H^\loc_{(s)}(X)$ is the set of distributions that are locally in the $L^2$ Sobolev space $H^s_{(s)}(X)$.

**Definition 2.7.** If $K \subset S^*X$ is a compact set, then we say that $P$ is microlocally solvable at $K$ if there exists an integer $N$ so that for every $f \in H^\loc_{(N)}(X)$ there exists $u \in \mathcal{D}'(X)$ such that $K \cap \WF(Pu - f) = \emptyset$.

Observe that solvability at a compact set $M \subset X$ is equivalent to solvability at $S^*X|_M$ by [4, Theorem 26.4.2], and that solvability at a set implies solvability at a subset. Also, by Proposition 26.4.4 in [4] the microlocal solvability is invariant under conjugation by elliptic Fourier integral operators and multiplication by elliptic pseudodifferential operators. The following is the main result of the paper.

**Theorem 2.8.** Let $P \in \Psi^\loc_{d}(X)$ have real principal symbol $\sigma(P) = p$, and subprincipal symbol $p_s$. Let $\{ \Gamma_j \}_{j=1}^\infty$ be a family of bicharacteristic intervals of $p$ in $S^*X$ so that (2.3) and (2.7) hold. Then $P$ is not microlocally solvable at any limit bicharacteristics of $\{ \Gamma_j \}_{j}$.

To prove Theorem 2.8 we shall use the following result. Let $\|u\|_{(k)}$ be the $L^2$ Sobolev norm of order $k$ for $u \in C_0^\infty$ and $P^*$ the $L^2$ adjoint of $P$.

**Remark 2.9.** If $P$ is microlocally solvable at $\Gamma \subset S^*X$, then Lemma 26.4.5 in [4] gives that for any $Y \subset X$ such that $\Gamma \subset S^*Y$ there exists an integer $\nu$ and a pseudodifferential operator $A$ so that $WF(A) \cap \Gamma = \emptyset$ and
\begin{equation}
\|u\|_{(-N)} \leq C(\|P^*u\|_{(\nu)} + \|u\|_{(-N-\nu)} + \|Au\|_{(0)}) \quad u \in C_0^\infty(Y)
\end{equation}
where $N$ is given by Definition 2.7.
We shall use Remark 2.9 in Section 7 to prove Theorem 2.8 by constructing approximate local solutions to $P^* u = 0$. We shall first prepare and get a microlocal normal form for the adjoint operator, which will be done in Section 4. Then we shall solve the eikonal equation in Section 5 and the transport equations in Section 6.

3. Examples

Example 3.1. Let $P$ have principal symbol $p = \prod_j p_j$ which is a product of real symbols $p_j$ of principal type, such that $p_j = 0$ on $\Sigma_2$, $\forall \ j$, and $p_j \neq p_k$ on $\Sigma \setminus \Sigma_2$ when $j \neq k$. We find that if $\Gamma \in p^{-1}_k(0)$ then

$$|H_p| = |H_{p_k}| \prod_{j \neq k} |p_j| = |H_{p_k}| q_k$$

where $q_k > 0$ on $\Gamma \setminus \Sigma_2$ close to $\Sigma_2$. Then $p$ satisfies (2.2) and (2.3) for any Lagrangean space by the invariance, since $\nabla p_k \neq 0$ and $\partial^p p_k = O(1)$, $\forall \alpha$. A bicharacteristic $\Gamma \subset \Sigma \setminus \Sigma_2$ of $p$ is a bicharacteristic for $p_k$ for some $k$. Then if $p_s$ is the subprincipal symbol and (2.7) is satisfied for a sequence of bicharacteristics of $p_k$ converging in $C^\infty$ to $\Sigma_2$, we obtain that the operator is not solvable at any limit of these bicharacteristics at $\Sigma_2$.

Example 3.2. Assume that $p(x, \xi)$ is real and vanishes of exactly order $k \geq 2$ at the involutive submanifold $\Sigma_2 = \{ \xi' = 0 \}, \xi = (\xi', \xi'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, such that the localization

$$\eta \mapsto \sum_{|\alpha| = k} \partial^{\xi}_\xi p(x, 0, \xi'') \eta^\alpha$$

is of principal type when $\eta \neq 0$. Then the bicharacteristics of $p$ satisfies (2.2) and (2.3) with $L_j = \{ \xi = 0 \}$. In fact, $|\partial^{\xi}_\xi p(x, \xi)| \cong |\xi'|^{|k-1}$ and $\partial^{x, \xi''}_\xi p(x, \xi) = O(|\xi'|^k)$ so this follows since $H_p = \partial^{\xi}_\xi p \partial_{x} + O(|\xi'|)$ and $\partial^{\xi}_\xi \nabla p = O(|\xi'|^{k-1})$ when $|\xi'| \ll 1$ and $|\xi| \cong 1$. The operator is not solvable if the imaginary part of the subprincipal symbol $\text{Im} p_s$ changes sign from $-$ to $+$ along a convergent sequence of bicharacteristics of the principal symbol and vanishes at most order $k - 2$ at $\Sigma_2$. In particular we obtain the results of [5] and [6].

Example 3.3. As in Example 2.3, let $P$ have real principal symbol $p = w^j - a(w')$ in the coordinates $(w_1, w') \in S^* \mathbb{R}^n$, where $w_1 = a(w') = 0$ at $\Sigma_2$. We find that (2.3) is satisfied if $dw_1|_{L_j} = 0$ and $d\nabla a|_{L_j} = O(|\nabla a|)$ when $w_1 = a = 0$. 

Example 3.4. Let $Q$ be a hyperbolic quadratic form on $T^* \mathbb{R}^n$. Then by [2, Theorem 1.4.6] we have the following normal form $Q_1(x, \xi) + Q_2(y, \eta)$ where

$$Q_1(x, \xi) = \sum_{j=1}^k \mu_j (x_j^2 + \xi_j^2) + \sum_{j=k+1}^m \xi_j^2 \quad \mu_j > 0 \quad \forall \ j$$

is positive semidefinite and $Q_2(y, \eta)$ is either $-\eta_1^2$, $\mu_0 y_1 \eta_1$ or $2 \eta_1 \eta_2 - y_2^2$. To simplify the notation, we will assume $\mu_j = 1$, $\forall \ j$.

Now the flow of the Hamilton vector field is a direct sum of the flows of $Q_1$ and $Q_2$. For $Q_1$ it is a direct sum of circles in $(x_j, \xi_j)$ of radius $r_j \to 0$, $j \leq k$, and $x_j$ lines, $k < j \leq m$. 

The circles can only converge to the origin if the radii goes to zero fast enough, see Example 2.2. The possible limits are points or lines in the $x_{k+1}, \ldots, x_m$ space.

In the case $Q_2(y, \eta) = -\eta_1^2$ we find that the limit bicharacteristics are given by

$$y_1 = \lambda t, \ x_j = \lambda_j t + a_j, \ k < j \leq m$$

where $\lambda^2 = \sum_{j=k+1}^m \lambda_j^2$. We can only find a Lagrangean space satisfying (2.3) if $\mu_j = 0, \forall j$, since it cannot be contained in the subspace $\{ x_j = \xi_j = 0 \}$, and then the Lagrangean space can be taken as $\{ (x, y; 0, 0) \}$. Theorem 2.8 then gives that the operator with principal symbol $Q(x, y; \xi, \eta)$ is not solvable when the imaginary part of the subprincipal symbol changes sign on the lines on $\Sigma_2$ given by (3.1), which also follows from the results in [6].

If $Q_2(y, \eta) = y_1 \eta_1$, then $Q^{-1}(0) = \{ y_1 \eta_1 = -Q_1(x, \xi) = -\lambda^2 \}$ where $H_{Q_1} \cong \lambda$. The bicharacteristic of $Q_2$ is given by $t \mapsto (y_1 e^{\lambda t}; \eta_1 e^{-t})$ where $y_1 \eta_1 = -\lambda^2 < 0$ which gives $|y_1| + |\eta_1| \gtrsim \lambda$. We find that $|H_{Q_1}| \cong L = |y_1| + |\eta_1|$, so in order for the bicharacteristic to converge in $C^\infty$, we find from (2.2) that $(|y_1| + |\eta_1|)/L^k \lesssim 1, \forall k$. Then $|y_1| + |\eta_1| \not\to 0$ so the bicharacteristics do not converge in $C^\infty$ to a limit bicharacteristic at $\Sigma_2$. Observe that a hyperbolic operator with Hessian of the principal symbol equal to $Q(x, y; \xi, \eta)$ is effectively hyperbolic and is solvable with any lower order terms, see [2].

Finally, when $Q_2(y, \eta) = 2\eta_1 \eta_2 - y_2^2$, then the characteristics in the $(y, \eta)$ variables are $\{ \eta_1 \eta_2 = (y_2^2 + \lambda^2)/2 \}$ where $\lambda^2 = Q_1$. We find $H_{Q_2} = 2\eta_2 \partial y_1 + 2\eta_1 \partial y_2 + 2y_2 \partial \eta_2$ so $\eta_1$ is constant on the orbits. Note that $y_2^2 + \lambda^2 = 2\eta_1 \eta_2 \leq \eta_1^2 + \eta_2^2$. Thus when $|\eta_1| \gtrsim |\eta_2|$ we find that $|\eta_1| \gtrsim |y_2| + \lambda$ on $\Sigma$. Now in order for (2.2) to hold, we find that $|\eta_1| \gtrsim 1$, so the characteristics will not converge in $C^\infty$ to any limit bicharacteristic at $\Sigma_2$. When $|\eta_2| \gg |\eta_1|$ we find that $|\eta_2| \gg |y_2| + \lambda$ on $\Sigma$. A straightforward computation shows by (2.2) that the bicharacteristics converge in $C^\infty$ to the $y_1$ lines on $\Sigma_2$ only if $|\eta_1| \lesssim |\eta_2|^3$, which implies that $|y_2| + \lambda \lesssim \eta_2^2$. Then the Lagrangean space can be taken as $\{ (x, y_1, 0; 0, 0, \eta_2) \}$ at every point of the bicharacteristics. An example is when $y_2 = \eta_1 = \lambda = 0$ and $\eta_2 \to 0$. Thus Theorem 2.8 gives that the operator with principal symbol $Q(x, y; \xi, \eta)$ is not solvable when the imaginary part of the subprincipal symbol changes sign on $y_1$ lines at $\Sigma_2$.

4. THE NORMAL FORM

First we shall put the adjoint operator $P^*$ on a normal form uniformly and microlocally near the bicharacteristics $\Gamma_j$ converging to $\Gamma$. This will present some difficulties since we only have conditions at the bicharacteristics. By the invariance, we may multiply with an elliptic operator so that the order of $P^*$ is $m = 1$ and $P^*$ has the symbol expansion $p + p_0 + \ldots$, where $p$ is the principal symbol. As before we may assume that $p_0$ is the subprincipal symbol. Observe that for the adjoint the signs in (2.7) are reversed and it changes to

$$\max_{\partial \Gamma_j} \int \text{Im } p_s|H_p|^{-1} ds |\log \kappa_j| \to -\infty \quad j \to \infty$$

(4.1)
Then by changing $w_j$ to the maximum of the integral in (4.1), we may assume that

$$\int \text{Im} p_0 / |H_p| \, ds \leq 0 \quad \text{on } \Gamma_j$$

with equality at $w_j \in \Gamma_j$. Since $\nabla \text{Im} p_0$ and $\nabla H_p$ are bounded on $S^*X$, we find that the ratio $\text{Im} p_0 / |H_p|$ only changes with a fixed term in an interval of length $\kappa_j$ given by (2.6). Because of (2.8), we may extend $\Gamma_j$ so that

$$|\Gamma_j| \gtrsim \kappa_j$$

and that condition (2.7) holds on intervals of at least length $\kappa_j$ at the endpoints of $\Gamma_j$.

Now we choose

$$1 \lesssim \lambda_j = \kappa_j^{-1/\varepsilon} \iff \kappa_j = \lambda_j^{-\varepsilon}$$

for some $\varepsilon > 0$ to be determined later. Then we may replace $|\log \kappa_j|$ with $\log \lambda_j$ in (2.7)–(2.8). By choosing a subsequence and renumbering, then we may assume by (2.7) that

$$\min_{\partial \Gamma_j} \int \text{Im} p_0 / |H_p| \, ds \leq -j \log \lambda_j$$

and that this holds on intervals of at least length $\kappa_j$ at the endpoints of $\Gamma_j$. Next, we introduce the normalized principal and subprincipal symbols

$$\tilde{p} = p / |H_p| \quad \text{and} \quad p_s = p_0 / |H_p|$$

then we have that $H_p|_{\Gamma_j} \in C^\infty$ uniformly, $|H_p| = 1$ on $\Gamma_j$ and $dH_p|_{L_j}$ is uniformly bounded at $\Gamma_j$. We find that condition (4.5) becomes

$$\min_{\partial \Gamma_j} \int \text{Im} p_s \, ds \leq -j \log \lambda_j$$

Observe that because of condition (2.8) we have that $\partial \Gamma_j$ has two components since $\text{Im} p_s$ has opposite sign there, so $\Gamma_j$ is a uniformly embedded curve.

In the following we shall consider a fixed bicharacteristic $\Gamma_j$ and suppress the index $j$, so that $\Gamma = \Gamma_j$, $L = L_j$ and $\kappa = \kappa_j = \lambda^{-\varepsilon}$. Observe that the preparation will be uniform in $j$. Now $H_{\tilde{p}} \in C^\infty$ uniformly on $\Gamma$ but not in a neighborhood. By (2.3) we may define the first jet of $\tilde{p}$ at $\Gamma$ uniformly. Since $\Gamma \in C^\infty$ uniformly, we can choose local coordinates uniformly so that $\Gamma = \{(t,0) : t \in I \subset \mathbb{R}\}$ locally. In fact, by taking a local parametrization $\gamma(t)$ of $\Gamma$ with respect to the arclength and choosing the orthogonal space $M$ to the tangent vector of $\Gamma$ at a point $w_0$ with respect to some local Riemannian metric. Then $\mathbb{R} \times M \ni (t,w) \mapsto \gamma(t) + w$ is uniformly bounded in $C^\infty$ with uniformly bounded inverse near $(t_0,0)$ giving local coordinates near $\Gamma$. We can then define the first order Taylor term of $\tilde{p}$ at $\Gamma$ by

$$\varrho(t,w) = \partial_w \tilde{p}(t,0) \cdot w \quad w = (x,\tau,\xi)$$

This can be done locally, and by using a uniformly bounded partition of unity we obtain this in a fixed neighborhood of $\Gamma$. Going back to the original coordinates, we find that $\varrho \in C^\infty$ uniformly near $\Gamma$ such that $\tilde{p} - \varrho = O(d^2)$ where $d$ is the distance to $\Gamma$, but
the error is not uniformly bounded. By condition (2.3) we find that the second order derivatives of \( \tilde{p} \) along the Lagrangean space \( L \) at \( \Gamma \) is uniformly bounded.

By completing \( \tau = \varrho \) in (4.7) to a uniformly bounded homogeneous symplectic coordinate system near \( \Gamma \) and conjugating with the corresponding uniformly bounded Fourier integral operators we may assume that

\[
\Gamma = \{ (t, 0; 0, \xi_0) : t \in I \}
\]

for \( |\xi_0| = 1 \), some fixed interval \( I \ni 0 \), and \( \tilde{p} \cong \tau \) modulo second order terms at \( \Gamma \). The second order terms are not uniformly bounded, but \( d\nabla \tilde{p} \big|_L \) is uniformly bounded at \( \Gamma \) by (2.3). Since \( H_{\tilde{p}} = D_t \) on \( \Gamma \) we may obtain that \( L = \{ (t, x; 0, 0) \} \) at any given point at \( \Gamma \) by choosing linear symplectic coordinates. Let \( D_{\tilde{p}} \) be the \( x \) component of \( H_{\tilde{p}} \), then \( D_{\tilde{p}} \big|_{\Gamma} = H_{\tilde{p}} \big|_{\Gamma} = D_t \).

Let

\[
q(t, w) = |\nabla p(t, w)| \geq \kappa = \lambda^{-\varepsilon} \quad \text{on } \Gamma
\]

and extended so that \( q \) is homogeneous of degree \( 0 \). We shall change variables so that \( w = 0 \) corresponds to the point \( (0, 0, \xi_0) \) \( \in \Gamma \) given by (4.8). We have \( |\nabla \tilde{p}| \equiv 1 \) at \( \Gamma \), higher derivatives are not uniformly bounded but can be estimated by the using the metric

\[
g_{\varepsilon} = (dt^2 + |dw|^2)\lambda^{2\varepsilon}
\]

according to the following result.

**Proposition 4.1.** We have that \( q \) is a weight for \( g_{\varepsilon} \), \( q \in S(q, g_{\varepsilon}) \) and \( \tilde{p}(t, w) \in S(\lambda^{-\varepsilon}, g_{\varepsilon}) \) when \( |w| \leq c\lambda^{-\varepsilon} \), \( |\xi| \equiv 1 \) and \( t \in I \), for some \( c > 0 \).

Here the symbol classes are defined by \( f \in S(m, g_{\varepsilon}) \) if \( \partial^{\alpha} f = \mathcal{O}(m\lambda^{\alpha|\varepsilon|}) \), \( \forall \alpha \). Observe that \( |\xi| \equiv 1 \) in this domain and that by homogeneity \( \tilde{p} \in S(\lambda^{1-\varepsilon}, g_{\varepsilon}) \) and \( p = q\tilde{p} \in S(q\lambda^{1-\varepsilon}, g_{\varepsilon}) \) in homogeneous coordinates \( (x, \xi/\lambda) \) when \( |\xi| \cong \lambda \geq 1 \). Observe that \( b \in S_{1-\varepsilon, \varepsilon}^\mu \) is and only if \( b \in S(\lambda^\mu, g_{\varepsilon}) \) in homogeneous coordinates when \( |\xi| \equiv \lambda \geq 1 \). In fact, if \( z = (x, \xi/\lambda) \) then this means that \( \partial_x^\mu b = \mathcal{O}(\lambda^{\mu+\varepsilon}) \) when \( |\xi| \equiv \lambda \).

**Proof.** We shall use the previous chosen coordinates \( (t, w) \) so that \( \Gamma = \{ (t, 0) : t \in I \} \). Since \( \partial^2 p = \mathcal{O}(1) \), \( q \geq \lambda^{-\varepsilon} \) at \( \Gamma \) and

\[
\partial q = \nabla p \cdot (\partial \nabla p) / q \quad \text{when } q \neq 0
\]

we find that \( q(s, w) \cong q(t, 0) \) when \( |s - t| + |w| \leq c\lambda^{-\varepsilon}, c > 0 \), so \( q \) is a weight for \( g_{\varepsilon} \) there. When \( |w| \leq c\lambda^{-\varepsilon} \) we find that \( |p(t, w)| \lesssim q(t, 0)\lambda^{-\varepsilon}, |\nabla p(t, w)| \lesssim q(t, 0) \) and \( 1 \gtrsim q_{\varepsilon} \), which gives \( p \in S(q\lambda^{-\varepsilon}, g_{\varepsilon}) \) when \( |w| \leq c\lambda^{-\varepsilon} \).

We find from (4.10) that \( \partial q = \mathcal{O}(q_{\varepsilon}) \) when \( |w| \leq c\lambda^{-\varepsilon} \), since \( \nabla p \in S(q, g_{\varepsilon}) \) in that domain. By induction we obtain that \( q \in S(q, g_{\varepsilon}) \), which gives \( q^{-1} \in S(q^{-1}, g_{\varepsilon}) \) when \( |w| \leq c\lambda^{-\varepsilon} \) and the result.

Next, we put \( Q(t, w) = \lambda^\varepsilon \tilde{p}(t\lambda^{-\varepsilon}, w\lambda^{-\varepsilon}) \) for \( t \in I_{\varepsilon} = \{ t\lambda^\varepsilon : t \in I \} \). Then by Proposition 4.1 we find that \( Q \in C^\infty \) uniformly when \( |w| \lesssim 1 \) and \( t \in I_{\varepsilon} \), \( \partial_t Q \equiv 1 \) and \( |\partial_{x, \xi} Q| \equiv 0 \) when \( w = 0 \) and \( t \in I_{\varepsilon} \). Thus we find \( |\partial_t Q| \neq 0 \) for \( |w| \lesssim 1 \) and \( t \in I_{\varepsilon} \).
By using Taylor’s formula when $|\xi| \cong 1$, we can write $Q(t, x; \tau, \xi) = \tau + h(t, x; \tau, \xi)$ when $|w| \lesssim 1$ and $t \in I_\epsilon$, where $h = |\nabla h| = 0$ at $w = 0$. By using the Malgrange preparation theorem, we find
\[
\tau = a(t, w)(\tau + h(t, w)) + s(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\epsilon
\]
where $a$ and $s \in C^\infty$ uniformly, $a \neq 0$, $a = 1$ and $s = |\nabla s| = 0$ on $\Gamma$. In fact, this can be done locally in $t$ and by a uniform partition of unity globally in $t$. This gives
\[
(4.11) \quad a(t, w)Q(t, w) = \tau - s(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\epsilon
\]
In the original coordinates, we find that
\[
\lambda^\epsilon \tilde{p}(t, w) = a^{-1}(t\lambda^\epsilon, w\lambda^\epsilon)(\tau \lambda^\epsilon - s(t\lambda^\epsilon, x\lambda^\epsilon, \xi\lambda^\epsilon))
\]
and thus
\[
(4.12) \quad \tilde{p}(t, w) = b(t, w)(\tau - r(t, x, \xi)) \quad |w| \lesssim \lambda^{-\epsilon} \quad t \in I
\]
where $b \in S(1, g_\epsilon)$, $r(t, x, \xi) = \lambda^{-\epsilon} s(t\lambda^\epsilon, x\lambda^\epsilon, \xi\lambda^\epsilon) \in S(\lambda^{-\epsilon}, g_\epsilon)$, $b = 1$ and $r = |\nabla r| = 0$ on $\Gamma$. By condition (2.3) we also find that
\[
(4.13) \quad |d\nabla r|_L \leq C \quad \text{when } w = 0 \text{ and } t \in I
\]
Now $\tilde{p} = p/q$, where $q \in S(q, g_\epsilon)$ when $|\xi| \cong 1$. In the following, we shall denote by $\Gamma$ the rays in $T^*X$ that goes through the bicharacteristic. By extending by homogeneity we obtain that
\[
b^{-1}q^{-1}p(t, x; \tau, \xi) = \tau - r(t, x, \xi)
\]
where $b^{-1} \in S^0_{1-\epsilon, \epsilon}$, $q^{-1} \in S^\epsilon_{1-\epsilon, \epsilon}$ and $\tau - r \in S^1_{1-\epsilon, \epsilon}$ when $|\xi| \cong \lambda$ and the homogeneous distance $d$ to $\Gamma$ is less than $c|\xi|^{-\epsilon}$, $c > 0$. Observe that $b^{-1}$ and $q^{-1}$ are homogeneous of degree $0$ and $r$ is homogeneous of degree $1$.

Next, we take a cut-off function $\chi \in S^0_{1-\epsilon, \epsilon}$ supported where $d \lesssim |\xi|^{-\epsilon}$ so that $\chi = 1$ when $d \leq c|\xi|^{-\epsilon}$, $c > 0$. Then we let $B = \chi b^{-1}q^{-1} \in S^0_{1-\epsilon, \epsilon}$ and compose the pseudodifferential operator $B$ with $P^*$. Since $P^* \in \Psi^1_{1, 0}$ we obtain an asymptotic expansion of $BP^*$ in $S^1_{1+\epsilon, (1-\epsilon)}$ for $j = 0, 1, 2, \ldots$ when the homogeneous distance $d \lesssim |\xi|^{-\epsilon}$. But actually the symbol is in a better class. The principal symbol is
\[
\chi(\tau - r(t, x, \xi)) \in S^1_{1-\epsilon, \epsilon} \quad |\xi| \cong \lambda \quad d \lesssim \lambda^{-\epsilon}
\]
and the calculus gives that the subprincipal symbol is equal to
\[
(4.14) \quad \frac{i}{2} H_p(\chi b^{-1}q^{-1}) + \chi b^{-1}q^{-1}p_0
\]
where $p_0$ is the subprincipal symbol of $P^*$. As before, we shall use homogeneous coordinates when $|\xi| \cong \lambda$. Then Proposition 4.1 gives $p = q\tilde{p} \in S(q\lambda^{1-\epsilon}, g_\epsilon)$ and since $\chi b^{-1}q^{-1} \in S(q^{-1}, g_\epsilon)$ we find that (4.14) is in $S(\lambda^\epsilon, g_\epsilon)$ when $d \lesssim \lambda^{-\epsilon}$ and $|\xi| \cong \lambda$. The value of $H_p$ at $\Gamma$ is equal to $q\partial_t$ so the value of (4.14) is equal to
\[
(4.15) \quad \frac{1}{2i} \partial_t q/q + p_0/q = \frac{D_t|\nabla p|}{2|\nabla p|} + \frac{p_0}{|\nabla p|} \quad \text{at } \Gamma
\]
where $|\nabla p| = \sqrt{|\partial_x p|^2/|\xi|^2 + |\partial_\xi p|^2}$ is the homogeneous gradient, and the error of this approximation is bounded by $\lambda^{2\varepsilon}$ times the homogeneous distance $d$ to $\Gamma$. In fact, by Proposition 4.1 we have $H_p q^{-1} \in S(\lambda^{-\varepsilon}, g_\varepsilon)$ and $p_0/q \in S(q^{-1}, g_\varepsilon)$. Observe that $p_0/|\nabla p|$ is equal to the normalized subprincipal symbol of $P^\star$. This preparation can only be done in a $g_\varepsilon$ neighborhood of $\Gamma$ and we have to estimate the error terms.

**Definition 4.2.** For $R \in \Psi_{p,\delta}^{\mu}$, $\rho > \varepsilon$ and $\delta < 1 - \varepsilon$, we say that $T^\star X \ni (x, \xi) \notin \text{WF}_\varepsilon(R)$ if the symbol of $R$ is $O(|\xi|^{-N})$, $\forall N$, when the homogeneous distance to the ray $(x, \xi) : \rho \in \mathbb{R}_+$ is less than $c|\xi|^{-\varepsilon}$ for some $c > 0$.

By the calculus, this means that there exists $A \in \Psi^{*}_{1-\varepsilon, \delta}$ that is non-vanishing at a neighborhood of the ray such that $AR \in \Psi^{-N}$ for any $N$. This neighborhood is in fact the points with fixed bounded homogeneous distance with respect to the metric $g_\varepsilon$ when $\lambda \approx |\xi|$. It also follows from the calculus that this definition is invariant under composition with classical pseudodifferential operators and conjugation with elliptic homogeneous Fourier integral operators since the conjugated symbol is given by an asymptotic expansion. We also have that $\text{WF}_\varepsilon(R) \subset \text{WF}(R)$ when $R \in \Psi_{p,\delta}^{\mu}$.

Now we can use the Malgrange division theorem in order to make the lower order terms independent on $\tau$ when $d \lesssim \lambda^{-\varepsilon}$, starting with the subprincipal symbol $p_0 \in S_{1-\varepsilon, \delta}^{\varepsilon}$ of $BP^\star$ given by (4.14). By using homogeneity it suffices to do this when $|\xi| \approx 1$. Then rescaling as before so that $\tilde{p}_0(t, w) = \lambda^{-\varepsilon}p_0(t, w, \lambda^{-\varepsilon})$ we obtain that

$$
\tilde{p}_0(t, w) = \tilde{c}(t, w)(\tau - s(t, x, \xi)) + \tilde{q}_0(t, x, \xi) \quad |w| \lesssim 1 \quad t \in I_\varepsilon
$$

where $s$ is given by (4.11), and $\tilde{c}$ and $\tilde{q}_0$ are uniformly in $C^{\infty}$. This can be done locally and by a partition of unity globally in $t$ when $(x, \tau, \xi) = w = \mathcal{O}(1)$. Extending by homogeneity we find in the original coordinates that

$$
p_0(t, w) = c(t, w)(\tau - r(t, x, \xi)) + q_0(t, x, \xi) \quad d \lesssim \lambda^{-\varepsilon} \quad t \in I
$$

where $q_0(t, w) = \lambda^\varepsilon\tilde{q}_0(t^{\varepsilon}, w^{\varepsilon})$ and $c(t, w) = \lambda^{2\varepsilon}\tilde{c}(t^{\varepsilon}, w^{\varepsilon})$ are homogeneous of degree 0 and $-1$ respectively. We find that $c \in S_{1-\varepsilon, \delta}^{2\varepsilon}$ and $q_0 \in S_{1-\varepsilon, \delta}^{\varepsilon}$ when $|\xi| \approx \lambda$ and that $q_0 = p_0$ at $\Gamma$. Now the composition of $c$ and $\tau - r$ gives error terms that are in $S_{1-\varepsilon, \delta}^{3\varepsilon-1}$ when $|\xi| \approx \lambda$. Thus if $\varepsilon < 1/3$ then by multiplication with a pseudodifferential operator with symbol $1 - c$ we can make the principal symbol independent of $\tau$. By iterating this procedure we can successively make any lower order terms independent of $\tau$.

By applying the cut-off function $\chi$ we obtain the following result.

**Proposition 4.3.** By conjugating with an elliptic homogeneous Fourier integral operator we may obtain that $\Gamma$ is given by (4.8). If $0 < \varepsilon < 1/3$ then by multiplying with an homogeneous elliptic operator $B \in \Psi^{*}_{1-\varepsilon, \delta}$ we may obtain that $BP^\star = Q + R \in \Psi^{1+\varepsilon}_{1+\varepsilon, \delta}$ where $\Gamma \cap \text{WF}_\varepsilon(R) = \emptyset$, and the symbol of $Q$ is equal to

$$
\tau - r(t, x, \xi) + q_0(t, x, \xi) + r_0(t, x, \xi) \quad \text{when } d \lesssim \lambda^{-\varepsilon} \text{ and } t \in I
$$
Here \( r \in S^{1-\varepsilon}_{1-\varepsilon,\varepsilon}, \quad q_0 \in S^1_{1-\varepsilon,\varepsilon} \) and \( r_0 \in S^{32-1}_{1-\varepsilon,\varepsilon} \) when \( |\xi| \equiv \lambda, \quad r = |\nabla r| = 0 \) on \( \Gamma \), and \( q_0 \) is equal to
\[
(4.18) \quad \frac{D_t[\nabla p(t,0)]}{2|\nabla p(t,0)|} + \frac{p_0(t,0)}{|\nabla p(t,0)|} + O(\lambda^{2e}d) \quad \text{when } |\xi| \equiv \lambda \text{ and } d \lesssim \lambda^{-\varepsilon}
\]
where \( d \) is the homogeneous distance to \( \Gamma \) and \( |\nabla p| = \sqrt{|\partial_x p|^2/|\xi|^2 + |\partial_\xi p|^2} \) is the homogeneous gradient.

Observe that the integration of the term \( D_t[\nabla p(t,0)]/2|\nabla p(t,0)| \) in (4.18) will as before give terms that are \( O(\log(|\nabla p(t,0)|) = O(|\log(k)|)) \). These will therefore not affect condition (2.7).

Recall that \( L \) is a smooth section of Lagrangean spaces \( L(w) \subset T\Sigma \subset T_w^*\mathbb{R}^n \), \( w \in \Gamma \), such that the linearization of the Hamilton vector field \( H_p \) is in \( TL \) at \( \Gamma \). (Here we identify \( T(T^*\mathbb{R}^n) \) with \( T^*\mathbb{R}^n \).) By Proposition 4.3 we may assume that \( p(t,x,\tau,\xi) = \tau - r(t,x,\xi) \) and \( \Gamma = \{(t,0;0,\xi_0) : t \in I\} \), and we shall denote \( L(t) = L(w) \) for \( w = (t,0,\xi_0) \). We may choose symplectic coordinates so that \( L(0) = \{(t,x,0,0) : (t,x) \in \mathbb{R}^n\} \), then by condition (2.3) we find that \( \partial_\xi^2 r(0,0,\xi_0) \) is uniformly bounded. In fact, since \( H_p \in L \) at \( \Gamma \), the \( t \) lines must be in \( L \) and the restriction of \( L(s) \) to \( t = 0 \) gives a Lagrangean subspace \( L_s \) in the \((x,\xi)\) variables. Since \( \partial_t p = 0 \) on \( \Gamma \) we find by continuity that for small \( s \)
\[
L(s) = \{(t,x;0,A(s)x) : (t,x) \in \mathbb{R}^n\}
\]
where \( A(s) \) is real, continuous and symmetric and \( A(0) = 0 \). Since the linearization of the Hamilton vector field \( H_p \) at \( \Gamma \) is tangent to \( L \), we find that \( L \) is invariant under the flow of that linearization. Also, if \( s(t,x,\xi) = r(t,x,\xi) \) on \( L \) then the linearization of the difference at \((t,0)\) is independent of \( \tau \) and vanishes on \( L \) so its Hamilton vector field is in \( L \) but is transversal to \( H_p \). Thus, only the restriction of \( r(t,w) \) to \( L \) determines the evolution of \( t \mapsto L(t) \). For (4.19) this restriction is given by
\[
R(t,x) = r(t,x,A(t)x)
\]
thus \( \partial_\xi^2 R(t,0) \) is uniformly bounded. The linearized Hamilton vector field is
\[
\partial_t + \langle \partial_\xi^2 R(t,0)x, \partial_\xi \rangle
\]
\[
= \partial_t + (\langle \partial^2_\xi^2 r(t,0,\xi_0) + 2 \text{Re}(\partial_\xi \partial_\xi r(t,0,\xi_0)A) + A \partial^2_\xi^2 r(t,0,\xi_0)A \rangle x, \partial_\xi \rangle
\]
where \( \text{Re} B = (B + B^t)/2 \) is the symmetric part of \( B \). Applying this on \( \xi - A(t)x \), which vanishes identically on \( L(t) \), we obtain that
\[
-A'(t) + \partial^2_\xi^2 r(t,0,\xi_0) + 2 \text{Re}(\partial_\xi \partial_\xi r(t,0,\xi_0)A(t)) + A(t) \partial^2_\xi^2 r(t,0,\xi_0)A(t) = 0
\]
which gives the evolution of \( L(t) \). The equation
\[
(4.20) \quad A'(t) = \partial^2_\xi^2 r(t,0,\xi_0) + 2 \text{Re}(\partial_\xi \partial_\xi r(t,0,\xi_0)A(t)) + A(t) \partial^2_\xi^2 r(t,0,\xi_0)A(t) \quad A(0) = 0
\]
is locally uniquely solvable and the right-hand side is uniformly bounded as long as \( A \) is bounded. Observe that if \( \partial^2_\xi^2 r(t,0,\xi_0) \equiv 0 \) then \( A(t) \equiv 0 \) by uniqueness. But since (4.20) is non-linear, the solution could become unbounded: \( \|A(s)\| \to \infty \) as \( s \to t_1 \in I \). This
means that the angle between $L(s) = \{ (t,x;0,A(s)x) : (t,x) \in \mathbb{R}^n \}$ and the vertical space $\{ (t;0,0,\xi) : (t,\xi) \in \mathbb{R}^n \}$ goes to zero, but this is just a coordinate singularity.

In general, we define for each $t$

$$R(t,x,\xi) = r(t,x,\xi_0 + \xi) \quad \text{when} \ (0,x;0,\xi) \in L(t)$$

so that $R$ is constant along the normal of $L_t$ in $\{ (0,x;0,\xi) : (x,\xi) \in T^*\mathbb{R}^{n-1} \}$, defined by some local choice of Riemannian metric. Then $R = r$ on $L$ and we find from (2.3) that

$$\tau - \langle R(t)z, z \rangle/2 \in C^\infty \quad \text{uniformly}$$

if $z = (x,\xi)$ and $R(t) = \partial^2_z R(t,0,0)|_L(t)$. Now we can complete $t$ and (4.22) to a uniform homogeneous symplectic coordinates system so that $\Gamma = \{ (t,0,\xi_0) : t \in I \}$ and $L(0) = \{ (t,x,0,0) : (t,x) \in \mathbb{R}^n \}$ In fact, we may let $x$ and $\xi$ have the same values when $t = 0$ and clearly $H_1$ is not changed on $\Gamma$. Then we find that $p = \tau - r_1$, where $r_1(t,x,\xi)$ is independent of $\tau$ and satisfies $\partial^2_z r_1(t,0,0)|_L(t) \equiv 0$. This follows since

$$p(t,x,\tau,\xi) = \tau - \langle R(t)z, z \rangle/2 - r_1(t,x,\xi)$$

and $\partial_z r_1 = -\{ t,r_1 \}$ is invariant under changes of symplectic coordinates. Similarly we find that the lower order terms $p_j(t,x,\xi)$ are independent of $\tau$ for $j \leq 0$. Since the evolution of $L$ is determined by the second order derivatives of the principal symbol along $L$ by Example 2.5, we find that $L(t) \equiv \{ (t,x,0,0) : (t,x) \in \mathbb{R}^n \}$. Changing notation so that $r = r_1$ and $p(t,x,\tau,\xi) = \tau - r(t,x,\xi)$ we obtain the following result.

**Proposition 4.4.** We may assume that the symplectic coordinates are chosen so that the grazing Lagrangean space $L(w) \equiv \{ (t,x,0,0) : (t,x) \in \mathbb{R}^n \}, \forall w \in \Gamma$, which implies that $\partial^2_z r(t,0) \equiv 0$.

We shall apply the adjoint $P^*$ of the operator on the form in Proposition 4.3 on approximate solutions on the form

$$u_\lambda(t,x) = \exp(i\lambda(t,x,\xi_0) + \omega(t,x))) \sum_{j=0}^M \varphi_j(t,x) \lambda^{-j\theta}$$

where the phase function $\omega(t,\cdot) \in S(\lambda^{-7\varepsilon},g_{3\varepsilon})$ is real valued such that $\partial^2_{xx} \omega(t,0) \equiv 0$ and $\varphi_j(t,x) \in S(1,g_{\delta})$ has support where $|x| \lesssim \lambda^{-\delta}$. Here $\delta$, $\varepsilon$ and $\theta$ are positive constants to be determined later. The phase function $\omega(t,x)$ will be constructed in Section 5, see Proposition 5.1. Observe that we have assumed that $\varepsilon < 1/3$ in Proposition 4.3, but we shall impose further restrictions on $\varepsilon$ later on. We shall assume that $\varepsilon + \delta < 1$, then if $p(t,x,\xi) \in \Psi^{1-\varepsilon}_1$, we obtain the formal expansion (see [7, Chapter VI, Theorem 3.1])

$$p(t,x,D_x)(\exp(i\lambda(t,x,\xi_0) + \omega(t,x)))\varphi(t,x)
\sim \exp(i\lambda(t,x,\xi_0) + \omega(t,x))) \sum_{\alpha} \partial^\alpha_x p(t,x,\lambda(\xi_0 + \partial_x \omega(t,x))) R_{\alpha}(\omega,\lambda,D)\varphi(t,x)/\alpha!$$

where $R_{\alpha}(\omega,\lambda,D)\varphi(t,x) = D_y^\alpha (\exp(i\lambda\omega(t,x,y))\varphi(t,y)|_{y=x}$ with

$$\omega(t,x,y) = \omega(t,y) - \omega(t,x) + (x-y)\partial_x \omega(t,x)$$
Using this we find by homogeneity that
\begin{equation}
P^s(t, x, D)(\exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x)))\varphi(t, x)) \\
\sim \exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x)))\left(\lambda(D_0\omega(t, x) - r(t, x, \xi_0 + \partial_x\omega))\varphi(t, x) \right) \\
+ D_0\varphi(t, x) - \sum_j \partial_{\xi_j} r(t, x, \xi_0 + \partial_x\omega)D_{x_j}\varphi(t, x) + g_0(t, x, \xi_0 + \partial_x\omega)\varphi(t, x) \\
+ \lambda^{-1}\sum_{j,k} \partial_{\xi_j}\partial_{\xi_k} r(t, x, \xi_0 + \partial_x\omega)(D_{x_j}D_{x_k}\varphi(t, x) + \varphi(t, x)\lambda D_{x_j}D_{x_k}\omega(t, x))/2 + \ldots)
\end{equation}

which gives an expansion in $S(\lambda^{1-\varepsilon-j(1-\delta-\varepsilon)}, g_3)$, $j \geq 0$, if $\delta + \varepsilon < 1$ and $\varepsilon \leq 1/4$. In fact, $|\xi| \approx \lambda$ every $\xi$ derivative on terms in $S(1, \varepsilon, \varepsilon)$ gives a factor that is $\mathcal{O}(\lambda^{-1})$ and every $x$ derivative of $\varphi$ gives a factor that is $\mathcal{O}(\lambda^0)$. A factor $\lambda D_0\omega$ requires $|\alpha| \geq 2$ number of $\xi$ derivatives of a term in the expansion of $P^s$. Since $|\alpha| \geq 2$ we get a factor that is $\mathcal{O}(\lambda^{1-\varepsilon-|\alpha|(1-4\varepsilon)}) = \mathcal{O}(\lambda^{\varepsilon-1})$ if $\varepsilon \leq 1/4$. Similarly, the expansion coming from terms in $P^s$ that have symbols in $S(1, \varepsilon, \varepsilon)$ gives an expansion in $S(1, \varepsilon, \varepsilon)$, $j \geq 0$. Thus, if $\delta + \varepsilon < 2/3$ and $\varepsilon \leq 1/4$ then the terms in the expansion have negative powers of $\lambda$ except the terms in (4.25), and for the last ones we find that
\begin{equation}
\lambda^{-1}\sum_{j,k} \partial_{\xi_j}\partial_{\xi_k} r(t, x, \xi_0 + \partial_x\omega)(D_{x_j}D_{x_k}\varphi + \lambda\varphi D_{x_j}D_{x_k}\omega) \\
= \sum_{j,k} \partial_{\xi_j}\partial_{\xi_k} r(t, x, \xi_0 + \partial_x\omega)(\lambda^{-1}D_{x_j}D_{x_k}\varphi + \varphi D_{x_j}D_{x_k}\omega) = \mathcal{O}(\lambda^{2\delta-\varepsilon-1} + \lambda^{3\varepsilon-\delta})
\end{equation}

In fact, $\partial_{\xi_j}\partial_{\xi_k} r(t, x, \xi_0 + \partial_x\omega) = \mathcal{O}(\lambda^\delta)$ and $D_{x_j}D_{x_k}\omega = \mathcal{O}(\lambda^{2\varepsilon}d)$ when $\varphi \neq 0$, since we have $D_{x_j}D_{x_k}\omega = 0$ when $x = 0$, and $d = \mathcal{O}(\lambda^{-\delta})$ in supp $\varphi$.

The error terms in (4.26) are of equal size if $2\delta + \varepsilon - 1 = 3\varepsilon - \delta$, i.e., $\delta = (1 + 2\varepsilon)/3$. We then obtain $3\varepsilon - \delta = (7\varepsilon - 1)/3 < 0$ if $\varepsilon < 1/7$. Observe that in this case $1 - \delta - \varepsilon = (2 - 5\varepsilon)/3$ and $\delta + \varepsilon < 2/3$ since $\varepsilon < 1/5$. Thus we obtain the following result.

**Proposition 4.5.** Assume that $\omega(t, \cdot) \in S(\lambda^{-7\varepsilon}, g_3\varepsilon)$ is real valued and $\partial_0^{2}\omega(t, 0) \equiv 0$, $\varphi_j(t, x) \in S(1, g_3)$ has support where $|x| \lesssim \lambda^{-\delta}$, for $\delta$, $\varepsilon > 0$. If $\delta = (1 + 2\varepsilon)/3$ and $\varepsilon < 1/7$, then (4.25) has an expansion in $S(\lambda^{-\varepsilon-7(2-5\varepsilon)/3}, g_3)$, $j \geq 0$, and is equal to
\begin{equation}
\exp(-i\lambda(\langle x, \xi_0 \rangle + \omega(t, x)))P^s(t, x, D)(\exp(i\lambda(\langle x, \xi_0 \rangle + \omega(t, x)))\varphi(t, x)) \\
\sim \lambda(D_0\omega(t, x) - r(t, x, \partial_x\omega))\varphi(t, x) \\
+ D_0\varphi(t, x) - \sum_j \partial_{\xi_j} r(t, x, \partial_x\omega)D_{x_j}\varphi(t, x) + g_0(t, x, \partial_x\omega)\varphi(t, x)
\end{equation}

modulo terms that are $\mathcal{O}(\lambda^{7\varepsilon-1})$.

In Section 6 we shall choose $\varepsilon = \varrho = 1/10$ which gives $\delta = 2/5$, $1 - \delta - \varepsilon = 1/2$ and $(7\varepsilon - 1)/3 = -1/10$.

5. **The eikonal equation**

The first term in the expansion is the eikonal equation
\begin{equation}
\partial_t\omega - r(t, x, \xi_0 + \partial_x\omega) = 0 \quad \omega(0, x) = 0
\end{equation}
where $|\xi_0| \equiv 1$. This we can solve by using the Hamilton-Jacobi equations:

$$
\begin{cases}
\partial_t x = \partial_t r(t, x, \xi_0 + \xi) \\
\partial_t \xi = -\partial_x r(t, x, \xi_0 + \xi)
\end{cases}
$$

(5.2)

with $(x(0), \xi(0)) = (x, 0)$, and letting $\partial_x \omega = \xi$ and $\partial_t \omega = r(t, x, \partial_x \omega)$.

We shall solve the Hamilton-Jacobi equations by scaling. Recall that $r(t, x, \xi) \in S(\lambda^{-e}, g_\varepsilon)$ for some chosen $0 < \varepsilon < 1/3$ in homogeneous coordinates by Proposition 4.3. By Proposition 4.4 we may assume that $L(t) \equiv \{ (t, x, 0, 0) \}, \forall t$, thus $\partial_x^2 r = 0$ on $\Gamma$. Since $r$, $\partial_r$ and $\partial_x^2 r$ vanish on $\Gamma$, Taylor’s formula gives

$$
\partial_t r(t, x, \xi_0 + \xi) = \partial_x \partial_t r(t, 0, \xi_0) x + \partial_x^2 r(t, 0, \xi_0) \xi + \langle \varrho_1(t, x, \xi) w, w \rangle
$$

(5.3)

where $w = (x, \xi)$, $\partial_x \partial_t r(t, 0, 0) = O(1)$ by (2.3), $\partial_x^2 r(t, 0, \xi_0) = O(\lambda^e)$ and $\varrho_1 \in S(\lambda^{2e}, g_\varepsilon)$.

Similarly, we find

$$
\partial_x r(t, x, \xi_0 + \xi) = \partial_x \partial_t r(t, 0, \xi_0) x + \langle \varrho_2(t, x, \xi) w, w \rangle
$$

(5.4)

where $\varrho_2 \in S(\lambda^{2e}, g_\varepsilon)$.

Now we put $(x, \xi) = (y \lambda^{-3e}, \eta \lambda^{-4e})$. Then by using (5.3) and (5.4) we find that (5.2) transforms into

$$
\begin{cases}
\partial_t y = B(t) y + C(t) \eta + \sigma_1(t, z) \\
\partial_t \eta = -B(t) \eta + \sigma_2(t, z)
\end{cases}
$$

(5.5)

where $z = (y, \eta)$, $B(t) = \partial_x \partial_x r(t, 0, \xi_0)$ and $C(t) = \lambda^{-e} \partial_x^2 r(t, 0, \xi_0)$ are uniformly bounded, and

$$
\begin{cases}
\sigma_1(t, y, \eta) = \lambda^{-3e} \langle \varrho_1(t, y \lambda^{-3e}, \eta \lambda^{-4e}) (y, \lambda^{-e} \eta), (y, \lambda^{-e} \eta) \rangle \\
\sigma_2(t, y, \eta) = -\lambda^{-2e} \langle \varrho_2(t, y \lambda^{-3e}, \eta \lambda^{-4e}) (y, \lambda^{-e} \eta), (y, \lambda^{-e} \eta) \rangle
\end{cases}
$$

(5.6)

are uniformly bounded in $C^\infty$ and vanishes of second order in $z$. Then (5.5) has a uniformly bounded $C^\infty$ solution if $z(0)$ is uniformly bounded. This means that if $x(0) = O(\lambda^{-3e})$ and $\xi(0) = 0$ then we find $x = O(\lambda^{-3e})$ and $\partial_x \omega = \xi = O(\lambda^{-4e})$ for any $t \in I$. The scaling gives that $\partial_x^a \partial_x \omega = O(\lambda^{-4+3a|t|})$ when $|x| \lesssim \lambda^{-3e}$. By choosing $\omega(t, 0) \equiv 0$ we obtain that $\omega = O(\lambda^{-7e})$ when $|x| \lesssim \lambda^{-3e}$, thus $\omega(t, \cdot) \in S(\lambda^{-7e}, g_{3e})$.

Since $\nabla r = 0$ on $\Gamma$ we find that $\partial_t x = \partial_t \xi = 0$ when $x = \xi = 0$ so $\partial_x \omega(t, 0) \equiv 0$. By differentiating (5.1) twice we find that

$$
\partial_t \partial_x^2 \omega(t, 0) = 2 \text{Re} \left( \partial_x \partial_x r(t, 0, \xi_0) \partial_x^2 \omega(t, 0) \right) + \partial_x^2 \omega(t, 0) \partial_x^2 r(t, 0, \xi_0) \partial_x^2 \omega(t, 0)
$$

because $\partial_x \omega(t, 0) = \partial_x r(t, 0, \xi_0) = \partial_x^2 r(t, 0, \xi_0) = 0$. Since $\partial_x^2 \omega(0, x) \equiv 0$ we find by uniqueness that $\partial_x^2 \omega(t, 0) \equiv 0$. This gives that $\partial_t \omega = O(\lambda^{-7e})$ when $|x| \lesssim \lambda^{-3e}$, and we obtain the following result.

**Proposition 5.1.** Let $0 < \varepsilon < 1/3$, and assume that Propositions 4.3 and 4.4 hold. Then we can find $\omega(t, \cdot) \in S(\lambda^{-7e}, g_{3e})$ satisfying $\partial_t \omega = r(t, x, \xi_0 + \partial_x \omega)$ when $|x| \lesssim \lambda^{-3e}$ and $t \in I$ such that $\partial_x^2 \omega(t, 0) \equiv 0$. We find that the values of $(t, x, \xi_0 + \partial_t x \omega(t, x))$ are in a $g_{3e}$ neighborhood of $\Gamma$ when $|x| \lesssim \lambda^{-3e}$ and $t \in I$. 

6. The transport equations

The next term in (4.27) is the transport equation, which is equal to

\[ D_p \varphi_0 + q_0 \varphi_0 = 0 \quad \text{at } \Gamma \]

where \( D_p = D_t - \sum_j \partial_{t_j} r(t, x, \xi_0 + \partial_x \omega) D_{x_j} = D_t \) when \( x = 0 \) and

\[ q_0(t) = D_t |\nabla p(t, 0, \xi_0)| / 2|\nabla p(t, 0, \xi_0)| + p_0(t, 0, \xi_0) / |\nabla p(t, 0, \xi_0)| = O(\lambda^\epsilon) \]

modulo \( O(\lambda^{2\epsilon} |x|) \) when \( |x| \lesssim \lambda^{-\delta} \) by (4.18).

**Lemma 6.1.** We have that

\[ D_p = D_t + \sum_j \langle a_j(t) \cdot x \rangle D_{x_j} + R(t, x, D) \]

where \( \mathbb{R}^{n-1} \ni a_j(t) = O(1) \) and \( R(t, x, D) \) is a first order differential operator in \( x \) with coefficients that are \( O(\lambda^{3\epsilon} |x|^2) \) when \( |x| \lesssim \lambda^{-3\epsilon} \).

**Proof.** Since \( \partial_x^2 \omega(t, 0) \equiv 0 \) we have from Taylor’s formula that \( a_j(t) = -\partial_x \partial_{t_j} r(t, 0, \xi_0) \) which is uniformly bounded by (4.13). The coefficients of the error term \( R \) are given by the second order \( x \) derivatives of the coefficients of \( D_p \) which are

\[ \partial_x^2 \partial_x r + 2 \text{Re} \left( \partial_x \partial_x^2 r \partial_x^2 \omega \right) + \partial_x \partial_x^2 \omega \partial_x^2 r \partial_x^2 \omega + \partial_x \partial_x^2 \omega \partial_x^2 \omega = O(\lambda^{3\epsilon}) \]

when \( |x| \lesssim \lambda^{-3\epsilon} \) by Propositions 4.3 and 5.1, which proves the result. \( \square \)

We shall change of variables by solving

\[ \partial_t x_j = \langle a_j(t) \cdot x \rangle \quad \forall j \]

then \( D_t + \sum_j \langle a_j(t) \cdot x \rangle D_{x_j} \) is transformed into \( D_t \). The change of variables is uniformly bounded since \( a_j = O(1) \), so it preserves the neighborhoods \( |x| \lesssim \lambda^{-\nu} \) and symbol classes \( S(\lambda^\mu, g_\nu), \forall \mu, \nu \). We shall then solve the approximate transport equation

\[ D_t \varphi_0 + q_0(t) \varphi_0 = 0 \]

where \( \varphi_0(0, x) \in S(1, g_\delta) \) is supported where \( |x| \lesssim \lambda^{-\delta} \) and \( q_0(t) \) is given by (6.2). If \( \lambda^{-\delta} \ll \lambda^{-3\epsilon} \) then by Lemma 6.1 the approximation errors will be in \( S(\lambda^{3\epsilon-\delta}, g_\delta) \), so we will assume \( \delta > 3\epsilon \). In fact, since \( \partial_x \) maps \( S(1, g_\delta) \) into \( S(\lambda^\delta, g_\delta) \) and \( |x| \lesssim \lambda^{-\delta} \), we find \( R \varphi_0 \in S(\lambda^{3\epsilon-\delta}, g_\delta) \). If we put \( \delta = 4\epsilon \) then the approximation errors in the transport equation will be \( O(\lambda^{-\epsilon}) \).

If we choose the initial data \( \varphi_0(0, x) = \phi_0(x) = \varphi(\lambda^\delta x) \), where \( \varphi \in C_0^\infty \) satisfies \( \varphi(0) = 1 \), we obtain the solution

\[ \varphi_0(t, x) = \phi_0(x) \exp(-iB(t)) \]

where \( B' = q_0 \) and \( B(0) = 0 \). By condition (4.2) we find that \( |\varphi_0(t, x)| \leq |\varphi(\lambda^\delta x)| \), so \( |x| \lesssim \lambda^{-\delta} \) in supp \( \varphi_0 \), which also holds in the original \( x \) coordinates. Observe that \( D_x^\alpha \varphi_0 = O(\lambda^{\delta|\alpha|}) \), \( \forall \alpha \), and we have from the transport equation that \( D_t \varphi_0 = -q_0 \varphi_0 = O(\lambda^\epsilon) \) by (6.2). Since \( D_t q_0 = O(\lambda^{(k+1)}) \) by Proposition 4.3 we find by induction that \( \varphi_0 \in S(1, g_\delta) \).
After solving the eikonal equation and the approximate transport equation, we find from Proposition 4.5 that the terms in the expansion (4.27) are $O(\lambda^{(7\varepsilon-1)/3}) + O(\lambda^{-\varepsilon})$, if $\varepsilon < 1/7$ and $\delta = (1 + 2\varepsilon)/3 = 4\varepsilon$, and all the terms contain the factor $\exp(-iB(t))$. We take $\varepsilon = 1/10$, $\delta = 2/5$ which gives $(7\varepsilon - 1)/3 = -\varepsilon = -1/10$. Then the expansion in Proposition 4.5 is in multiples of $\lambda^{-1/2}$, but since the terms of (4.27) are $O(\lambda^{-1/10})$ we will take $\varrho = 1/10$.

Thus the approximate transport equation for $\varphi_1$ is
\begin{equation}
(6.5) \quad D_t\varphi_1 + q_0(t)\varphi_1 = \lambda^{1/10}R_1 \exp(-iB(t)) \quad \text{at } \Gamma
\end{equation}
where $R_1$ is uniformly bounded in the symbol class $S(\lambda^{-1/10},g_{2/5})$ and is supported where $|x| \lesssim \lambda^{-2/5}$. In fact, $R_1$ contains both the error terms from the transport equation (6.1) for $\varphi_0$ and the terms that are $O(\lambda^{-1/10})$ in (4.27). By putting
\[ \varphi_1(t,x) = \exp(-iB(t))\phi_1(t,x) \]
the transport equation reduces to solving
\begin{equation}
(6.6) \quad D_t\phi_1 = \lambda^{1/10}R_1
\end{equation}
with initial values $\phi_1(0,x) = 0$. Then $\phi_1 \in S(1,g_{2/5})$ will have support where $|x| \lesssim \lambda^{-2/5}$.

Similarly, the general term in the expansion is $\varphi_k\lambda^{-k/10}$ where $\varphi_k$ will solve the approximate transport equation
\begin{equation}
(6.7) \quad D_t\varphi_k + q_0(t)\varphi_k = \lambda^{k/10}R_k \exp(iB(t)) \quad k \geq 1
\end{equation}
with $R_k$ is uniformly bounded in the symbol class $S(\lambda^{-k/10},g_{2/5})$ and is supported where $|x| \lesssim \lambda^{-2/5}$. In fact, $R_k$ contains the error terms from the transport equation (6.1) for $\varphi_{k-1}$ and also the terms that are $O(\lambda^{-k/10})$ in (4.27). Taking $\varphi_k = \exp(-iB(t))\phi_k$ we obtain the equation
\begin{equation}
(6.8) \quad D_t\phi_k = \lambda^{k/10}R_k \in S(1,g_{2/5})
\end{equation}
with initial values $\phi_k(0,x) = 0$, which can be solved with $\phi_k \in S(1,g_{2/5})$ uniformly having support where $|x| \lesssim \lambda^{-2/5}$. Proceeding we obtain an solution modulo $O(\lambda^{-N/10})$ for any $N$.

**Proposition 6.2.** Choosing $\delta = 2/5$, $\varepsilon = 1/10$ and $\varrho = 1/10$ we can solve the transport equations (6.1) and (6.7) with $\varphi_k \in S(1,g_{2/5})$, $\forall k$, such that $\varphi_0(0,0) = 1$ and $\varphi_k(0,x) \equiv 0$, $k \geq 1$.

Now, we get localization in $x$ from the initial values and the transport equation. To get localization in $t$ we use that $\text{Im } B(t) \leq 0$. Then we find that $\text{Re}(-iB) \leq 0$ with equality at $t = 0$. Near $\partial \Gamma$ we have for large $j$ that that $\text{Re}(-iB(t)) \ll -\log \lambda$ in an interval of length $O(\lambda^{-\varepsilon}) = O(\lambda^{-1/10})$ by (4.3) and (4.6). Thus by applying a cut-off function
\[ \chi(t) \in S(1,\lambda^{1/5}dt^2) \subset S(1,g_{2/5}) \]
such that $\chi(0) = 1$ and $\chi'(t)$ is supported where (4.5) holds, i.e., where $\varphi_k = O(\lambda^{-N})$, $\forall k$, we obtain a solution modulo $O(\lambda^{-N})$ for any $N$. In fact, if $w_\lambda$ is defined by (4.23)
and $Q$ by Proposition 4.3 then
\[ Q\chi u_\lambda = \chi Qu_\lambda + [Q, \chi]u_\lambda \]
where $[Q, \chi] = D_t\chi$ is supported where $u_\lambda = O(\lambda^{-N})$ which gives terms that are $O(\lambda^{-N})$, \forall $N$. Thus, by solving the eikonal equation (5.1) for $\omega$ and the transport equations (6.7) for $\varphi_k$ we obtain that $Q\chi u_\lambda = O(\lambda^{-N})$ for any $N$ and we get the following result.

**Remark 6.3.** In Proposition 6.2 we may assume that $\varphi_k(t, x) = \phi_k(\lambda^{1/10}t, \lambda^{2/5}x) \in S(1, g_{2/5})$, $k \geq 0$, where $\phi_k \in C^\infty_0$ has support where $|x| \lesssim 1$ and $|t| \lesssim \lambda^{1/10} \leq \lambda^{2/5}$, $\lambda \geq 1$.

### 7. The proof of Theorem 2.8

For the proof we will need the following modification of [4, Lemma 26.4.14]. Recall that $\mathcal{D}'_\Gamma^\dagger = \{ u \in \mathcal{D}' : \text{WF}(u) \subset \Gamma \}$, and that $\|u\|_{(k)}$ is the $L^2$ Sobolev norm of order $k$ of $u \in C^\infty_0$.

**Lemma 7.1.** Let
\[ u_\lambda(x) = \lambda^{(n-1)\delta/2} \exp(i\lambda^e \omega(\lambda^\varepsilon x)) \sum_{j=0}^M \varphi_j(\lambda^\delta x) \lambda^{-j\kappa} \]
with $\omega \in C^\infty(\mathbb{R}^n)$ satisfying $\text{Im} \omega \leq 0$ and $|d\omega| \geq c > 0$, $\varphi_j \in C^\infty_0(\mathbb{R}^n)$, $\lambda \geq 1$, $\varepsilon$, $\delta$, $\kappa$ and $\varrho$ are positive such that $\delta < \varepsilon + \varrho$. Here $\omega$ and $\varphi_j$ may depend on $\lambda$ but uniformly, and $\varphi_j$ has fixed compact support in all but one of the variables, for which the support is bounded by $C\lambda^\delta$. Then for any integer $N$ we have
\[ \|u_\lambda\|_{(-N)} \leq C\lambda^{-N(\varepsilon+\varrho)} \]
If $\varphi_0(x_0) \neq 0$ and $\text{Im} \omega(x_0) = 0$ for some $x_0$ then there exists $c > 0$ so that
\[ \|u_\lambda\|_{(-N)} \geq c\lambda^{-(N+N/2)(\varepsilon+\varrho)+(n-1)\delta/2} \]
Let $\Sigma = \bigcap_{\lambda \geq 1} \bigcup_j \text{supp } \varphi_j(\lambda^\delta x)$ and let $\Gamma$ be the cone generated by
\[ \{ (x, \partial_\omega(x)), \ x \in \Sigma, \ \text{Im} \omega(x) = 0 \} \]
then for any real $m$ we find $\lambda^m u_\lambda \to 0$ in $\mathcal{D}'_\Gamma^\dagger$ so $\lambda^m Au_\lambda \to 0$ in $C^\infty$ if $A$ is a pseudodifferential operator such that $\text{WF}(A) \cap \Gamma = \emptyset$. The estimates are uniform if $\omega \in C^\infty$ with fixed lower bound on $|d\text{Re} \omega|$, and $\varphi_j \in C^\infty$ uniformly.

We shall use Lemma 7.1 for $u_\lambda$ in (4.23), then $\omega$ will be real valued and $\Gamma$ in (7.4) will be the bicharacteristic $\Gamma_j$ converging to a limit bicharacteristic.

**Proof of Lemma 7.1.** We shall adapt the proof of [4, Lemma 26.4.14] to this case. By making the change of variables $y = \lambda^\varepsilon x$ we find that
\[ \hat{u}_\lambda(\xi) = \lambda^{(n-1)\delta/2-n\varepsilon} \sum_{j=0}^M \lambda^{-j\kappa} \int e^{i(\lambda^\varepsilon \omega(y) - (\xi/\lambda^\varepsilon))} \varphi_j(\lambda^{\delta-\varepsilon} y) \, dy \]
Let $U$ be a neighborhood of the projection on the second component of the set in (7.4). When $\xi/\lambda^{\varepsilon+\varrho} \notin U$ then for $\lambda \gg 1$ we have that

$$ \bigcup_j \text{supp } \varphi_j(\lambda^\delta x) \ni x \mapsto (\lambda^\varepsilon \omega(y) - \langle y, \xi/\lambda^\gamma \rangle)/(\lambda^\varepsilon + |\xi|/\lambda^\gamma) $$

$$ = (\omega(y) - \langle y, \xi/\lambda^{\varepsilon+\varrho} \rangle)/(1 + |\xi|/\lambda^{\varepsilon+\varrho}) $$

is in a compact set of functions with non-negative imaginary part with a fixed lower bound on the gradient of the real part. Thus, by integrating in part in (7.5) we find for any positive integer $m$ that

$$(7.6) \quad |\hat{u}_\lambda(\xi)| \leq C_m \lambda^{(n-1)\delta/2 + m(\delta-\varepsilon)}(\lambda^\varepsilon + |\xi|/\lambda^\gamma)^{-m} \quad |\xi/\lambda^{\varepsilon+\varrho}| \notin U \quad \lambda \gg 1$$

This gives any negative power of $\lambda$ for $m$ large enough since $\delta < \varepsilon + \varrho$. If $V$ is bounded and $0 \notin V$ then since $u_\lambda$ is uniformly bounded in $L^2$ we find

$$ \int_\tau u_\lambda(\xi)|^2(1 + |\xi|^2)^{-N} d\xi \leq C_V \tau^{-2N} $$

Using this estimate with $\tau = \lambda^{\varepsilon+\varrho}$ together with the estimate (7.6) we obtain (7.2). If $\chi \in C_0^\infty$ then we may apply (7.6) to $\chi u_\lambda$, thus we find for any positive integer $j$ that

$$ |\hat{\chi u_\lambda}(\xi)| \leq C_j \lambda^{(n-1)\delta/2 + j(\delta-\varepsilon)}(\lambda^\varepsilon + |\xi|/\lambda^\gamma)^{-j} \quad \chi \in W \quad \lambda \gg 1 $$

if $W$ is any closed cone with $\Gamma \cap (\text{supp } \chi \times W) = \emptyset$. Thus we find that $\lambda^m u_\lambda \to 0$ in $D'_\Gamma$ for every $m$. To prove (7.3) we assume $x_0 = 0$ and take $\psi \in C_0^\infty$. If $\omega(0) = 0$ and $\varphi(0) \neq 0$ we find

$$ \lambda^{n(\varepsilon+\varrho)-(n-1)\delta/2} \langle u_\lambda, \psi(\lambda^{\varepsilon+\varrho}) \rangle = \int e^{\lambda^\varepsilon \omega(x/\lambda^\gamma)}\psi(x) \sum_j \varphi_j(x/\lambda^{\varepsilon+\varrho})\lambda^{-j\kappa} dx $$

$$ \to \int e^{i\langle \Re \partial_x \omega(0), x \rangle} \psi(x) \varphi(0) dx $$

which is not equal to zero for some suitable $\psi \in C_0^\infty$. In fact, we have $\varphi_j(x/\lambda^{\varepsilon+\varrho}) = \varphi_j(0) + O(\lambda^{\delta-\varepsilon}) \to \varphi_j(0)$ when $\lambda \to \infty$, because $\delta < \varepsilon + \varrho$. Since

$$ \|\psi(\lambda^{\varepsilon+\varrho})\|_{(N)} \leq C\lambda^{(N-n/2)(\varepsilon+\varrho)} $$

we obtain that $c \leq \lambda^{(N+\frac{n}{2})(\varepsilon+\varrho)-(n-1)\delta/2}$ which gives (7.3) and the lemma. \hfill $\square$

**Proof of Theorem 2.8.** Assume that $\Gamma$ is a limit bicharacteristic of $P$. We are going to show that (2.10) does not hold for any $\nu$, $N$ and pseudodifferential operator $A$ such that $\Gamma \cap \text{WF}(A) = \emptyset$. This means that there exists approximate solutions $u_j \in C_0^\infty$ to $P^* u_j \equiv 0$ such that

$$(7.7) \quad \|u_j\|_{(-N)} + \|u_j\|_{(-N-n)} + \|Au_j\|_{(0)} \to \infty \quad \text{when } j \to \infty$$

which will contradict the local solvability of $P$ at $\Gamma$ by Remark 2.9.

Let $\Gamma_j$ be a sequence of bicharacteristics of $p$ that converges to $\Gamma \subset \Sigma_2$. Since the conditions and conclusions are invariant under symplectic changes of homogeneous coordinates
and multiplication by elliptic pseudodifferential operators, we may by Proposition 4.3 assume that the coordinates are chosen so that $\Gamma_j = I \times (0, 0, \xi), |\xi_j| = 1,$ and for any $0 < \varepsilon < 1/3$ we can write $P^* = Q + R$ where $\Gamma_j \cap \text{WF}_\varepsilon(R) = \emptyset$ and $Q$ has symbol

\begin{equation}
(7.8) \quad \tau - r(t, x, \xi) + q_0(t, x, \xi) + r_0(t, x, \xi)
\end{equation}

in a $g_\varepsilon$ neighborhood of $\Gamma_j.$ When $|\xi| \equiv \lambda$ we have $R \in S^{1+\varepsilon}_{1-\varepsilon, \varepsilon},$ $r_0 \in S^{k-1}_{1-\varepsilon, \varepsilon},$ $q_0 \in S^{\varepsilon}_{1-\varepsilon, \varepsilon}$ is given by (4.18), and $r \in S^1_{1-\varepsilon, \varepsilon}$ vanishes of second order at $\Gamma_j.$ By using Proposition 4.4 we may assume that the grazing Lagrangean space $L_j(w) = \{ (t, x; 0, 0) : (t, x) \in \mathbb{R}^n \}.$ 

Now, we may replace the norms $\| u \|_{(s)}$ in (7.7) by the norms given by

\[ \| u \|_{s}^2 = \| (D_x)_s u \|^2 = \int \langle \xi \rangle^{2s} |\hat{u}(\xi, \tau)|^2 d\tau d\xi. \]

In fact, the quotient $\langle \xi \rangle^{2s}/\langle (\tau, \xi) \rangle^{2s}$ is bounded in a conical neighborhood of $\Gamma$ so replacing the norms in the estimate (7.7) only changes the constant and the operator $A$ in the estimate.

Let $\lambda_j$ be given by (4.4) with $\varepsilon = 1/10.$ By choosing $\delta = 2/5,$ $\varepsilon = 1/10$ and $\rho = 1/10$ and using Propositions 4.5, 5.1, 6.2 and Remark 6.3, we can for each $\Gamma_j$ construct approximate solution $u_{\lambda_j}$ on the form (4.23) so that $Qu_{\lambda_j} = O(\lambda^k),$ for any $k.$ The real valued phase function is $\langle x, \xi_j \rangle + \omega_j(t, x)$ where $\omega_j(t, x) \in S(\lambda_j^{-7/10}, g_{3/10})$ and $(t, x; \partial_t \omega_j(t, x), \xi_j + \partial_x \omega_j(t, x))$ is in a $g_{2/5}$ neighborhood of $\Gamma_j$ when $|x| \lesssim \lambda_j^{-2/5}. \quad \text{Then} \quad \omega_j(t, x) = \lambda_j^{-7/10} \omega_j(\lambda_j^{-3/10} t, \lambda_j^{-3/10} x), \quad \text{where} \quad \omega_j \in C^\infty \quad \text{uniformly so} \quad \partial_x \omega = O(\lambda_j^{-2/5})$ when $x = O(\lambda_j^{-2/5})$ and

\[ \lambda_j(\langle x, \xi_j \rangle + \omega_j(t, x)) = \lambda_j^{7/10}(\langle \lambda_j^{-3/10} x, \xi_j \rangle + \lambda_j^{-4/10} \omega_j(\lambda_j^{-3/10} t, \lambda_j^{-3/10} x)) \]

Thus $\delta = 2/5,$ $\varepsilon = 3/10,$ $\kappa = 1/10$ and $\rho = 7/10$ in (7.1) so we find $\varepsilon + \rho = 1 > \delta = 2/5.$

The amplitude functions $\varphi_{k,j}(t, x) = \phi_{k,j}(\lambda_j^{-2/5} t, \lambda_j^{-2/5} x)$ where $\phi_{k,j} \in C_0^\infty$ uniformly in $j$ with fixed compact support in $x$ but in $t$ the support is bounded by $C\lambda_j^{2/5},$ so $u_{\lambda_j}$ will satisfy the conditions in Lemma 7.1 uniformly. Clearly differentiation of $Qu_{\lambda_j}$ can at most give a factor $\lambda_j$ since $\varepsilon + \rho = 1$ and $\delta < 1.$ Because of the bound on the support of $u_{\lambda_j}$ we can obtain that

\begin{equation}
(7.9) \quad \| Qu_{\lambda_j} \|_{(\nu)} = O(\lambda_j^{-N-n})
\end{equation}

for any chosen $\nu.$

Since $x = O(\lambda_j^{-2/5})$ in supp $u_{\lambda_j},$ we find that $(x, \xi_j + \partial \omega_j(x))$ is contained in a $g_{2/5}$ neighborhood of $\Gamma_j$ for $x \in \text{supp} u_{\lambda_j},$ and this converges to $\Gamma.$ Thus, if $R \in S^{11/10}_{9/10, 1/10}$ such that $\text{WF}_{1/10}(R) \cup \Gamma_j = \emptyset$ then we find from the expansion (4.24) that all the terms of $Ru_{\lambda_j}$ vanish for large enough $\lambda_j.$ In fact, since $\lambda_j^{-2/5} \ll \lambda_j^{-1/10}$ for $j \gg 1,$ we find for any $\alpha$ and $K$ that

\[ \partial^\alpha R(t, x; \lambda_j(\xi_0 + \partial \omega_j)) = O(\lambda_j^{-K}) \]
in \( \bigcup_k \text{supp} \varphi_{k,j} \). As before, we find that \( \| Ru_{\lambda_j} \|_{(0)} = \mathcal{O}(\lambda_j^{-N-n}) \) by the bound on the support of \( u_{\lambda_j} \), so we obtain from (7.9) that
\[
\| P^* u_{\lambda_j} \|_{(\nu)} = \mathcal{O}(\lambda_j^{-N-n})
\]
for any chosen \( \nu \).

If \( \text{WF}(A) \cap \Gamma = \emptyset \), then we find \( \text{WF}(A) \cap \Gamma_j = \emptyset \) for large \( j \), so Lemma 7.1 gives \( \| Au_{\lambda_j} \|_{(0)} = \mathcal{O}(\lambda_j^{-N-n}) \). Since \( \varepsilon + \varrho = 1 \) we also find from Lemma 7.1 that
\[
\lambda_j^{-N} = \lambda_j^{-N(\varepsilon+\varrho)} \gtrsim \| u_{\lambda_j} \|_{-N} \approx \| u_{\lambda_j} \|_{(-N)} \gtrsim \lambda_j^{-(N+N/2)(\varepsilon+\varrho)+(n-1)\delta/2} = \lambda_j^{-N-n/2+(n-1)/5} \geq \lambda_j^{-N-n/2}
\]
when \( \lambda_j \geq 1 \). We obtain that (7.7) holds for \( u_j = u_{\lambda_j} \) when \( j \to \infty \), so Remark 2.9 gives that \( P \) is not solvable at the limit bicharacteristic \( \Gamma \). \( \square \)

References


Centre for Mathematical Sciences, University of Lund, Box 118, SE-221 00 LUND, SWEDEN

E-mail address: dencker@maths.lth.se