Exercise 1
Let $1 \leq p \leq \infty$ and consider the space $\ell^p$ consisting of sequences $(x_n)$ of complex numbers with
\[ \sum_n |x_n|^p < \infty \text{ if } p < \infty, \]
and
\[ \sup_n |x_n| < \infty \text{ if } p = \infty. \]
Show that the function defined by
\[ d_p(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p} \text{ if } p < \infty \]
and
\[ d_{\infty}(x, y) = \sup_{n \geq 1} |x_n - y_n| \]
is a metric on $\ell^p$. Also prove that the space $(\ell^p, d_p)$ is complete.

Exercise 2
Prove that for $1 \leq p < \infty$, $\ell^p$ is separable, but $\ell^\infty$ is not separable.
Exercise 3

Consider the function $d : C([0, 1]) \times C([0, 1]) \to [0, \infty)$ given by

$$d(f, g) = \int_0^1 |f(x) - g(x)| \, dx.$$ 

Show that $d$ is a metric but $(C([0, 1]), d)$ is not complete. What is the completion of $C([0, 1])$ with respect to this metric?

Everybody should try to solve this exercise!

Exercise 4

Let $(X_n, d_n), n = 1, 2, \ldots$ be a countable family of metric spaces. Show that the function

$$d : \prod_{n \geq 1} X_n \times \prod_{n \geq 1} X_n \to [0, \infty)$$

defined by

$$d((x_n), (y_n)) = \sum_{n=1}^{\infty} 2^{-n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}$$

defines a metric on $\prod_{n \geq 1} X_n$. Also show that if each $(X_n, d_n)$ is complete, then $(\prod_{n \geq 1} X_n, d)$ is complete.

Exercise 5

Consider the set

$$Y = \{ f \in C([0, 1]); \ f \ is \ differentiable \ at \ 1/2 \}.$$ 

Prove that $Y$ is of the first category in $C([0, 1])$.

Exercise 6

a) Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function, i.e.,

$$f(x + y) = f(x) + f(y), \quad \forall x, y \in \mathbb{R}.$$ 

Show that if $f$ is continuous at a point, then it must be of the form $f(x) = ax$ where $a$ is a fixed real number.

b) Consider $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$ of rational numbers and let $B$ be a Hamel basis (i.e. every $x \in \mathbb{R}$ can be written uniquely as a finite $\mathbb{Q}$-linear combination of elements of $B$). Define a function $g : \mathbb{R} \to \mathbb{R}$ in the following way. If $x \in \mathbb{R}$ is written as

$$x = \sum_{\text{finite}} r_k b_k$$

where $r_k \in \mathbb{Q}$ and $b_k \in B$, then

$$g(x) = \sum r_k.$$

Show that $g$ is additive, but for every non-empty open interval $I \subset \mathbb{R}$ we have $\overline{g(I)} = \mathbb{R}$.

Presentation 1

Theorem 3 and Theorem 4 in Section 3.2.