Plan for 20./11.

Exercises

Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces and let $T : \mathcal{H} \to \mathcal{K}$ be a bounded linear operator. We say that $T$ is compact, if the closure of $T(B_1(0))$ is compact in $\mathcal{K}$. The rank of $T$ is the dimension of range $T$ in $\mathcal{K}$.

Exercise 1
Show that for every separable infinite-dimensional Hilbert space $\mathcal{H}$, there exists a unitary map

$$U : \mathcal{H} \to \ell^2.$$

Exercise 2
Let $T : \mathcal{H} \to \mathcal{K}$ be of finite rank. Show that $T$ is compact.

Exercise 3
Let $T : \mathcal{H} \to \mathcal{K}$ be of rank 1. Show that there exist $e \in \mathcal{H}$, $d \in \mathcal{K}$ such that $|d| = |T|$, $|e| = 1$, and $Tx = \langle x, e \rangle d$ for all $x \in \mathcal{H}$. (This is often written as $T = d \otimes e$).

Exercise 4
Show that each compact idempotent linear map between Hilbert spaces is of finite rank.

Exercise 5
Let $T : \mathcal{H} \to \mathcal{H}$ be of rank 1 and self-adjoint. Show that $T$ is a real multiple of an orthogonal projection.