Let $X, Y$ be Banach spaces and let $T : X \to Y$ be a bounded linear operator. We say that $T$ is compact, if the closure of $T(B_1(0))$ is compact in $Y$.

Plan for 27/11.

Presentation 1
Section 22.2.

Presentation 2
Section 22.5.

Exercise 1
Let $X$ be an infinite-dimensional Banach space. Show that a compact linear map $C : X \to X$ with closed range is of finite rank (i.e., the range of $C$ is finite-dimensional).

Exercise 2
Let $A, A_n \in L(H)$. We say that $A_n \to A$ in the strong operator topology (SOT), if $\|A_n h - Ah\| \to 0$ for each $h \in H$. Show that for $H$ separable, any $A \in L(H)$ is the SOT-limit of a sequence $(A_n)$ of finite rank operators.

Exercise 3
Let $u : \mathbb{R} \to \mathbb{C}$ be a bounded, continuous function. Consider the bounded operator $M_u$ of multiplication by $u$ on $L^2(\mathbb{R})$,

$$M_u f = uf.$$

Show that $M_u$ is compact only if $u = 0$. Prove also that the same is true when $u \in L^\infty(\mathbb{R})$ in the sense that $M_u$ is compact if and only if $u = 0$ almost everywhere.
Exercise 4

Let $H$ be a separable Hilbert space and let $T : H \to H$ be a linear map with the property that there exists an orthonormal basis $\{e_n\}_{n \geq 1}$ of $H$ such that

$$\sum_{n \geq 1} \|Te_n\|^2 < \infty.$$ 

Operators with this property are known as Hilbert-Schmidt operators.

(i) Prove that if $\{f_n\}$ is an arbitrary orthonormal basis for $H$ then

$$\sum_{n \geq 1} \|Tf_n\|^2 < \infty.$$ 

In fact, the value of the sum is independent of the basis.

(ii) Show that $T$ is compact.

(iii) Show that the Hilbert-Schmidt operators form a two-sided ideal in $L(H)$.

Exercise 5

Show that the class of Hilbert-Schmidt operators on $L^2(\mathbb{R})$ coincides with the integral operators with square-integrable kernels, i.e., the operators of the form

$$f \mapsto \int_{\mathbb{R}} K(x, y)f(y) \, dy,$$

where

$$\iint_{\mathbb{R}^2} |K(x, y)|^2 \, dx \, dy < \infty.$$