**Some Example Questions**

**Question 1 (Q4 from the Exam Jan 2017).** Let $X$ be a Banach space such that $X^*$ is separable. Show that $X$ is also separable.

**Proof:** Let $\{f_n\}_{n=1}^{\infty}$ be a dense sequence in the unit ball $B$ in $X^*$.

For each $n \in \mathbb{N}$, pick $x_n$, $x_n \in X$, $\|x_n\| = 1$, such that $|f_n(x_n)| > 1/2\|f_n\|$. Let $Y = \text{span}\{x_n\}$ and observe that $Y$ is separable, since finite rational combinations of the $\{x_n\}$ are dense in $Y$. It is now sufficient to show that $X = Y$. We proceed by contradiction. Suppose that $X \neq Y$. Then there is an $f \in X^*$, with $\|f\| = 1$, such that $f(x) = 0$ for all $x \in Y$. Now choose $n$ such that $\|f_n - f\| < 1/4$, in particular $\|f_n\| > 3/4$. Then

$$0 = \|f(x_n)\| \geq \|f_n(x_n)\| - \|f_n(x_n) - f(x_n)\| > \frac{1}{2}\|f_n\| - \frac{1}{4} > \frac{3}{8} - \frac{1}{4} > 0.$$ 

Hence we have a contradiction, and $X = Y$. $\square$

**Question 2 (Baire’s Theorem).** Let $X$ be the Banach space $C([0,1])$, equipped with the sup norm $\| \cdot \|_{\infty}$. Let

$$\mathcal{N} = \{f \in X : f \text{ is monotonic on some interval } [a,b], \text{ where } 0 \leq a < b \leq 1\}.$$ 

Show that $\mathcal{N}$ is of first category in $X$.

**Proof:** For $0 \leq a < b \leq 1$, let $\mathcal{N}_{a,b} = \{f \in X : f \text{ is monotonic on } [a,b]\}$. Clearly $\mathcal{N}_{a,b}$ is closed, since the uniform limit of monotonic functions is monotonic. Moreover, the open interior of $\mathcal{N}_{a,b}$ is empty. To see this, let $f \in \mathcal{N}_{a,b}$ and $\varepsilon > 0$. Say $f$ is increasing. Let $x \in (a,b)$. Since $f$ is continuous, there exists $\delta > 0$ such that $f(y) - f(x) < \varepsilon/2$ for $x < y < b$, $y - x < \delta$. Choose such a $y$, and let $\phi \in X$, $\|\phi\|_{\infty} < \varepsilon$, $\phi$ supported on $[x,b]$ and $\phi(y) \leq -\varepsilon/2$. Then $f(x) + \phi(x) = f(x)$, $f(y) + \phi(y) \leq f(y) - \varepsilon/2 < f(x)$, so $f + \phi$ is not increasing on $[a,b]$. Since $f(b) + \phi(b) = f(b) \geq f(x) > f(y) + \phi(y)$, it is not decreasing either. Hence $B_{\varepsilon}(f)$ is not contained in $\mathcal{N}_{a,b}$, so $\mathcal{N}_{a,b}$ has empty open interior and is thus nowhere dense.

Note that each $\mathcal{N}_{a,b}$ is contained in $\mathcal{N}_{p,q}$, where $p,q$ are rational numbers with $a < p < q < b$. Hence

$$\mathcal{N} = \bigcup_{(a,b): 0 \leq a < b \leq 1} \mathcal{N}_{a,b} = \bigcup_{(p,q): 0 \leq p < q \leq 1, p,q \in \mathbb{Q}} \mathcal{N}_{p,q}$$

is of first category in $X$.

**Question 3 (Krein-Milman Theorem).** Let $S$ be the set of selfadjoint $3 \times 3$ matrices of norm less or equal to one, where the norm is the operator norm of matrices as bounded linear maps $\mathbb{C}^3 \to \mathbb{C}^3$. Let $\mathcal{E} = \{A \in S, \text{ all eigenvalues of } A \text{ are in } \{-1,1\}\}$. Show that $S$ is the closed convex hull of $\mathcal{E}$.

**Proof:** Clearly $S$ is compact and convex, so by the Krein-Milman Theorem, we only have to show that $\mathcal{E}$ contains all extreme points of $S$. Let $A \in S \setminus \mathcal{E}$. It suffices to show that $A$ is not an extreme point of $S$. Since $A$ is selfadjoint and of norm less or equal to one, its spectrum is contained in $[-1,1]$. Since
A is not in $\mathcal{E}$, it has to have an eigenvalue $\lambda \in (-1, 1)$. Since $A$ is selfadjoint, it is (by the Spectral Theorem) diagonal in a suitable orthonormal basis of $\mathbb{C}^3$, and $\lambda$ is one of the entries on the diagonal. Choose $0 < \gamma < 1 - |\lambda|$. Then, in a suitable orthonormal basis,

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} = \frac{1}{2} \left( \begin{pmatrix} \lambda + \gamma & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} + \begin{pmatrix} \lambda - \gamma & 0 & 0 \\ 0 & a_1 & 0 \\ 0 & 0 & a_2 \end{pmatrix} \right)$$

with $|a_1|, |a_2| \leq 1$. Hence the matrices in the convex combination on the right are in $\mathcal{S}$, and $A$ is not an extreme point.