Integration Theory MATM18, Spring 2016

Assignment

1. Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_k)$ be a sequence of sets in $\mathcal{A}$. Let
   
   \[ B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k, \quad C = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k. \]

   1. Show:
      
      \[ B = \{ x \in X : x \in A_k \text{ for all but finitely many } k \in \mathbb{N} \}, \]
      
      \[ C = \{ x \in X : x \in A_k \text{ for infinitely many } k \in \mathbb{N} \}. \]

   2. Show that
      
      \[ \liminf_k \chi_{A_k} = \chi_B, \quad \limsup_k \chi_{A_k} = \chi_C \]

   B is therefore called the lim inf of the sequence of sets $(A_k)$, and $C$ is called the lim sup.

   3. Show: If $\sum_{k=1}^{\infty} \mu(A_k) < \infty$, then $\mu(C) = 0$.

   This result is known as the Borel-Cantelli Lemma.

2. Let $K$ be the Cantor set and let $f : [0, 1] \to [0, 1]$ be the Cantor singular function constructed at the end of Ch. 2.1. For a finite union of pairwise disjoint half-open intervals in $[0, 1]$, 
   
   \[ A = \bigcup_{i=1}^{N} (a_i, b_i), \]

   we define
   
   \[ \mu(A) = \sum_{i=1}^{N} f(b_i) - f(a_i). \]
1. Show that $\mu$ extends to a measure on $([0,1], \mathcal{L}([0,1]))$ with $\mu([0,1]) = 1$. Here, $\mathcal{L}([0,1])$ denotes the $\sigma$-algebra of Lebesgue-measurable subsets of $[0,1]$. Hint: You can use the Caratheodory Extension Theorem, which we proved in Exercises 1.3. Question 5d, or Prop 1.3.10. for this.

2. Show that $\mu(K^c) = 0$.

3. Show that $\mu(x) = 0$ for each $x \in [0,1]$.

$\mu$ is an example of a non-atomic singular measure on $\mathbb{R}$ - it has all its mass on a set of Lebesgue measure 0 (singular), but each single point has measure 0 (non-atomic).

3. Let $f, g : [0,1] \to \mathbb{R}$ be defined by

$$ f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases} $$

and

$$ g(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q}, p, q \text{ relatively prime} \\ 0 & \text{if } x = 0 \text{ or } x \notin \mathbb{Q}. \end{cases} $$

1. Show that $f$ and $g$ are measurable.

2. Show that $\int_{[0,1]} f \, d\lambda = 0$ and $\int_{[0,1]} g \, d\lambda = 0$.

3. Show that $f$ is nowhere continuous, and that $g$ is continuous $\lambda$-almost everywhere on $[0,1]$.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space, and let $f : X \to [0, +\infty]$ be measurable.

1. Suppose that $f$ takes values only in $\mathbb{N}_0$. Show that

$$ \int_X f \, d\mu = \sum_{k=1}^{\infty} \mu(\{x \in X : f(x) \geq k\}). $$

2. Now fix $n \in \mathbb{N}$ and suppose that $f$ takes values only in $\left\{ \frac{k}{2^n} : k \in \mathbb{N}_0 \right\}$. Show that

$$ \int_X f \, d\mu = \sum_{k=1}^{\infty} \frac{1}{2^n} \mu(\{x \in X : f(x) \geq \frac{k}{2^n}\}). $$

3. Show that in the general case of $f$ taking values in $[0, +\infty]$,

$$ \int_X f \, d\mu = \int_{[0,\infty]} \mu(\{x \in X : f(x) \geq t\}) \, d\lambda(t). $$

Hint: Look at the proof of Prop. 2.1.8.

Please hand in on Monday, March 7, before the lecture.