Harmonic Morphisms from Lie Groups and Symmetric Spaces
- Some Existence Theory -

Sigmundur Gudmundsson

Department of Mathematics
Faculty of Science
Lund University

Max-Planck Institut - Bonn - 27 February 2020
Harmonic Morphisms

1. The Origins - Jacobi 1848
2. Riemannian Geometry - Fuglede 1978, Ishihara 1979
4. Existence?
Outline

1 Harmonic Morphisms
   • The Origins - Jacobi 1848
   • Riemannian Geometry - Fuglede 1978, Ishihara 1979
   • Geometric Motivation - Baird-Eells 1981
   • Existence ?

2 The Conjecture
   • The Conjecture
   • Relevant History
Outline

1 Harmonic Morphisms
   - The Origins - Jacobi 1848
   - Riemannian Geometry - Fuglede 1978, Ishihara 1979
   - Geometric Motivation - Baird-Eells 1981
   - Existence ?

2 The Conjecture
   - The Conjecture
   - Relevant History

3 Constructions by Eigenfamilies
   - Definition
   - Useful Machinery
   - The Classical Semisimple Lie Groups
Outline

1. Harmonic Morphisms
   - The Origins - Jacobi 1848
   - Riemannian Geometry - Fuglede 1978, Ishihara 1979
   - Geometric Motivation - Baird-Eells 1981
   - Existence ?

2. The Conjecture
   - The Conjecture
   - Relevant History

3. Constructions by Eigenfamilies
   - Definition
   - Useful Machinery
   - The Classical Semisimple Lie Groups

4. Constructions by Orthogonal Harmonic Families
   - Another Useful Machine
   - Symmetric Spaces $G/K$ of Non-Compact Type
   - Nilpotent and Solvable Lie Groups
   - Symmetric Spaces $U/K$ of Compact Type
   - Examples
   - Homogeneous Spaces of Positive Curvature
Harmonic Morphisms

The Origins - Jacobi 1848
Riemannian Geometry - Fuglede 1978, Ishihara 1979
Geometric Motivation - Baird-Eells 1981
Existence ?

The Conjecture
The Conjecture
Relevant History

Constructions by Eigenfamilies
Definition
Useful Machinery
The Classical Semisimple Lie Groups

Constructions by Orthogonal Harmonic Families
Another Useful Machine
Symmetric Spaces $G/K$ of Non-Compact Type
Nilpotent and Solvable Lie Groups
Symmetric Spaces $U/K$ of Compact Type
Examples
Homogeneous Spaces of Positive Curvature

Low-Dimensional Classifications

References
**Definition 1.1 (Harmonic Morphisms (Jacobi 1848))**

A map $\phi = u + iv : U \subset \mathbb{R}^3 \rightarrow \mathbb{C}$ is said to be a **harmonic morphism** if the composition $f \circ \phi$ with any **holomorphic** function $f : W \subset \mathbb{C} \rightarrow \mathbb{C}$ is **harmonic**.
Definition 1.1 (Harmonic Morphisms (Jacobi 1848))

A map \( \phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C} \) is said to be a **harmonic morphism** if the composition \( f \circ \phi \) with any **holomorphic** function \( f : W \subset \mathbb{C} \to \mathbb{C} \) is harmonic.

Theorem 1.2 (Jacobi 1848)

A map \( \phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C} \) is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal i.e.

\[
\Delta u = \Delta v = 0, \quad \langle \nabla u, \nabla v \rangle = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla v|^2.
\]
Definition 1.1 (Harmonic Morphisms (Jacobi 1848))

A map \( \phi = u + iv : U \subset \mathbb{R}^3 \rightarrow \mathbb{C} \) is said to be a harmonic morphism if the composition \( f \circ \phi \) with any holomorphic function \( f : W \subset \mathbb{C} \rightarrow \mathbb{C} \) is harmonic.

Theorem 1.2 (Jacobi 1848)

A map \( \phi = u + iv : U \subset \mathbb{R}^3 \rightarrow \mathbb{C} \) is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal i.e.

\[
\Delta u = \Delta v = 0, \quad \langle \nabla u, \nabla v \rangle = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla v|^2.
\]

Proof.

\[
\Delta(f \circ \phi) = \left[ \frac{\partial f}{\partial z} \right] \cdot \Delta \phi + \left[ \frac{\partial^2 f}{\partial z^2} \right] \cdot \langle \nabla \phi, \nabla \phi \rangle_\mathbb{C} = 0
\]
Theorem 1.3 (Jacobi 1848)

Let \( f, g : W \subset \mathbb{C} \rightarrow \mathbb{C} \) be holomorphic functions, then every local solution \( z : U \subset \mathbb{R}^3 \rightarrow \mathbb{C} \) to the equation

\[
\langle f(z(x)) \left[ 1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x)) \right], x \rangle_{\mathbb{C}} = 1
\]

is a harmonic morphism.
Theorem 1.3 (Jacobi 1848)

Let \( f, g : W \subset \mathbb{C} \to \mathbb{C} \) be holomorphic functions, then every local solution \( z : U \subset \mathbb{R}^3 \to \mathbb{C} \) to the equation

\[
\langle f(z(x)) \left[ 1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x)) \right], x \rangle \mathbb{C} = 1
\]

is a harmonic morphism.

Theorem 1.4 (Baird-Wood 1988)

Every harmonic morphism \( z : U \to \mathbb{C} \) defined locally on the Euclidean \( \mathbb{R}^3 \) is obtained this way.
Example 1.5 (The Outer Disc Example)

Let \( r \in \mathbb{R}^+ \) and choose \( g(z) = z \), \( f(z) = -1/2irz \) then we yield

\[
(x_1 - ix_2)z^2 - 2(x_3 + ir)z - (x_1 + ix_2) = 0
\]

with the two solutions

\[
z_r^\pm = \frac{-(x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}.
\]
Definition 1.6 (Harmonic Morphisms (Fuglede 1978, Ishihara 1979))

A map \( \phi : (M^m, g) \to (N^n, h) \) between Riemannian manifolds is called a harmonic morphism if, for any harmonic function \( f : U \to \mathbb{R} \) defined on an open subset \( U \) of \( N \) with \( \phi^{-1}(U) \) non-empty, \( f \circ \phi : \phi^{-1}(U) \to \mathbb{R} \) is a harmonic function.
Definition 1.6 (Harmonic Morphisms (Fuglede 1978, Ishihara 1979))

A map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

Theorem 1.7 (Fuglede 1978, Ishihara 1979)

A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.
(Harmonicity)

For local coordinates $x$ on $(M, g)$ and $y$ on $(N, h)$, we have the non-linear system

$$
\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left( \frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}_{ij}^k \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha, \beta=1}^{n} \Gamma_{\alpha \beta}^\gamma \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) \Bigg|_{x} = 0,
$$

where $\phi^\alpha = y_\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.
Harmonic Morphisms
The Conjecture
Constructions by Eigenfamilies
Constructions by Orthogonal Harmonic Families
Low-Dimensional Classifications
References

The Origins - Jacobi 1848
Riemannian Geometry - Fuglede 1978, Ishihara 1979
Geometric Motivation - Baird-Eells 1981
Existence ?

(Harmonicity)

For local coordinates $x$ on $(M, g)$ and $y$ on $(N, h)$, we have the **non-linear** system

$$
\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left( \frac{\partial^{2} \phi^{\gamma}}{\partial x_{i} \partial x_{j}} - \sum_{k=1}^{m} \hat{\Gamma}_{ij}^{k} \frac{\partial \phi^{\gamma}}{\partial x_{k}} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\beta}^{\gamma} \circ \phi \frac{\partial \phi^{\alpha}}{\partial x_{i}} \frac{\partial \phi^{\beta}}{\partial x_{j}} \right) = 0,
$$

where $\phi^{\alpha} = y_{\alpha} \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.

(Horizontal (weak) Conformality)

There exists a continuous function $\lambda : M \to \mathbb{R}_{0}^{+}$ such that for all $\alpha, \beta = 1, 2, \ldots, n$

$$
\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial \phi^{\alpha}}{\partial x_{i}}(x) \frac{\partial \phi^{\beta}}{\partial x_{j}}(x) = \lambda^{2}(x) h^{\alpha\beta}(\phi(x)).
$$

This is a first order **non-linear** system of $\left[\binom{n+1}{2} - 1\right]$ equations.
Theorem 1.8 (Baird, Eells 1981)

Let $\phi : (M, g) \rightarrow (N^2, h)$ be a horizontally conformal map from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if its fibres are minimal at regular points $\phi$. 
Theorem 1.8 (Baird, Eells 1981)

Let $\phi : (M, g) \rightarrow (N^2, h)$ be a \textbf{horizontally conformal} map from a Riemannian manifold to a surface. Then $\phi$ is \textbf{harmonic} if and only if its fibres are \textbf{minimal} at regular points $\phi$.

The problem is \textbf{invariant} under \textbf{isometries} on $(M, g)$. If the codomain $(N, h)$ is a surface ($n = 2$) then it is also invariant under \textbf{conformal changes} $\sigma^2 h$ of the metric on $N^2$. This means, at least for local studies, that $(N^2, h)$ can be chosen to be the \textbf{standard complex plane} $\mathbb{C}$. 
Example 1.9 (The Nilpotent Lie Group Nil$^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$
Example 1.9 (The Nilpotent Lie Group $\text{Nil}^3$)

\[(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).\]

The left-invariant metric, with orthonormal basis

\[\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \} \]

at the neutral element $e = (0, 0, 0)$, is given by

\[ds^2 = dx^2 + dy^2 + (dz - xdy)^2.\]

(Baird, Wood 1990): Every local solution is a restriction of the globally defined harmonic morphism $\phi : \text{Nil}^3 \to \mathbb{C}$ with

\[\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto x + iy.\]
Example 1.10 (The Solvable Lie Group Sol³)

\[(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in SL_3(\mathbb{R}).\]

The left-invariant metric, with orthonormal basis

\[
\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \} \]

at the neutral element \(e = (0, 0, 0)\), is given by

\[ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.\]
Example 1.10 (The Solvable Lie Group $\text{Sol}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial/\partial x, \; Y = \partial/\partial y, \; Z = \partial/\partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$  

(Baird, Wood 1990): No solutions exist, not even locally.

$$e^{-2z} \frac{\partial^2 \phi}{\partial x^2} + e^{2z} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

$$e^{-2z} \left( \frac{\partial \phi}{\partial x} \right)^2 + e^{2z} \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 = 0.$$
Conjecture 1 (SG 1995)

Let \((M, g)\) be an irreducible Riemannian symmetric space of dimension \(m \geq 2\). For each point \(p \in M\) there exists a non-constant complex-valued harmonic morphism \(\phi : U \subset M \to \mathbb{C}\) defined on an open neighbourhood \(U\) of \(p\). If \(M\) is of non-compact type then the domain \(U\) can be chosen to be the whole of \(M\).

Definition 2.1 (Symmetric Space)

A Riemannian manifold \((M, g)\) is said to be a symmetric space if for each point \(p \in M\) there exists a global geodesic reflective isometry \(\sigma : (M, g) \to (M, g)\) i.e. such that its differential \(d\sigma_p : T_p M \to T_p M\) at \(p\) satisfies

\[
    d\sigma_p = -\text{id}_{T_p M}.
\]
Baird-Eells (1981): $S^3 = \text{SO}(1+3)/\text{SO}(1) \times \text{SO}(3)$. The Hopf map $\phi : S^3 \to S^2 \cong \mathbb{C}$ with

$$\phi : (x_1, x_2, x_3, x_4) \mapsto (x_1 + ix_2)/(x_3 + ix_4).$$

Baird-Wood (1989): $H^3 = \text{SO}_o(1, 3)/\text{SO}(1) \times \text{SO}(3)$

Wood (1991): $S^4 = \text{SO}(1+4)/\text{SO}(1) \times \text{SO}(4)$

Baird (1992): $H^4 = \text{SO}_o(1, 4)/\text{SO}(1) \times \text{SO}(4)$

SG (1994): $\mathbb{C}P^q = \text{U}(1+q)/\text{U}(1) \times \text{U}(q)$

SG (1994): $\mathbb{H}P^q = \text{Sp}(1+q)/\text{Sp}(1) \times \text{Sp}(q)$

SG (1995): $H^{2n+1} = \text{SO}_o(1, 2n+1)/\text{SO}(1) \times \text{SO}(2n+1)$. The ”dual” Hopf map $\phi : H^3 \to \mathbb{C}$ with

$$\phi : (x_1, x_2, x_3, x_4) \mapsto (x_1 + ix_2)/(x_3 - x_4).$$
Definition 3.1 (The Laplacian - The Conformality Operator)

For complex-valued functions $\phi, \psi : (M, g) \to \mathbb{C}$ on a Riemannian manifold we have the complex-valued \textbf{Laplacian} $\tau(\phi)$ and the symmetric bilinear \textbf{conformality operator} $\kappa$ given by

$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$
Definition 3.1 (The Laplacian - The Conformality Operator)

For complex-valued functions \( \phi, \psi : (M, g) \to \mathbb{C} \) on a Riemannian manifold we have the complex-valued Laplacian \( \tau(\phi) \) and the symmetric bilinear conformality operator \( \kappa \) given by

\[
\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi).
\]

The harmonicity and the horizontal conformality of \( \phi : (M, g) \to \mathbb{C} \) are then given by the following relations

\[
\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.
\]
Definition 3.1 (The Laplacian - The Conformality Operator)

For complex-valued functions $\phi, \psi : (M, g) \rightarrow \mathbb{C}$ on a Riemannian manifold we have the complex-valued **Laplacian** $\tau(\phi)$ and the symmetric bilinear **conformality operator** $\kappa$ given by

$$\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi).$$

The **harmonicity** and the **horizontal conformality** of $\phi : (M, g) \rightarrow \mathbb{C}$ are then given by the following relations

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$

Definition 3.2 (Eigenfamilies)

A set $\mathcal{E} = \{\phi_\alpha : (M, g) \rightarrow \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an **eigenfamily** on $(M, g)$ if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu\phi\psi.$$
Theorem 3.3 (SG, Sakovich 2008)

Let \((M, g)\) be a Riemannian manifold and \(\mathcal{E} = \{\phi_1, \ldots, \phi_n\}\) be a finite eigenfamily of complex-valued functions on \(M\). If \(P, Q : \mathbb{C}^n \to \mathbb{C}\) are linearly independent homogeneous polynomials of the same positive degree then the quotient

\[
    P(\phi_1, \ldots, \phi_n)/Q(\phi_1, \ldots, \phi_n)
\]

is a non-constant harmonic morphism on the open and dense subset

\[
    \{p \in M \mid Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.
\]
Theorem 3.3 (SG, Sakovich 2008)

Let \((M, g)\) be a Riemannian manifold and \(\mathcal{E} = \{\phi_1, \ldots, \phi_n\}\) be a finite eigenfamily of complex-valued functions on \(M\). If \(P, Q : \mathbb{C}^n \to \mathbb{C}\) are linearly independent homogeneous polynomials of the same positive degree then the quotient

\[
P(\phi_1, \ldots, \phi_n)/Q(\phi_1, \ldots, \phi_n)
\]

is a non-constant harmonic morphism on the open and dense subset

\[
\{p \in M \mid Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.
\]

The authors apply this machinery to construct solutions on the classical semisimple Lie groups \(SO(n), SU(n), Sp(n), SL_n(\mathbb{R}), SU^*(2n)\) and \(Sp(n, \mathbb{R})\) equipped with their standard Riemannian metrics.

They also develop a duality principle and use this to construct solutions from the semisimple Lie groups \(SO(n), SU(n), Sp(n), SL_n(\mathbb{R}), SU^*(2n), Sp(n, \mathbb{R}), SO^*(2n), SO(p, q), SU(p, q)\) and \(Sp(p, q)\) equipped with their standard dual semi-Riemannian metrics.
 Equip the special orthogonal group

\[ \text{SO}(n) = \{ x \in \text{GL}_n(\mathbb{R}) \mid x^t \cdot x = I_n, \ \det x = 1 \} \]

with the standard Riemannian metric \( g \) induced by the Euclidean scalar product \( g(X, Y) = \text{trace}(X^t \cdot Y) \) on the Lie algebra

\[ \mathfrak{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}. \]
Equip the special orthogonal group

\[
\text{SO}(n) = \{ x \in \text{GL}_n(\mathbb{R}) \mid x^t \cdot x = I_n, \det x = 1 \}
\]

with the standard Riemannian metric \( g \) induced by the Euclidean scalar product \( g(X, Y) = \text{trace}(X^t \cdot Y) \) on the Lie algebra

\[
\text{so}(n) = \{ X \in \text{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}.
\]

**Lemma 3.4 (SG, Sakovich 2008)**

For \( 1 \leq i, j \leq n \), let \( x_{ij} : \text{SO}(n) \to \mathbb{R} \) be the real valued coordinate functions given by \( x_{ij} : x \mapsto \langle e_i, x \cdot e_j \rangle \) where \( \{ e_1, \ldots, e_n \} \) is the canonical basis for \( \mathbb{R}^n \). Then the following relations hold

\[
\tau(x_{ij}) = -\frac{(n-1)}{2} x_{ij}, \quad \kappa(x_{ij}, x_{kl}) = -\frac{1}{2} (x_{il} x_{kj} - \delta_{jl} \delta_{ik}).
\]
Theorem 3.5 (SG, Sakovich 2008)

Let \( p \in \mathbb{C}^n \) be a non-zero isotropic element i.e. \( \langle p, p \rangle_{\mathbb{C}} = 0 \). Then the following is an eigenfamily on \( \text{SO}(n) \)

\[ \mathcal{E}_p = \{ \phi_a : \text{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle_{\mathbb{C}}, \ a \in \mathbb{C}^n \} . \]
Theorem 3.5 (SG, Sakovich 2008)

Let \( p \in \mathbb{C}^n \) be a non-zero isotropic element i.e. \( \langle p, p \rangle_\mathbb{C} = 0 \). Then the following is an eigenfamily on \( \text{SO}(n) \)

\[
\mathcal{E}_p = \{ \phi_a : \text{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle_\mathbb{C}, \ a \in \mathbb{C}^n \}.
\]

Example 3.6 (Eigenfamilies on \( \text{SO}(4) \))

For \( z, w \in \mathbb{C} \), let \( p \) be the isotropic element of \( \mathbb{C}^4 \) given by

\[
p(z, w) = (1 + zw, i(1 - zw), i(z + w), z - w).
\]

This gives us the complex 2-dimensional deformation of eigenfamilies \( \mathcal{E}_p \) each consisting of functions \( \phi_a : \text{SO}(4) \to \mathbb{C} \) with

\[
\phi_a(x) = (1 + zw)(x_{11}a_1 + x_{21}a_2 + x_{31}a_3 + x_{41}a_4) \\
+ i(1 - zw)(x_{12}a_1 + x_{22}a_2 + x_{32}a_3 + x_{42}a_4) \\
+ i(z + w)(x_{13}a_1 + x_{23}a_2 + x_{33}a_3 + x_{43}a_4) \\
+ (z - w)(x_{14}a_1 + x_{24}a_2 + x_{34}a_3 + x_{44}a_4).
\]
Definition 4.1 (Orthogonal Harmonic Family)

A set \( \Omega = \{ \phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I \} \) of complex-valued functions is called an orthogonal harmonic family on \((M, g)\) if for all \( \phi, \psi \in \Omega \)

\[ \tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0. \]
Definition 4.1 (Orthogonal Harmonic Family)

A set $\Omega = \{\phi_\alpha : (M, g) \rightarrow \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an orthogonal harmonic family on $(M, g)$ if for all $\phi, \psi \in \Omega$

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$  

Example 4.2

Let $\Omega = \{\phi_\alpha : (M, g, J) \rightarrow \mathbb{C} \mid \alpha \in I\}$ be a collection of holomorphic functions on a Kähler manifold. Then $\Omega$ is an orthogonal harmonic family.
Theorem 4.3 (SG 1997)

Let \((M, g)\) be a Riemannian manifold and \(U\) be an open subset of \(\mathbb{C}^n\) containing the image of \(\Phi = (\phi_1, \ldots, \phi_n) : M \to \mathbb{C}^n\). Further let

\[
H = \{ F_\alpha : U \to \mathbb{C} \mid \alpha \in I \}
\]

be a collection of holomorphic functions defined on \(U\). If the finite set

\[
\Omega = \{ \phi_k : (M, g) \to \mathbb{C} \mid k = 1, \ldots, n \}
\]

is an orthogonal harmonic family on \((M, g)\) then

\[
\Omega_H = \{ \psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \ldots, \phi_n), \ F \in H \}
\]

is again an orthogonal harmonic family.
Let \((M, g)\) be an irreducible Riemannian \textbf{symmetric space} of \textbf{non-compact} type presented as the quotient \(G/K\) where \(G\) a connected semisimple Lie group and \(K\) its maximal compact subgroup.
Let \((M, g)\) be an irreducible Riemannian **symmetric space** of **non-compact** type presented as the quotient \(G/K\) where \(G\) a connected semisimple Lie group and \(K\) its maximal compact subgroup.

Let \(G = NAK\) be the **Iwasawa decomposition** of \(G\), where \(N\) is nilpotent and \(A\) is abelian.
Let \((M, g)\) be an irreducible Riemannian \textbf{symmetric space} of \textbf{non-compact} type presented as the quotient \(G/K\) where \(G\) a connected semisimple Lie group and \(K\) its maximal compact subgroup.

Let \(G = NAK\) be the \textbf{Iwasawa decomposition} of \(G\), where \(N\) is \textbf{nilpotent} and \(A\) is \textbf{abelian}.

\textbf{Fact 4.4 (solvable Lie group - rank)}

\begin{quote}
\textit{The non-compact symmetric space} \((M, g)\) \textit{can be identified with the solvable subgroup} \(S = NA\) \textit{of} \(G\) \textit{and its rank} \(r\) \textit{is the dimension of abelian subgroup} \(A\).
\end{quote}
Let \((M, g)\) be an irreducible Riemannian symmetric space of non-compact type presented as the quotient \(G/K\) where \(G\) a connected semisimple Lie group and \(K\) its maximal compact subgroup.

Let \(G = NAK\) be the Iwasawa decomposition of \(G\), where \(N\) is nilpotent and \(A\) is abelian.

**Fact 4.4 (solvable Lie group - rank)**

The non-compact symmetric space \((M, g)\) can be identified with the solvable subgroup \(S = NA\) of \(G\) and its rank \(r\) is the dimension of abelian subgroup \(A\).

Let \(\mathfrak{s}, \mathfrak{n}, \mathfrak{a}\) be the Lie algebras of \(S, N, A\), respectively. For this situation we have \(\mathfrak{s} = \mathfrak{a} + \mathfrak{n} = \mathfrak{a} + [\mathfrak{s}, \mathfrak{s}]\), hence

\[\mathfrak{a} = \mathfrak{s}/[\mathfrak{s}, \mathfrak{s}]\.]
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a **natural group epimorphism** $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$. 
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a **natural group epimorphism** $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.

**Fact 4.5 (semisimple - solvable - nilpotent)**

*If the group $G$ is **semisimple** then $d = 0$, if $G$ is **solvable** then $d \geq 1$ and if $G$ is **nilpotent** then $d \geq 2$.***
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a **natural group epimorphism** $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.

**Fact 4.5 (semisimple - solvable - nilpotent)**

*If the group $G$ is **semisimple** then $d = 0$, if $G$ is **solvable** then $d \geq 1$ and if $G$ is **nilpotent** then $d \geq 2$.***

Equip $\mathbb{R}^d$ with its standard Euclidean metric and the Lie group $G$ with a left-invariant Riemannian metric $g$ such that the natural group epimorphism $\Phi : G \to \mathbb{R}^d$ is a **Riemannian submersion**. Then the kernel $[\mathfrak{g}, \mathfrak{g}]$ of the linear map $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ generates a left-invariant Riemannian foliation $\mathcal{V}$ on $(G, g)$ with orthogonal distribution $\mathcal{H} = [\mathfrak{g}, \mathfrak{g}]^\perp$. 
Theorem 4.6 (SG, Svensson 2009)

Let \( \mathcal{B} = \{X_1, \ldots, X_d\} \) be an ONB for the horizontal subspace \([\mathfrak{g}, \mathfrak{g}]^\perp\) of \( \mathfrak{g} \) and \( \xi \in \mathbb{C}^d \) be given by \( \xi = (\text{trace ad}_{X_1}, \ldots, \text{trace ad}_{X_d}) \). For a maximal isotropic subspace \( V \) of \( \mathbb{C}^d \) put

\[
V_\xi = \{v \in V \mid \langle \xi, v \rangle_{\mathbb{C}} = 0\}.
\]

If the real dimension of the isotropic subspace \( V_\xi \) is at least 2 then

\[
\Omega = \{\phi_v(x) = \langle \Phi(x), v \rangle_{\mathbb{C}} \mid v \in V_\xi\}
\]

is an orthogonal harmonic family on \((G, \mathfrak{g})\).
Theorem 4.6 (SG, Svensson 2009)

Let $\mathcal{B} = \{X_1, \ldots, X_d\}$ be an ONB for the horizontal subspace $[g, g]^\perp$ of $g$ and $\xi \in \mathbb{C}^d$ be given by $\xi = (\text{trace ad}_{X_1}, \ldots, \text{trace ad}_{X_d})$. For a maximal isotropic subspace $V$ of $\mathbb{C}^d$ put

$$V_\xi = \{v \in V \mid \langle \xi, v \rangle_{\mathbb{C}} = 0\}.$$

If the real dimension of the isotropic subspace $V_\xi$ is at least 2 then

$$\Omega = \{\phi_v(x) = \langle \Phi(x), v \rangle_{\mathbb{C}} \mid v \in V_\xi\}$$

is an orthogonal harmonic family on $(G, g)$.

Proof.

The tension field of natural group epimorphism $\Phi : G \to \mathbb{R}^d$ satisfies

$$\tau(\Phi)(p) = \sum_{k=1}^{d} (\text{trace ad}_{X_k})d\Phi_e(X_k).$$
Example 4.7

For the nilpotent Riemannian Lie group

\[
N_n = \begin{cases}
(1 \quad x_{12} \quad \cdots \quad x_{1,n-1} \quad x_{1n}) \\
0 \quad 1 \quad \ddots \quad \vdots \\
\vdots \quad \ddots \quad \ddots \quad \vdots \\
\vdots \quad \ddots \quad 1 \quad x_{n-1,n} \\
0 \quad \cdots \quad \cdots \quad 0 \quad 1
\end{cases} \quad \in \text{SL}_n(\mathbb{R}) \mid x_{ij} \in \mathbb{R}
\]

the natural group epimorphism \( \Phi : N_n \rightarrow \mathbb{R}^{n-1} \) is given by

\[
\Phi(x) = (x_{12}, \ldots, x_{n-1,n})
\]

and the vector \( \xi \in \mathbb{C}^n \) satisfies \( \xi = 0 \).
Example 4.8

For the **solvable** Riemannian Lie group

\[
S_n = \left\{ \begin{pmatrix}
 e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\
 0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\
 0 & \cdots & 0 & 0 & e^{t_n}
\end{pmatrix} \in \text{GL}_n(\mathbb{R}) \mid x_{ij}, t_i \in \mathbb{R}\right\}
\]

the **natural group epimorphism** \( \Phi : S_n \rightarrow \mathbb{R}^n \) is given by

\[
\Phi(x) = (t_1, t_2, \ldots, t_n)
\]

and the vector \( \xi \in \mathbb{C}^n \) satisfies

\[
\xi = ((n + 1) - 2, (n + 1) - 4, \ldots, (n + 1) - 2n).
\]
As an immediate consequence of Theorem 4.6 we now have the existence of globally defined harmonic morphisms from any simply connected symmetric space $G/K$ of non-compact type and rank $r \geq 3$.

With a series of other additional methods we have the following result.

**Theorem 4.9 (SG, Svensson 2009)**

Let $(M,g)$ be an irreducible Riemannian symmetric space of non-compact type other than $G^*/SO(4)$. Then there exists a non-constant globally defined complex-valued harmonic morphism $\phi : M \to \mathbb{C}$. 
This can be extended to the following result.

**Theorem 4.10 (SG, Svensson 2009)**

Let \((M, g)\) be an irreducible Riemannian **symmetric space** other than \(G_2^*/SO(4)\) or its compact dual \(G_2/SO(4)\). Then for each point \(p \in M\) there exists a non-constant complex-valued **harmonic morphism** \(\phi : U \rightarrow \mathbb{C}\) defined on an open neighbourhood \(U\) of \(p\). If the space \((M, g)\) is of non-compact type then the domain \(U\) can be chosen to be the whole of \(M\).
This can be extended to the following result.

**Theorem 4.10 (SG, Svensson 2009)**

Let \((M, g)\) be an irreducible Riemannian **symmetric space** other than \(G_2^*/SO(4)\) or its compact dual \(G_2/\text{SO}(4)\). Then for each point \(p \in M\) there exists a non-constant complex-valued **harmonic morphism** \(\phi : U \to \mathbb{C}\) defined on an open neighbourhood \(U\) of \(p\). If the space \((M, g)\) is of non-compact type then the domain \(U\) can be chosen to be the whole of \(M\).

An essential tool is the following **Duality Principle**:

**Theorem 4.11 (SG, Svensson 2006)**

Let \(\mathcal{F}\) be a family of local maps \(\phi : W \subset G/K \to \mathbb{C}\) and \(\mathcal{F}^*\) be the dual family consisting of the local maps \(\phi^* : W^* \subset U/K \to \mathbb{C}\). Then \(\mathcal{F}^*\) is a local orthogonal harmonic family on \(U/K\) if and only if \(\mathcal{F}\) is a local orthogonal harmonic family on \(G/K\).
The Duality Principle explains the following.

**Example 4.12 (Baird, Eells 1981)**

The map \( \phi : U \subset S^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 + ix_4}
\]

is a locally defined harmonic morphism.
The **Duality Principle** explains the following.

**Example 4.12 (Baird, Eells 1981)**

The map \( \phi : U \subset S^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \rightarrow \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 + ix_4}
\]

is a **locally defined** harmonic morphism.

**Example 4.13 (SG 1996)**

The map \( \phi : H^3 \subset \mathbb{R}_1^4 \rightarrow \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 - x_4}
\]

is a **globally defined** harmonic morphism.
Our existence result for symmetric spaces has the following interesting consequence:

**Theorem 4.14 (SG, Svensson 2013)**

*Let* \((M, g)\) *be a Riemannian homogeneous space of positive curvature other than the Berger space* \(\text{Sp}(2)/\text{SU}(2)\). *Then* \(M\) *admits local complex-valued harmonic morphisms.*
Fact 5.1

Every Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.
Fact 5.1

Every Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.

(SG, Svensson 2011): Give a classification for 3-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing \(\text{Sol}^3\), not carrying any left-invariant metric admitting complex-valued harmonic morphisms.
Fact 5.1

Every Riemannian homogeneous space $(M, g)$ of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.

(SG, Svensson 2011): Give a classification for 3-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing $\text{Sol}^3$, not carrying any left-invariant metric admitting complex-valued harmonic morphisms.

(SG, Svensson 2013): Give a classification for 4-dimensional Riemannian Lie groups admitting left-invariant solutions. Most of the solutions constructed are NOT holomorphic with respect to any (integrable) Hermitian structure.
Fact 5.1

Every Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.

(SG, Svensson 2011): Give a classification for 3-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing \(\text{Sol}^3\), not carrying any left-invariant metric admitting complex-valued harmonic morphisms.

(SG, Svensson 2013): Give a classification for 4-dimensional Riemannian Lie groups admitting left-invariant solutions. Most of the solutions constructed are NOT holomorphic with respect to any (integrable) Hermitian structure.

(SG 2016): Gives a large collection of 5-dimensional Riemannian Lie groups admitting left-invariant solutions.


