Harmonic Morphisms from Homogeneous Spaces

- Some Existence Theory -

Sigmundur Gudmundsson

Department of Mathematics
Faculty of Science
Lund University

Sigmundur.Gudmundsson@math.lu.se

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Outline

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Definition 1.1 (harmonic morphism)

A map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called a **harmonic morphism** if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

Definition 1.1 (harmonic morphism)

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Theorem 1.2 (Fuglede 1978, Ishihara 1979)

A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is **harmonic** *and* **horizontally (weakly) conformal**.

(harmonicity)

For local coordinates x on M and y on N, we have the **non-linear** system

$$\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}_{ij}^{k} \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\beta}^{\gamma} \circ \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) = 0,$$

where $\phi^\alpha = y_\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on M, N, resp.

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(horizontal conformality)

There exists a continuous function $\lambda : M \to \mathbb{R}_0^+$ such that for all $\alpha, \beta = 1, 2, \dots, n$

$$\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial \phi^\alpha}{\partial x_i}(x) \frac{\partial \phi^\beta}{\partial x_j}(x) = \lambda^2(x) h^{\alpha\beta}(\phi(x)).$$

This is a first order **non-linear** system of $[\binom{n+1}{2} - 1]$ equations.

Theorem 1.3 (Baird, Eells 1981)

Let $\phi : (M, g) \to (N^2, h)$ be a **horizontally conformal** *map from a Riemannian manifold to a surface. Then ϕ is* **harmonic** *if and only if its fibres are* **minimal** *at regular points ϕ.*

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The problem is **invariant** under **isometries** on (M, g). If the codomain is a surface ($n = 2$) then it is also invariant under **conformal changes** $\sigma^2 h$ of the metric on (N^2, h). This means, at least for local studies, that (N^2, h) can be chosen to be the **standard complex plane** $\mathbb{C}$.

Definition 1.4 (Riemannian homogeneous space)

A Riemannian manifold (M, g) is said to be **homogeneous** if it possesses a transitive group G of isometries i.e. if for all $p, q \in M$ there exists an isometry $\phi_{qp} : M \to M$ such that $\phi_{qp}(p) = q$.

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Every **Lie group** (G, g) equipped with a left-invariant Riemannian metric acts transitively on itself.

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Every **Lie group** (G, g) equipped with a left-invariant Riemannian metric acts transitively on itself.

Example 1.6 (Riemannian symmetric space)

Every Riemannian **symmetric space** $M = (G/K, g)$ is homogeneous.

Example 1.7 (the nilpotent Lie group Nil^3)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{\ X = \partial/\partial x,\ Y = \partial/\partial y,\ Z = \partial/\partial z\ \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$

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(Baird, Wood 1990): Every solution is a restriction of the globally defined harmonic morphism $\phi : \mathrm{Nil}^3 \to \mathbb{C}$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto x + iy.$$

Example 1.8 (the solvable Lie group Sol^3)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{SL}_3(\mathbb{R}).$$

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(Baird, Wood 1990): No solutions exist, not even locally.

$$e^{-2z} \frac{\partial^2 \phi}{\partial x^2} + e^{2z} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

$$e^{-2z} \left(\frac{\partial \phi}{\partial x}\right)^2 + e^{2z} \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 = 0.$$

Definition 2.1 (Laplacian, conformality operator)

For functions $\phi, \psi : (M, g) \to \mathbb{C}$ the metric g induces the complex-valued **Laplacian** $\tau(\phi)$ and the symmetric bilinear **conformality operator** κ by

$$\kappa(\phi, \psi) = g(\nabla\phi, \nabla\psi).$$

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The **harmonicity** and the **horizontal conformality** of $\phi : (M, g) \to \mathbb{C}$ are then given by the following relations

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Definition 2.2 (eigenfamily)

A set $\mathcal{E} = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an **eigenfamily** on (M, g) if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$\tau(\phi) = \lambda\phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu\phi\psi.$$

Theorem 2.3 (SG, Sakovich 2008)

Let (M,g) be a Riemannian manifold and $\mathcal{E}=\{\phi_1,\dots,\phi_n\}$ be a **finite eigenfamily** *of complex-valued functions on M. If $P,Q:\mathbb{C}^n\to\mathbb{C}$ are linearily independent homogeneous polynomials of the same positive degree then the quotient*

$$P(\phi_1,\dots,\phi_n)/Q(\phi_1,\dots,\phi_n)$$

is a non-constant **harmonic morphism** *on the open and dense subset*

$$\{p\in M|\ Q(\phi_1(p),\dots,\phi_n(p))\neq 0\}.$$

The authors apply this machine to construct solutions on the **classical semisimple** Lie groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$, $\mathbf{SL}_n(\mathbb{R})$, $\mathbf{SU}^*(2n)$ and $\mathbf{Sp}(n,\mathbb{R})$ equipped with their standard Riemannian metrics.

They also develop a **duality principle** and use this to construct solutions from the **semisimple** Lie groups $\mathbf{SO}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$, $\mathbf{SL}_n(\mathbb{R})$, $\mathbf{SU}^*(2n)$, $\mathbf{Sp}(n,\mathbb{R})$, $\mathbf{SO}^*(2n)$, $\mathbf{SO}(p,q)$, $\mathbf{SU}(p,q)$ and $\mathbf{Sp}(p,q)$ equipped with their standard dual semi-Riemannian metrics.

Equip the special orthogonal group

$$\mathbf{SO}(n) = \{x \in \mathbf{GL}_n(\mathbb{R}) \mid x^t \cdot x = I_n,\ \det x = 1\}$$

with the standard Riemannian metric g induced by the Euclidean scalar product $g(X, Y) = \text{trace}(X^t \cdot Y)$ on the Lie algebra

$$\mathfrak{so}(n) = \{X \in \mathfrak{gl}_n(\mathbb{R}) |\ X^t + X = 0\}.$$

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Lemma 2.4 (SG, Sakovich 2008)

For $1 \leq i, j \leq n$, let $x_{ij} : \mathbf{SO}(n) \to \mathbb{R}$ be the real valued coordinate functions given by $x_{ij} : x \mapsto \langle e_i, x \cdot e_j \rangle$ where $\{e_1, \dots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. Then the following relations hold

$$\tau(x_{ij}) = -\frac{(n-1)}{2} x_{ij}, \quad \kappa(x_{ij}, x_{kl}) = -\frac{1}{2}(x_{il}x_{kj} - \delta_{jl}\delta_{ik}).$$

Theorem 2.5 (SG, Sakovich 2008)

Let $p \in \mathbb{C}^n$ *be a non-zero isotropic element i.e.* $\langle p, p \rangle = 0$. *Then*

$$\mathcal{E}_p = \{\phi_a : \mathbf{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle, \ a \in \mathbb{C}^n\}.$$

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Example 2.6 (eigenfamilies on SO(n))

For $z, w \in \mathbb{C}$, let p be the isotropic element of $\mathbb{C}^4$ given by

$$p(z,w) = (1 + zw, i(1 - zw), i(z + w), z - w).$$

This gives us the complex 2-dimensional deformation of eigenfamilies $\mathcal{E}_p$ each consisting of functions $\phi_a : \mathbf{SO}(4) \to \mathbb{C}$ with

$$\begin{aligned} \phi_a(x) = &(1 + zw)(x_{11}a_1 + x_{21}a_2 + x_{31}a_3 + x_{41}a_4) \\ &+i(1 - zw)(x_{12}a_1 + x_{22}a_2 + x_{32}a_3 + x_{42}a_4) \\ &+i(z + w)(x_{13}a_1 + x_{23}a_2 + x_{33}a_3 + x_{43}a_4) \\ &+(z - w)(x_{14}a_1 + x_{24}a_2 + x_{34}a_3 + x_{44}a_4) \end{aligned}$$

Definition 3.1 (orthogonal harmonic family)

A set $\Omega = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an **orthogonal harmonic family** on (M, g) if for all $\phi, \psi \in \Omega$

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Example 3.2

Let $\Omega = \{\phi_\alpha : (M, g, J) \to \mathbb{C} \mid \alpha \in I\}$ be a collection of **holomorphic** functions on a Kähler manifold. Then Ω is an **orthogonal harmonic family**.

Theorem 3.3 (SG 1997)

Let (M, g) be a Riemannian manifold and U be an open subset of $\mathbb{C}^n$ containing the image of $\Phi = (\phi_1, \dots, \phi_n) : M \to \mathbb{C}^n$. Further let

$$H = \{F_\alpha : U \to \mathbb{C} \mid \alpha \in I\}$$

be a collection of **holomorphic** *functions defined on U. If the finite set*

$$\Omega = \{\phi_k : (M, g) \to \mathbb{C} \mid k = 1, \dots, n\}$$

is an **orthogonal harmonic family** *on (M, g) then*

$$\Omega_H = \{\psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \dots, \phi_n),\ \ F \in H\}$$

is again an **orthogonal harmonic family**.

Let G be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a **natural group epimorphism** $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.

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Fact 3.4 (semisimple - solvable - nilpotent)

If the group G is **semisimple** *then $d = 0$, if G is* **solvable** *then $d \geq 1$ and if G is* **nilpotent** *then $d \geq 2$.*

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Equip $\mathbb{R}^d$ with its standard Euclidean metric and the Lie group G with a left-invariant Riemannian metric g such that the natural group epimorphism $\Phi : G \to \mathbb{R}^d$ is a **Riemannian submersion**. Then the kernel $[\mathfrak{g}, \mathfrak{g}]$ of the linear map $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ generates a left-invariant Riemannian foliation $\mathcal{V}$ on (G, g) with orthogonal distribution $\mathcal{H} = [\mathfrak{g}, \mathfrak{g}]^{\perp}$.

Theorem 3.5 (SG, Svensson 2009)

Let $\mathcal{B} = \{X_1, \dots, X_d\}$ be an ONB for the horizontal subspace $[\mathfrak{g}, \mathfrak{g}]^\perp$ of $\mathfrak{g}$ and $\xi \in \mathbb{C}^d$ be given by $\xi = (\operatorname{trace} \operatorname{ad}_{X_1}, \dots, \operatorname{trace} \operatorname{ad}_{X_d})$. For a maximal isotropic subspace V of $\mathbb{C}^d$ put

$$V_\xi = \{v \in V \mid \langle \xi, v \rangle = 0\}.$$

If the real dimension of the isotropic subspace V_ξ is at least 2 then

$$\Omega = \{\phi_v(x) = \langle \Phi(x), v \rangle \mid v \in V_\xi\}$$

is an **orthogonal harmonic family** *on (G, g).*

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Proof.

The tension field of Φ satisfies

$$\tau(\Phi)(p) = \sum_{k=1}^{d} (\operatorname{trace} \operatorname{ad}_{X_k}) d\Phi_e(X_k).$$

□

Example 3.6

For the **nilpotent** Riemannian Lie group

$$N_n = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & x_{n-1,n} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbf{SL}_n(\mathbb{R}) \mid x_{ij} \in \mathbb{R} \right\}.$$

the **natural group epimorphism** $\Phi : N_n \to \mathbb{R}^{n-1}$ is given by

$$\Phi(x) = (x_{12}, \ldots, x_{n-1,n}).$$

Example 3.7

For the **solvable** Riemannian Lie group

$$S_n = \left\{ \begin{pmatrix} e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\ 0 & \cdots & 0 & 0 & e^{t_n} \end{pmatrix} \in \mathbf{GL}_n(\mathbb{R}) \mid x_{ij}, t_i \in \mathbb{R} \right\}$$

the **natural group epimorphism** $\Phi : S_n \to \mathbb{R}^n$ is given by

$$\Phi(x) = (t_1, t_2, \ldots, t_n)$$

and the vector $\xi \in \mathbb{C}^n$ satisfies

$$\xi = ((n+1) - 2, (n+1) - 4, \ldots, (n+1) - 2n).$$

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Fact 3.8 (solvable Lie group - rank)

The **symmetric space** (M, g) *can be identified with the* **solvable subgroup** $S = NA$ *of G and its* **rank** r *is the dimension of abelian subgroup* A*.*

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Let $\mathfrak{s}, \mathfrak{n}, \mathfrak{a}$ be the Lie algebras of S, N, A, respectively. For this situation we have $\mathfrak{s} = \mathfrak{a} + \mathfrak{n} = \mathfrak{a} + [\mathfrak{s}, \mathfrak{s}]$, hence

$$\mathfrak{a} = \mathfrak{s}/[\mathfrak{s}, \mathfrak{s}].$$

With a series of different methods we have obtained the following result:

Theorem 3.9 (SG, Svensson 2009)

Let (M, g) be an irreducible Riemannian **symmetric space** *other than $G_2^*/\mathbf{SO}(4)$ or its compact dual $G_2/\mathbf{SO}(4)$. Then for each point $p \in M$* **there exists** *a non-constant complex-valued* **harmonic morphism** *$\phi : U \to \mathbb{C}$ defined on an open neighbourhood U of p. If the space (M, g) is of non-compact type then the domain U can be chosen to be the whole of M.*

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An essential tool is the following **Duality Principle**:

Theorem 3.10 (SG, Svensson 2006)

Let $\mathcal{F}$ be a family of local maps $\phi : W \subset G/K \to \mathbb{C}$ and $\mathcal{F}^*$ be the dual family consisting of the local maps $\phi^* : W^* \subset U/K \to \mathbb{C}$. Then $\mathcal{F}^*$ is a local orthogonal harmonic family on U/K **if and only if** *$\mathcal{F}$ is a local orthogonal harmonic family on G/K.*

The **Duality Principle** explains the following.

Example 3.11 (Baird, Eells 1981)

The map $\phi : U \subset S^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{C}$ given by

$$\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 + ix_4}$$

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Example 3.12 (SG 1996)

The map $\phi : H^3 \subset \mathbb{R}^4_1 \to \mathbb{C}$ given by

$$\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 - x_4}$$

is a **globally defined** harmonic morphism.

Our existence result for symmetric spaces has the following **interesting consequence**:

Theorem 3.13 (SG, Svensson 2013)

Let (M, g) *be a Riemannian* **homogeneous** *space of* **positive curvature** *other than the Berger space* $\mathbf{Sp}(2)/\mathbf{SU}(2)$. *Then* M *admits local complex-valued harmonic morphisms.*

Fact 4.1

Every Riemannian **homogeneous** *space* (M, g) *of dimension* **3** *or* **4** *is either* **symmetric** *or a* **Lie group** *with a left-invariant metric.*

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(SG, Svensson 2011): Give a **classification** for **3**-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing Sol^3, not carrying any left-invariant metric admitting complex-valued harmonic morphisms.

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(SG, Svensson 2013): Give a **classification** for **4**-dimensional Riemannian Lie groups admitting **left-invariant** solutions. Most of the solutions constructed are **NOT holomorphic** with respect to any (integrable) Hermitian structure.

Fact 4.1

Every Riemannian **homogeneous** *space* (M, g) *of dimension* **3** *or* **4** *is either* **symmetric** *or a* **Lie group** *with a left-invariant metric.*

(SG, Svensson 2011): Give a **classification** for **3**-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing Sol^3, not carrying any left-invariant metric admitting complex-valued harmonic morphisms.

(SG, Svensson 2013): Give a **classification** for **4**-dimensional Riemannian Lie groups admitting **left-invariant** solutions. Most of the solutions constructed are **NOT holomorphic** with respect to any (integrable) Hermitian structure.

(SG 2016): Gives a large collection of **5**-dimensional Riemannian Lie groups admitting **left-invariant** solutions.

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