Harmonic Morphisms from Homogeneous Spaces
- Some Existence Theory -

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   • Basics
   • Geometric Motivation
   • Homogeneous Spaces
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Definition 1.1 (harmonic morphism)

A map $\phi : (M^m, g) \rightarrow (N^n, h)$ between Riemannian manifolds is called a **harmonic morphism** if, for any harmonic function $f : U \rightarrow \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \rightarrow \mathbb{R}$ is a harmonic function.
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Theorem 1.2 (Fuglede 1978, Ishihara 1979)

A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.
For local coordinates $x$ on $M$ and $y$ on $N$, we have the non-linear system

$$
\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left( \frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}^k_{ij} \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma_{\alpha\beta} \circ \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) = 0,
$$

where $\phi^\alpha = y_\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.
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where $\phi^\alpha = y^\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.

There exists a continuous function $\lambda : M \to \mathbb{R}_0^+$ such that for all $\alpha, \beta = 1, 2, \ldots, n$

$$
\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial \phi^\alpha}{\partial x_i}(x) \frac{\partial \phi^\beta}{\partial x_j}(x) = \lambda^2(x) h^{\alpha\beta}(\phi(x)).
$$

This is a first order non-linear system of $\left(\binom{n+1}{2} - 1\right)$ equations.
Theorem 1.3 (Baird, Eells 1981)

Let $\phi : (M, g) \rightarrow (N^2, h)$ be a \textbf{horizontally conformal} map from a Riemannian manifold to a surface. Then $\phi$ is \textbf{harmonic} if and only if its fibres are \textbf{minimal} at regular points $\phi$. 
Theorem 1.3 (Baird, Eells 1981)

Let $\phi : (M, g) \rightarrow (N^2, h)$ be a horizontally conformal map from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if its fibres are minimal at regular points $\phi$.

The problem is invariant under isometries on $(M, g)$. If the codomain is a surface ($n = 2$) then it is also invariant under conformal changes $\sigma^2 h$ of the metric on $(N^2, h)$. This means, at least for local studies, that $(N^2, h)$ can be chosen to be the standard complex plane $\mathbb{C}$. 
Definition 1.4 (Riemannian homogeneous space)

A Riemannian manifold \((M, g)\) is said to be \textbf{homogeneous} if it possesses a transitive group \(G\) of isometries i.e. if for all \(p, q \in M\) there exists an isometry \(\phi_{qp} : M \to M\) such that \(\phi_{qp}(p) = q\).
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Example 1.5 (Riemannian Lie group)

Every \textbf{Lie group} \((G, g)\) equipped with a left-invariant Riemannian metric acts transitively on itself.
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Every **Lie group** \((G, g)\) equipped with a left-invariant Riemannian metric acts transitively on itself.

Example 1.6 (Riemannian symmetric space)

Every Riemannian **symmetric space** \(M = (G/K, g)\) is homogeneous.
Example 1.7 (the nilpotent Lie group $\text{Nil}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = dx^2 + dy^2 + (dz - x dy)^2.$$
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(Baird, Wood 1990): Every solution is a restriction of the globally defined harmonic morphism $\phi : \text{Nil}^3 \rightarrow \mathbb{C}$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto x + iy.$$
Example 1.8 (the solvable Lie group $\text{Sol}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

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*(Baird, Wood 1990)*: No solutions exist, not even locally.

$$e^{-2z} \frac{\partial^2 \phi}{\partial x^2} + e^{2z} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

$$e^{-2z} \left( \frac{\partial \phi}{\partial x} \right)^2 + e^{2z} \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 = 0.$$
Definition 2.1 (Laplacian, conformality operator)

For functions $\phi, \psi : (M, g) \to \mathbb{C}$ the metric $g$ induces the complex-valued Laplacian $\tau(\phi)$ and the symmetric bilinear conformality operator $\kappa$ by

$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$
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The harmonicity and the horizontal conformality of $\phi : (M, g) \rightarrow \mathbb{C}$ are then given by the following relations

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$
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Definition 2.2 (eigenfamily)

A set $\mathcal{E} = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an eigenfamily on $(M, g)$ if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$\tau(\phi) = \lambda \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \phi \psi.$$
Theorem 2.3 (SG, Sakovich 2008)

Let \((M, g)\) be a Riemannian manifold and \(\mathcal{E} = \{\phi_1, \ldots, \phi_n\}\) be a finite eigenfamily of complex-valued functions on \(M\). If \(P, Q : \mathbb{C}^n \to \mathbb{C}\) are linearly independent homogeneous polynomials of the same positive degree then the quotient

\[
P(\phi_1, \ldots, \phi_n)/Q(\phi_1, \ldots, \phi_n)
\]

is a non-constant harmonic morphism on the open and dense subset

\[
\{p \in M | Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.
\]

The authors apply this machine to construct solutions on the classical semisimple Lie groups \(\text{SO}(n), \text{SU}(n), \text{Sp}(n), \text{SL}_n(\mathbb{R}), \text{SU}^*(2n)\) and \(\text{Sp}(n, \mathbb{R})\) equipped with their standard Riemannian metrics.

They also develop a duality principle and use this to construct solutions from the semisimple Lie groups \(\text{SO}(n), \text{SU}(n), \text{Sp}(n), \text{SL}_n(\mathbb{R}), \text{SU}^*(2n), \text{Sp}(n, \mathbb{R}), \text{SO}^*(2n), \text{SO}(p,q), \text{SU}(p,q)\) and \(\text{Sp}(p,q)\) equipped with their standard dual semi-Riemannian metrics.
Equip the special orthogonal group
\[ \text{SO}(n) = \{ x \in \text{GL}_n(\mathbb{R}) \mid x^t \cdot x = I_n, \det x = 1 \} \]
with the standard Riemannian metric \( g \) induced by the Euclidean scalar product \( g(X, Y) = \text{trace}(X^t \cdot Y) \) on the Lie algebra
\[ \mathfrak{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}. \]
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$$\mathfrak{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}.$$  

**Lemma 2.4 (SG, Sakovich 2008)**

For $1 \leq i, j \leq n$, let $x_{ij}: \text{SO}(n) \to \mathbb{R}$ be the real valued coordinate functions given by $x_{ij}: x \mapsto \langle e_i, x \cdot e_j \rangle$ where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$. Then the following relations hold

$$\tau(x_{ij}) = -\frac{(n-1)}{2}x_{ij}, \quad \kappa(x_{ij}, x_{kl}) = -\frac{1}{2}(x_{il}x_{kj} - \delta_{jl}\delta_{ik}).$$
Theorem 2.5 (SG, Sakovich 2008)

Let \( p \in \mathbb{C}^n \) be a non-zero isotropic element i.e. \( \langle p, p \rangle = 0 \). Then

\[
E_p = \{ \phi_a : \text{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle, \ a \in \mathbb{C}^n \}.
\]

is an eigenfamily on \( \text{SO}(n) \).
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Let \( p \in \mathbb{C}^n \) be a non-zero isotropic element i.e. \( \langle p, p \rangle = 0 \). Then

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\]

is an eigenfamily on SO\((n)\)

Example 2.6 (eigenfamilies on SO\((n)\))

For \( z, w \in \mathbb{C} \), let \( p \) be the isotropic element of \( \mathbb{C}^4 \) given by

\[
p(z, w) = (1 + zw, i(1 - zw), i(z + w), z - w).
\]

This gives us the complex 2-dimensional deformation of eigenfamilies \( \mathcal{E}_p \) each consisting of functions \( \phi_a : \text{SO}(4) \to \mathbb{C} \) with

\[
\phi_a(x) = (1 + zw)(x_{11}a_1 + x_{21}a_2 + x_{31}a_3 + x_{41}a_4)
+ i(1 - zw)(x_{12}a_1 + x_{22}a_2 + x_{32}a_3 + x_{42}a_4)
+ i(z + w)(x_{13}a_1 + x_{23}a_2 + x_{33}a_3 + x_{43}a_4)
+ (z - w)(x_{14}a_1 + x_{24}a_2 + x_{34}a_3 + x_{44}a_4)
\]
Definition 3.1 (orthogonal harmonic family)

A set $\Omega = \{\phi_{\alpha} : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an orthogonal harmonic family on $(M, g)$ if for all $\phi, \psi \in \Omega$

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$
Definition 3.1 (orthogonal harmonic family)

A set $\Omega = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an orthogonal harmonic family on $(M, g)$ if for all $\phi, \psi \in \Omega$

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Example 3.2

Let $\Omega = \{\phi_\alpha : (M, g, J) \to \mathbb{C} \mid \alpha \in I\}$ be a collection of holomorphic functions on a Kähler manifold. Then $\Omega$ is an orthogonal harmonic family.
Theorem 3.3 (SG 1997)

Let \((M, g)\) be a Riemannian manifold and \(U\) be an open subset of \(\mathbb{C}^n\) containing the image of \(\Phi = (\phi_1, \ldots, \phi_n) : M \to \mathbb{C}^n\). Further let

\[ H = \{ F_{\alpha} : U \to \mathbb{C} \mid \alpha \in I \} \]

be a collection of holomorphic functions defined on \(U\). If the finite set

\[ \Omega = \{ \phi_k : (M, g) \to \mathbb{C} \mid k = 1, \ldots, n \} \]

is an orthogonal harmonic family on \((M, g)\) then

\[ \Omega_H = \{ \psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \ldots, \phi_n), \ F \in H \} \]

is again an orthogonal harmonic family.
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \rightarrow \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a **natural group epimorphism** $\Phi : G \rightarrow \mathbb{R}^d$ with $d = \dim \mathfrak{a}$. 
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a natural group epimorphism $\Phi : G \to \mathbb{R}^d$ with $d = \text{dim} \mathfrak{a}$.

**Fact 3.4 (semisimple - solvable - nilpotent)**

*If the group $G$ is semisimple then $d = 0$, if $G$ is solvable then $d \geq 1$ and if $G$ is nilpotent then $d \geq 2$.***
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a natural group epimorphism $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.

**Fact 3.4 (semisimple - solvable - nilpotent)**

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Equip $\mathbb{R}^d$ with its standard Euclidean metric and the Lie group $G$ with a left-invariant Riemannian metric $g$ such that the natural group epimorphism $\Phi : G \to \mathbb{R}^d$ is a **Riemannian submersion**. Then the kernel $[\mathfrak{g}, \mathfrak{g}]$ of the linear map $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ generates a left-invariant Riemannian foliation $\mathcal{V}$ on $(G, g)$ with orthogonal distribution $\mathcal{H} = [\mathfrak{g}, \mathfrak{g}]^\perp$. 
Theorem 3.5 (SG, Svensson 2009)

Let \( B = \{X_1, \ldots, X_d\} \) be an ONB for the horizontal subspace \([g, g] \perp\) of \( g\) and \( \xi \in \mathbb{C}^d \) be given by \( \xi = (\text{trace } \text{ad} \, X_1, \ldots, \text{trace } \text{ad} \, X_d) \). For a maximal isotropic subspace \( V \) of \( \mathbb{C}^d \) put

\[
V_\xi = \{ v \in V \mid \langle \xi, v \rangle = 0 \}.
\]

If the real dimension of the isotropic subspace \( V_\xi \) is at least 2 then

\[
\Omega = \{ \phi_v(x) = \langle \Phi(x), v \rangle \mid v \in V_\xi \}
\]

is an orthogonal harmonic family on \((G, g)\).
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Proof.

The tension field of \(\Phi\) satisfies

\[
\tau(\Phi)(p) = \sum_{k=1}^{d} (\text{trace ad}X_k) d\Phi_e(X_k).
\]
Example 3.6

For the nilpotent Riemannian Lie group

\[
N_n = \left\{ \begin{pmatrix}
1 & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\
0 & 1 & \ddots & \vdots & \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & 1 & x_{n-1,n} \\
0 & \ldots & \ldots & 0 & 1
\end{pmatrix} \in \text{SL}_n(\mathbb{R}) \mid x_{ij} \in \mathbb{R} \right\}.
\]

the natural group epimorphism \( \Phi : N_n \to \mathbb{R}^{n-1} \) is given by

\[
\Phi(x) = (x_{12}, \ldots, x_{n-1,n}).
\]
Example 3.7

For the **solvable** Riemannian Lie group

$$S_n = \left\{ \begin{pmatrix} e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\ 0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\ 0 & \cdots & 0 & 0 & e^{t_n} \end{pmatrix} \in \text{GL}_n(\mathbb{R}) \mid x_{ij}, t_i \in \mathbb{R} \right\}$$

the **natural group epimorphism** $\Phi : S_n \to \mathbb{R}^n$ is given by

$$\Phi(x) = (t_1, t_2, \ldots, t_n)$$

and the vector $\xi \in \mathbb{C}^n$ satisfies

$$\xi = ((n + 1) - 2, (n + 1) - 4, \ldots, (n + 1) - 2n).$$
Let \((M, g)\) be an irreducible Riemannian symmetric space of non-compact type and write \(M = G/K\) with \(G\) a semisimple, connected and simply connected Lie group and \(K\) a maximal compact subgroup of \(G\).
Let \((M, g)\) be an irreducible Riemannian **symmetric space** of **non-compact type** and write \(M = G/K\) with \(G\) a semisimple, connected and simply connected Lie group and \(K\) a maximal compact subgroup of \(G\).

Let \(G = NAK\) be an **Iwasawa decomposition** of \(G\), where \(N\) is **nilpotent** and \(A\) is **abelian**.
Let \((M, g)\) be an irreducible Riemannian symmetric space of non-compact type and write \(M = G/K\) with \(G\) a semisimple, connected and simply connected Lie group and \(K\) a maximal compact subgroup of \(G\).

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**Fact 3.8 (solvable Lie group - rank)**

The symmetric space \((M, g)\) can be identified with the solvable subgroup \(S = NA\) of \(G\) and its rank \(r\) is the dimension of abelian subgroup \(A\).
Let $(M, g)$ be an irreducible Riemannian symmetric space of non-compact type and write $M = G/K$ with $G$ a semisimple, connected and simply connected Lie group and $K$ a maximal compact subgroup of $G$.

Let $G = NAK$ be an Iwasawa decomposition of $G$, where $N$ is nilpotent and $A$ is abelian.

**Fact 3.8 (solvable Lie group - rank)**

*The symmetric space $(M, g)$ can be identified with the solvable subgroup $S = NA$ of $G$ and its rank $r$ is the dimension of abelian subgroup $A$.*

Let $\mathfrak{s}, \mathfrak{n}, \mathfrak{a}$ be the Lie algebras of $S, N, A$, respectively. For this situation we have $\mathfrak{s} = \mathfrak{a} + \mathfrak{n} = \mathfrak{a} + [\mathfrak{s}, \mathfrak{s}]$, hence

$$\mathfrak{a} = \mathfrak{s}/[\mathfrak{s}, \mathfrak{s}].$$
With a series of different methods we have obtained the following result:

**Theorem 3.9 (SG, Svensson 2009)**

Let $(M, g)$ be an irreducible Riemannian **symmetric space** other than $G_2^*/SO(4)$ or its compact dual $G_2/ SO(4)$. Then for each point $p \in M$ there exists a non-constant complex-valued **harmonic morphism** $\phi : U \to \mathbb{C}$ defined on an open neighbourhood $U$ of $p$. If the space $(M, g)$ is of non-compact type then the domain $U$ can be chosen to be the whole of $M$. 
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Let \((M, g)\) be an irreducible Riemannian **symmetric space** other than \(G_2^*/\SO(4)\) or its compact dual \(G_2/\SO(4)\). Then for each point \(p \in M\) **there exists** a non-constant complex-valued **harmonic morphism** \(\phi : U \to \mathbb{C}\) defined on an open neighbourhood \(U\) of \(p\). If the space \((M, g)\) is of non-compact type then the domain \(U\) can be chosen to be the whole of \(M\).

An essential tool is the following **Duality Principle**:

**Theorem 3.10 (SG, Svensson 2006)**

Let \(\mathcal{F}\) be a family of local maps \(\phi : W \subset G/K \to \mathbb{C}\) and \(\mathcal{F}^*\) be the dual family consisting of the local maps \(\phi^* : W^* \subset U/K \to \mathbb{C}\). Then \(\mathcal{F}^*\) is a local orthogonal harmonic family on \(U/K\) **if and only if** \(\mathcal{F}\) is a local orthogonal harmonic family on \(G/K\).
The **Duality Principle** explains the following.

**Example 3.11 (Baird, Eells 1981)**

The map \( \phi : U \subset S^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \to \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 + ix_4}
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is a **locally defined** harmonic morphism.

**Example 3.12 (SG 1996)**

The map \( \phi : H^3 \subset \mathbb{R}^4_1 \to \mathbb{C} \) given by
\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 - x_4}
\]
is a **globally defined** harmonic morphism.
Our existence result for symmetric spaces has the following interesting consequence:

**Theorem 3.13 (SG, Svensson 2013)**

Let $(M, g)$ be a Riemannian homogeneous space of positive curvature other than the Berger space $\text{Sp}(2)/\text{SU}(2)$. Then $M$ admits local complex-valued harmonic morphisms.
Fact 4.1

Every Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.
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(SG 2016): Gives a large collection of 5-dimensional Riemannian Lie groups admitting left-invariant solutions.


