Outline

1 Harmonic Maps in Gaussian Geometry
   - Holomorphic Functions in One Variable
   - Minimal Surfaces in $\mathbb{R}^3$
   - Harmonic Morphisms in $\mathbb{R}^3$
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1 Harmonic Maps in Gaussian Geometry
   • Holomorphic Functions in One Variable
   • Minimal Surfaces in $\mathbb{R}^3$
   • Harmonic Morphisms in $\mathbb{R}^3$

2 Harmonic Maps in Riemannian Geometry
   • Minimal Surfaces
   • Holomorphic Functions in Several Variables
   • Harmonic Morphisms
Let $W$ be an open subset of the complex plane $\mathbb{C}$. A function

$$\phi = u + iv : W \to \mathbb{C}$$

is said to be **holomorphic** if it satisfies the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$
Let $W$ be an open subset of the complex plane $\mathbb{C}$. A function
\[ \phi = u + iv : W \to \mathbb{C} \]
is said to be holomorphic if it satisfies the Cauchy-Riemann equations
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \]

As a direct consequence of these equations we see that
\[ \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 v}{\partial y^2}. \]

This implies that the functions $u, v : W \to \mathbb{R}$ are harmonic, hence
\[ \Delta \phi = \Delta (u + iv) = \Delta u + i\Delta v = 0. \]
Theorem 1.1

Let $u : W \rightarrow \mathbb{R}$ be a real-valued $C^2$-function defined on an open simply connected subset $W$ of $\mathbb{C}$. Then the following are equivalent,

i) $u$ is the real part of a **holomorphic** $\phi = u + iv : W \rightarrow \mathbb{C},$

ii) $u$ is **harmonic** i.e. $\Delta u = 0.$
Theorem 1.1

Let \( u : W \rightarrow \mathbb{R} \) be a real-valued \( C^2 \)-function defined on an open simply connected subset \( W \) of \( \mathbb{C} \). Then the following are equivalent,

i) \( u \) is the real part of a holomorphic \( \phi = u + iv : W \rightarrow \mathbb{C} \),

ii) \( u \) is harmonic i.e. \( \Delta u = 0 \).

For the gradients \( \nabla u = (\partial u/\partial x, \partial u/\partial y) \), \( \nabla v = (\partial v/\partial x, \partial v/\partial y) \) we have

\[
\langle \nabla u, \nabla v \rangle = \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} = 0,
\]

and

\[
|\nabla u|^2 = \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2 = \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 = |\nabla v|^2.
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and

$$|\nabla u|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = |\nabla v|^2.$$

This means that the two gradients are orthogonal and of the same length at each point in $W$ i.e. the map $\phi : W \to \mathbb{C}$ is conformal, or equivalently,

$$\langle \nabla \phi, \nabla \phi \rangle = \langle \nabla (u + iv), \nabla (u + iv) \rangle = 0.$$
Definition 1.2 (minimal surface)

Let $\Sigma_t$ with $t \in (-\epsilon, \epsilon)$ be a family of surfaces with a common boundary curve i.e. $\partial \Sigma_0 = \partial \Sigma_t$ for all $t \in (-\epsilon, \epsilon)$. Then $\Sigma_0$ is said to be a **minimal surface** if it is a critical point for the area functional i.e.

$$\frac{d}{dt} \left( \int_{\Sigma_t} dA \right) |_{t=0} = 0.$$
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Theorem 1.3 (zero mean curvature)

A surface $\Sigma$ in $\mathbb{R}^3$ is **minimal** if and only if its mean curvature vanishes i.e.

$$H = \frac{k_1 + k_2}{2} \equiv 0.$$
There is an interesting connection between the theory of minimal surfaces, harmonic functions and hence complex analysis.

**Theorem 1.4 (the Weierstrass representation - 1866)**

*Every minimal surface Σ in \( \mathbb{R}^3 \) can locally be parametrized by a conformal and harmonic map \( \phi : W \subset \mathbb{C} \to \mathbb{R}^3 \) of the form

\[
\phi : z \mapsto \text{Re} \int_{z_0}^{z} f(w) \left[ 1 - g^2(w), i(1 + g^2(w)), 2g(w) \right] dw,
\]

where \( f, g : W \subset \mathbb{C} \to \mathbb{C} \) are two holomorphic functions.*
Definition 1.5 (harmonic morphism)

The map $\phi = u + iv : U \subset \mathbb{R}^3 \rightarrow \mathbb{C}$ is said to be a \textbf{harmonic morphism} if the composition $f \circ \phi$ with any \textbf{holomorphic} function $f : W \subset \mathbb{C} \rightarrow \mathbb{C}$ is \textbf{harmonic}.
Definition 1.5 (harmonic morphism)

The map $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is said to be a **harmonic morphism** if the composition $f \circ \phi$ with any **holomorphic** function $f : W \subset \mathbb{C} \to \mathbb{C}$ is **harmonic**.

Theorem 1.6 (Jacobi 1848)

*The map $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal i.e.*

$$\Delta u = \Delta v = 0, \quad \langle \nabla u, \nabla v \rangle = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla v|^2.$$

Proof.\[\Delta (f \circ \phi) = \partial f / \partial z \cdot \Delta \phi + \partial^2 f / \partial z^2 \cdot \langle \nabla \phi, \nabla \phi \rangle = 0.\]
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Theorem 1.6 (Jacobi 1848)

The map \( \phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C} \) is a harmonic morphism if and only if it is **harmonic** and horizontally (weakly) **conformal** i.e.

\[
\Delta u = \Delta v = 0, \quad \langle \nabla u, \nabla v \rangle = 0 \quad \text{and} \quad |\nabla u|^2 = |\nabla v|^2.
\]

Proof.

\[
\Delta (f \circ \phi) = \frac{\partial f}{\partial z} \cdot \Delta \phi + \frac{\partial^2 f}{\partial z^2} \cdot \langle \nabla \phi, \nabla \phi \rangle = 0.
\]
Theorem 1.7 (the Jacobi representation - 1848)

Let $f, g : W \subset \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic functions, then every local solution $z : U \subset \mathbb{R}^3 \rightarrow \mathbb{C}$ to the equation

$$\langle f(z(x)) \left[ 1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x)) \right], x \rangle = 1$$

is a harmonic morphism.
Theorem 1.7 (the Jacobi representation - 1848)

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\langle f(z(x)) \left[ 1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x)) \right], x \rangle = 1
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is a harmonic morphism.

Theorem 1.8 (Baird, Wood 1988)

Every local harmonic morphism \( z : U \to \mathbb{C} \) in the Euclidean \( \mathbb{R}^3 \) is obtained this way.
Example 1.9 (the outer disc example)

Let \( r \in \mathbb{R}^+ \) and choose \( g(z) = z \), \( f(z) = -1/2irz \) then we obtain

\[
(x_1 - ix_2)z^2 - 2(x_3 + ir)z - (x_1 + ix_2) = 0
\]

with the two solutions

\[
z_r^\pm = \frac{- (x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}.
\]
Definition 2.1 (harmonic)

A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be harmonic if it is a critical point for the energy functional

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dvol_M.$$
Definition 2.1 (harmonic)

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The Euler-Lagrange equation for harmonic maps can be expressed, in local coordinates \( x \) on \( M \) and \( y \) on \( N \), as the following \textbf{non-linear} system of PDEs

\[
\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left( \frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}^k_{ij} \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma^\gamma_{\alpha\beta} \circ \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) = 0
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where \( \phi^\alpha = y_\alpha \circ \phi \) and \( \hat{\Gamma} \) and \( \Gamma \) the Christoffel symbols on \( M \) and \( N \), respectively.
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A map \( \phi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is said to be **harmonic** if it is a critical point for the **energy functional**

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where \( \phi^\alpha = y_\alpha \circ \phi \) and \( \hat{\Gamma} \) and \( \Gamma \) the Christoffel symbols on \( M \) and \( N \), respectively.

Example 2.2 (geodesic)
A curve \( \gamma : I \rightarrow (N, h) \) is **harmonic** if and only if it is a **geodesic**.
Theorem 2.3 (Eells, Sampson 1964)

A conformal immersion \( \phi : (M^2, g) \to (N, h) \) of a surface is harmonic if and only if the image \( \phi(M) \) is an immersed minimal surface in \( N \).
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Definition 2.4 (constant mean curvature)

A surface \( \Sigma \) in \( \mathbb{R}^3 \) is said to be of **constant mean curvature** if there exists a constant \( c \in \mathbb{R} \) such that the mean curvature \( H \) of \( \Sigma \) satisfies

\[
H = \frac{k_1 + k_2}{2} = c.
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Theorem 2.3 (Eells, Sampson 1964)

A conformal immersion $\phi : (M^2, g) \to (N, h)$ of a surface is harmonic if and only if the image $\phi(M)$ is an immersed minimal surface in $N$.

Definition 2.4 (constant mean curvature)

A surface $\Sigma$ in $\mathbb{R}^3$ is said to be of constant mean curvature if there exists a constant $c \in \mathbb{R}$ such that the mean curvature $H$ of $\Sigma$ satisfies

$$H = \frac{k_1 + k_2}{2} \equiv c.$$

Theorem 2.5 (Ruh, Vilms 1970)

A surface $\Sigma$ in $\mathbb{R}^3$ is of constant mean curvature if and only if its Gauss map $N : \Sigma \to S^2$ is harmonic.
Let \( \phi = u + iv : U \rightarrow \mathbb{C} \) be a \textbf{holomorphic} function defined on an open subset \( U \) of \( \mathbb{C}^m \). Then it satisfies the Cauchy-Riemann equations i.e. if for each \( k = 1, 2, \ldots, m \)

\[
\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k} \quad \text{and} \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}.
\]

As a direct consequence of these equations we get

\[
\frac{\partial^2 u}{\partial x_k^2} = \frac{\partial^2 v}{\partial x_k \partial y_k} = -\frac{\partial^2 u}{\partial y_k^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x_k^2} = -\frac{\partial^2 u}{\partial x_k \partial y_k} = -\frac{\partial^2 v}{\partial y_k^2}.
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Let $\phi = u + iv : U \to \mathbb{C}$ be a **holomorphic** function defined on an open subset $U$ of $\mathbb{C}^m$. Then it satisfies the Cauchy-Riemann equations i.e. if for each $k = 1, 2, \ldots, m$

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This implies that the functions $u, v : U \to \mathbb{R}$ are **harmonic**, hence

$$\Delta \phi = \Delta (u + iv) = \Delta u + i \Delta v = 0.$$  

For the two gradients $\nabla u, \nabla v$ we have the following

$$\langle \nabla \phi, \nabla \phi \rangle = \langle \nabla (u + iv), \nabla (u + iv) \rangle = 0.$$  

If $\psi : U \to \mathbb{C}$ is another **holomorphic** function then the polar identity

$$4\langle \nabla \phi, \nabla \psi \rangle = \langle \nabla (\phi + \psi), \nabla (\phi + \psi) \rangle - \langle \nabla (\phi - \psi), \nabla (\phi - \psi) \rangle$$

gives

$$\langle \nabla \phi, \nabla \psi \rangle = 0.$$
Definition 2.6 (harmonic morphism)

A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is said to be a **harmonic morphism** if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.
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Theorem 2.7 (Fuglede 1978)

Let $\phi : (M, g) \to (N^n, h)$ be a non-constant harmonic morphism. Then $m \geq n$ and

$$M^* = \{ x \in M | \text{rank } d\phi_x = n \}$$

is an **open and dense** subset of $M$. 

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is an **open and dense** subset of $M$.

**Theorem 2.8 (Fuglede 1978)**

Let $\phi : (M, g) \to (N, h)$ be a non-constant harmonic morphism. If $M$ is **compact**, then $N$ is **compact**.
Theorem 2.9 (Fuglede 1978, Ishihara 1979)

A map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal.
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Definition 2.10 (horizontally (weakly) conformal)

A map \( \phi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is said to be horizontally (weakly) conformal at \( x \in M \) if either \( d\phi_x = 0 \), or \( d\phi_x \) maps the horizontal space \( \mathcal{H}_x = (\text{Ker}(d\phi_x))^\perp \) conformally onto \( T_{\phi(x)}N \) i.e. there exist a function \( \lambda : M \rightarrow \mathbb{R}^+_0 \) such that for all \( X, Y \in \mathcal{H}_x \) we have

\[
h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x) g(X, Y).\]
Example 2.11 (Fuglede 1978)

Let \( \phi = u + iv : U \rightarrow \mathbb{C} \) be a \textbf{holomorphic} function defined on an open subset \( U \) of \( \mathbb{C}^m \). Then we know that \( \phi \) is \textbf{harmonic} i.e.

\[
\Delta \phi = \Delta (u + iv) = \Delta u + i\Delta v = 0.
\]

Furthermore

\[
\langle \nabla \phi, \nabla \phi \rangle = \langle \nabla (u + iv), \nabla (u + iv) \rangle = 0.
\]

This means that the two gradients are orthogonal and of the same length at each point i.e. the map \( \phi : U \rightarrow \mathbb{C} \) is \textbf{horizontally conformal}, hence a harmonic morphism.
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Let $\phi = u + iv : U \to \mathbb{C}$ be a holomorphic function defined on an open subset $U$ of $\mathbb{C}^m$. Then we know that $\phi$ is harmonic i.e.

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Furthermore

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This means that the two gradients are orthogonal and of the same length at each point i.e. the map $\phi : U \to C$ is horizontally conformal, hence a harmonic morphism.

Theorem 2.12 (Baird-Eells 1981)

Let $\phi : (M, g, J) \to \mathbb{C}$ be a holomorphic function from a Kähler manifold. Then $\phi$ is a harmonic morphism.