spherical and hyperbolic space. It is divided into two equal-sized parts: the first is devoted to the two-dimensional case, where much more is known than in the n-dimensional setting, which is discussed in the second part. In addition, there is an appendix providing some important background information, essentially from convex geometry. Many of the sections end with interesting and stimulating open problems, and each chapter closes with a brief survey of related problems.

The material is presented in a clear and concise way. The author has succeeded in providing a unified treatment of all these different threads of finite packing and covering problems. The sometimes technical and condensed proofs are introduced by brief descriptions of their underlying basic ideas, which is a very helpful aid to understanding them. It is a pity, however, that the book contains no figures, particularly in the two-dimensional part. Also, whilst the bibliography is huge and impressive, and contains many of the less well-known and earlier references, the index and the list of notation are unfortunately too short.

All in all, however, this book is a unique and indispensable source for everyone interested in finite packing and covering of convex bodies.

Reference

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Otto-von-Guericke-Universität Magdeburg — Martin Henk

HARMONIC MORPHISMS BETWEEN RIEMANNIAN MANIFOLDS

(London Mathematical Society Monographs: New Series 29)

By Paul Baird and John C. Wood: 520 pp.,
£74.00 (LMS members' price £51.80), isbn 0-19-850362-8
(Oxford University Press, 2003).

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The impressive monograph under review is the first documentation, in book form, of results within the modern theory of harmonic morphisms between Riemannian manifolds. This research field has developed rapidly during the last three decades, and brings together the distinctive areas of differential geometry, topology and analysis to produce a fascinating subject.

1. *Introduction*

In the context of Riemannian geometry, a *harmonic morphism* is a map

$$\phi : (M, g) \longrightarrow (N, h)$$

between Riemannian manifolds with the property that its composition with any locally defined harmonic function on N is a locally defined harmonic function on M.

The first study in this direction [**2**] was conducted in 1848 by Jacobi, whose aim was to construct real-valued harmonic functions defined locally in the Euclidean $\mathbb{R}^3$.

For this purpose Jacobi was interested in maps $\phi : U \to \mathbb{C}$ defined on open subsets of $\mathbb{R}^3$ such that their composition $f \circ \phi$ with any holomorphic function f is harmonic.

In their paper [**1**], dating from 1965, the potential theorists Constantinescu and Cornea studied maps between so-called *Brelot harmonic spaces*. These are topological spaces on which the notion of a local harmonic function has been defined axiomatically to emulate the properties characterizing harmonic functions in Euclidean spaces. The authors studied those maps that preserve these local harmonic functions.

The name *harmonic morphism* was coined in 1978 by Fuglede, in his seminal study [**3**] of the special case when the Brelot harmonic spaces are Riemannian manifolds with their standard local harmonic functions. A similar but independent investigation [**4**] was carried out at the same time by Ishihara. The main result proved by Fuglede and Ishihara was the following fundamental characterization, which gives the theory a strong geometric flavour.

THEOREM. *A map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is a harmonic morphism if and only if it is both a harmonic map and horizontally* (*weakly*) *conformal.*

In general, the two conditions of harmonicity and horizontal conformality constitute a highly non-linear, over-determined system of partial differential equations, which indicates the difficulty of, and explains the interest in, the theory of harmonic morphisms.

Today – almost thirty years since the introduction of harmonic morphisms in Riemannian geometry – the subject has grown, and studies have proliferated into a broad range of settings, such as metric graphs, Riemannian polyhedra and Weyl spaces. Interesting applications, with considerable potential, are also found within probability theory.

2. *Content*

The book is divided into four main parts. The first part, 'Basic facts on harmonic morphisms', begins with a nice elementary study of the Euclidean situation in $\mathbb{R}^3$ investigated by Jacobi. The subsequent chapters provide background material for the general case, presenting conformality, harmonic maps and the most fundamental properties of harmonic morphisms. Closely related to harmonic morphisms is the theory of *conformal foliations*, the definitions and main properties of which are also explained here. The study of harmonic morphisms has sparked new interest in conformal foliations, and many early results on harmonic morphisms have since been reformulated using terminology from this field. The final chapter of this introductory part deals with harmonic morphisms between Euclidean spaces defined by *polynomials*. While there is an abundance of non-polynomial examples of harmonic morphisms from the Euclidean $\mathbb{R}^m$ to $\mathbb{R}^n$ when $n = 2$, the situation becomes far more rigid when $n > 2$. In this case, any harmonic morphism, defined off a closed *polar set*, is in fact polynomial. Such polynomials have been classified through an intriguing connection to Clifford algebras, and provide a rich source of examples.

The second part, 'Twistor methods', is a comprehensive description of the impact that twistor theory and its methods have had on the study of harmonic morphisms. Beginning with the most simple of cases, harmonic morphisms from three-dimensional space forms to surfaces, the reader is carefully guided through the construction of the *mini-twistor space* for the generic space forms. Harmonic morphisms are then easily constructed through a process similar to the well-known Weierstrass representation of minimal surfaces. The familiar twistor space of all orthogonal complex structures is then investigated for the study of harmonic morphisms from manifolds of higher dimensions. The emphasis is here on the four-dimensional situation, which is thoroughly worked out in the text. According to a now-classical result by Wood, later strengthened by Ville, *any* harmonic morphism from an Einstein four-manifold to a surface is in fact holomorphic with respect to some integrable Hermitian structure on its domain and, moreover, has *superminimal* fibres with respect to this structure. This leads to a profound insight into the mechanics governing harmonic morphisms in this situation, and the authors carefully demonstrate how this provides results about existence, non-existence and, in some cases, even complete classifications of harmonic morphisms.

In the third part, 'Topological and curvature considerations', the authors study necessary conditions on the topology and curvature for the existence of harmonic morphisms. Horizontally (weakly) conformal maps generalize Riemannian submersions, and conformal foliations generalize Riemannian foliations. During the development of harmonic morphisms, classical results using the O'Neill tensors have been reformulated and re-proved to fit this more general situation. On the other hand, there is a well-known Weitzenböck formula for harmonic maps which simplifies significantly for harmonic morphisms. For the first time, all these results and their subsequent conclusions are presented in a unified manner. The considerable number of results on harmonic morphisms with one-dimensional fibres is also comprehensively collected here. Bryant was the first to find a classification of these maps on space forms. His results were later generalized to Einstein manifolds, and are presented here in the most general form.

The fourth and last part of the book, 'Further developments', begins with a chapter on harmonic maps and morphisms between semi-Riemannian manifolds, which are rather more delicate than in the Riemannian situation. Then follow three appendices with proofs of important analytical results that have been used in the text. For example, notions such as *capacity* and *polar sets* are explained, and the existence of local harmonic functions with a given 2-jet is proved.

Of particular value is that every chapter ends with a carefully planned section giving notes and comments on the material. The results are placed in their historical context, and readers can glean useful related information and find valuable references that provide signposts to further reading.

The book also contains an extensive bibliography, to help readers navigate their way within the vast number of papers on harmonic maps, harmonic morphisms and related topics.

3. *Conclusion*

Published in 2003, this informative and inspiring book gathers the most important results on harmonic morphisms into a single volume, presenting them in a unified and modern way. In its breadth and comprehensiveness it provides an invaluable tool for both experts in the field, as well as newcomers.

The book is written by two of the foremost experts on harmonic maps and harmonic morphisms. Serious dedication and commitment to the quality and scope of the work have resulted in this veritable opus. The exposition is lucid and authoritative, making it a highly enjoyable reading, as well as a powerful reference tool. We warmly recommend this book.

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Lund University — Sigmundur Gudmundsson

University of Southern Denmark — Martin Svensson

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