THE GEOMETRY OF HARMONIC MORPHISMS

by

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Submitted in accordance with the regulations for the degree of
Ph.D. in Pure Mathematics.

University of Leeds,
Department of Pure Mathematics,

The candidate confirms that the work submitted is his own and that
appropriate credit has been given where reference has been made to the work of
others.
This thesis is dedicated to my parents

Gudmundur and Ólafina
The objects under consideration are harmonic morphisms $\pi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds. They are maps which pull back germs of real valued harmonic functions on $N$ to germs of harmonic functions on $M$. They have been characterized as being those harmonic maps which are horizontally conformal, so they form a special class of harmonic maps.

We generalize O'Neill’s fundamental curvature equations for Riemannian submersions to the case of horizontal conformality. These equations relate the curvature tensors on $M$ and $N$ and give necessary conditions for the existence of horizontally conformal maps $\pi : (M, g) \rightarrow (N, h)$. From these conditions we derive some new non-existence results for harmonic morphisms.

We observe a connection linking harmonic morphisms between space forms and isoparametric systems. From this we obtain a classification of a natural class of harmonic morphisms between open subsets of space forms.

We study the connection between horizontally conformal submersions $\pi : (M, g) \rightarrow (N, h)$ with minimal fibres and minimal submanifolds. We find conditions on such maps, which imply the equivalence of the minimality of submanifolds $L$ of $N$ and their inverse images $\pi^{-1}(L)$ in $M$. We then use this to give examples of minimal foliations of real hypersurfaces in $(\mathbb{C}^*)^m$ for any $m \geq 2$.

We describe how the classical theory of multivalued holomorphic functions can be generalized to harmonic morphisms $\pi : (M^m, g) \rightarrow (N^2, h)$ onto surfaces. We give examples of such maps for the cases of $M^3 = \mathbb{R}^3$ and $M^3 = S^3$.

We use a result of L.Bérard Bergery to give many new examples of homogeneous harmonic morphisms $\pi : (G/K, g) \rightarrow (G/H, \bar{g})$ between reductive homogeneous spaces. We then show that some well known examples from 3-manifolds to surfaces can be obtained in this way.
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0.1. Background.

In this section we very briefly mention some known results related to this work. For a more detailed account see chapter 1.

In the late 70’s Fuglede and Ishihara published two papers [Fug-1] and [Ish], where they discuss their results on “harmonic morphisms” or “mappings preserving harmonic functions”. They characterize non-constant harmonic morphisms \( \pi : (M, g) \to (N, h) \) between Riemannian manifolds as those harmonic maps, which are horizontally conformal. This means that we are dealing with a special class of harmonic maps.

As in many areas of Differential Geometry the theory becomes especially rich if we restrict our attention to the case, when one of the manifolds involved is a surface. It is observed in [Fug-1], as a direct consequence of the Cauchy-Riemann equations, that every holomorphic function \( f : U \subset \mathbb{C}^m \to \mathbb{C} \) is a harmonic morphism. A regular fibre of such a function is of course a minimal submanifold of \( \mathbb{C}^m \).

This is generalized by Baird and Eells in [Bai-Eel], where they show that a regular fibre of a harmonic morphism \( \pi : (M^m, g) \to (N^2, h) \) from any Riemannian manifold to a surface is minimal in \( M \). Furthermore, they prove that if \( n \geq 3 \), then a harmonic morphism \( \pi : (M^m, g) \to (N^n, h) \) has minimal fibres if and only if it is horizontally homothetic.

It is noted in [Bai-Eel], that the dilation of a horizontally homothetic harmonic morphism \( \pi : (M^m, g) \to (N^n, h) \) of codimension 1 is an isoparametric function on \( M \). Conversely, Baird proves in [Bai-1] that totally umbilic isoparametric foliations of hypersurfaces in space forms give rise to harmonic morphisms.

In a series of papers [Bai-2] and [Bai-Woo-1-2-3-4], Baird and Wood study harmonic morphisms from 3-dimensional manifolds to surfaces. They show that any non-constant harmonic morphism \( \pi : (M^3, g) \to (N^2, h) \) determines a conformal foliation of geodesics on \( M^3 \).

For the cases when \( M^3 = \mathbb{R}^3, S^3 \) or \( H^3 \) they prove that any submersive harmonic morphism from an open subset of \( M^3 \) to a Riemann surface \( N^2 \) can be represented by two meromorphic functions on the surface \( N^2 \). This is used to classify such
maps which are either globally defined on $M^3$ or have an isolated singularity.

So far, less attention has been given to harmonic morphisms between manifolds of higher dimensions. The only such result related to this work is given by Kasue and Washio in [Kas-Was]. They prove that if $n \geq 3$ and $\pi : \mathbb{R}^m \to (N^n, h)$ is a harmonic morphism with totally geodesic fibres, then $N^n = \mathbb{R}^n$ and $\pi$ is an orthogonal projection followed by a homothety.

0.2. Main Results.

In this section we state the most important results presented in this thesis. Some of them have appeared in [Gud-1], [Gud-2], [Gud-Woo] and [Bai-Gud].

By exploiting the fact that any non-constant harmonic morphism is horizontally conformal we obtain:

**Proposition 2.1.6.** Let $m > n \geq 3$ and $\pi : (M^m, g) \to (N^n, h)$ be a non-constant harmonic morphism with totally geodesic fibres. If the horizontal distribution $\mathcal{H}$ is integrable, then the resulting foliation $\mathcal{F}_\mathcal{H}$ is spherical.

We generalize O’Neill’s well known fundamental curvature equations for Riemannian submersions to the case of horizontal conformality. As a direct consequence we have the following:

**Corollary 2.2.6.** Let $m > n \geq 2$ and $(M^m, g), (N^n, h)$ be two Riemannian manifolds with $K_M(\mathcal{H}) \geq 0$ and $K_N \leq 0$. If $\pi : M \to N$ is a horizontally homothetic map, then

1. $K_N \equiv 0$ and $K_M(\mathcal{H}) \equiv 0$,
2. the dilation $\lambda : M \to \mathbb{R}^+$ is constant, and
3. the horizontal distribution $\mathcal{H}$ is integrable.

For space forms this leads to the following non-existence result:

**Theorem 3.3.1.** Let $m > n \geq 2$ and $(M, N) = (S^m, \mathbb{R}^n), (S^m, H^n)$ or $(\mathbb{R}^m, H^n)$. If $U$ is an open subset of $M$, then there exists no horizontally homothetic map $\pi : U \to N$.

We show that if $(M, g)$ has constant curvature, then Proposition 2.1.6 implies:

**Corollary 3.3.3.** Let $m > n \geq 2$, $(M^m, g), (N^n, h)$ be simply connected space forms and $U$ an open and connected subset of $M$. Let $\pi : U \to N$ be a horizontally
homothetic harmonic morphism with totally geodesic fibres and integrable horizontal distribution. Then $F_H$ is a totally umbilic isoparametric foliation on $U$.

For totally umbilic isoparametric foliations on open and connected subsets of space forms there exists a well known classification, see appendix A. Using this together with Corollary 3.3.3 we obtain a new classification for a natural class of harmonic morphisms given by Corollary 3.3.4 and Theorem 3.3.5. The latter can be regarded as a local characterization of the standard harmonic morphisms.

**Corollary 3.3.4.** Let $m > n \geq 3$ and $(M, N) = (S^m, S^n)$, $(\mathbb{R}^m, S^n)$ or $(H^m, S^n)$. Then there exists no harmonic morphism $\pi : M \to N$ with totally geodesic fibres and integrable horizontal distribution.

**Theorem 3.3.5.** Let $m > n \geq 2$, $(M, N) = (S^m, S^n)$, $(\mathbb{R}^m, S^n)$, $(H^m, S^n)$, $(H^m, \mathbb{R}^n)$, $(H^m, H^n)$ or $(\mathbb{R}^m, \mathbb{R}^n)$, and $\hat{\pi}_i : \hat{U}_i \to N$ the corresponding standard harmonic morphism (see section 3.2). Further let $U$ be an open and connected subset of $M$ and $\pi : U \to N$ be a horizontally homothetic harmonic morphism. If $\pi$ has totally geodesic fibres and integrable horizontal distribution, then up to isometries of $M$ and $N$, $\pi = \hat{\pi}_i|_{\hat{U}_i}$.

Note that the missing pairs $(M, N)$ in Theorem 3.3.5 are covered by Theorem 3.3.1.

In some cases we are able to relax the assumptions, as an example we get:

**Corollary 3.3.7.** Let $m > n \geq 3$, $U$ be an open and connected subset of $\mathbb{R}^m$ and $\pi : U \to \mathbb{R}^n$ be a harmonic morphism, with totally geodesic fibres. Then $\pi$ is an orthogonal projection, followed by a homothety.

We show by Theorem 4.1.4 a link between harmonic morphisms and minimal submanifolds.

**Theorem 4.1.4.** Let $m > n \geq 2$, $\pi : (M^m, g) \to (N^n, h)$ be a horizontally homothetic harmonic morphism and $L$ be a submanifold of $N$. Then the following conditions are equivalent.

(1) $L$ is minimal in $N$, and

(2) $\pi^{-1}(L)$ is minimal in $M$.

In Example 5.3.4 we discuss a variant of the disk example of a harmonic morphism from a dense open subset of $\mathbb{R}^3$ to the sphere, which include the radial
projection as a limit case. These examples are both surjective and have connected fibres, no two of which are parallel as oriented line segments. Then in Theorem 5.3.5 we characterize these examples as the only harmonic morphisms from an open subset of \( \mathbb{R}^3 \) to a closed Riemann surface satisfying the above conditions (up to isometries of \( \mathbb{R}^3 \) and conformal transformations of the surface).

We use a result of L.Bérand Bergery to give many new examples of homogeneous harmonic morphisms \( \pi : (G/K, g) \to (G/H, \bar{g}) \) between reductive homogeneous spaces. We then show that some well known examples from 3-manifolds to surfaces can be obtained in this way.

0.3. Acknowledgements.

First and foremost I thank my supervisor John C. Wood for his encouragement, extremely useful ideas and constructive criticism. It has been a great privilege to work with him. I also thank his family for their warm hospitality while I stayed at their home in Bures-Sur-Yvette during their stay in France.

I thank Paul Baird and Chris Wood for many useful discussions and their continuing interest in this work.

This research was supported by grants from The Foreign and Commonwealth Office and The Committee of Vice-Chancellors and Principals of the Universities of the United Kingdom and a loan from The Icelandic Government’s Student Loan Fund (LÍN).

Last but not least I would like to thank my best friend Gudrún for all her patience and understanding, whilst I was so often lost in mathematical orbit.
CHAPTER 1. HARMONIC MORPHISMS.

The main purpose of this chapter is to set up the necessary framework for this thesis. We introduce the notions of harmonic maps and harmonic morphisms between Riemannian manifolds. We then make the reader familiar with the basic facts and examples needed later on.

1.1. Definitions.

Throughout we assume, unless otherwise stated, that all our objects such as manifolds, metrics and maps are smooth, that is, in the $C^\infty$-category. By $(M^m, g, \nabla)$ and $(N^n, h, \nabla^N)$ we denote two Riemannian manifolds with their Levi-Civita connections. The positive integers $m$ and $n$ denote the dimensions of $M^m$ and $N^n$ respectively.

For a map $\phi: (M^m, g) \to (N^n, h)$ its energy density is a function $e_\phi : M \to \mathbb{R}_0^+$, given by

$$e_\phi := \frac{1}{2} |d\phi|^2 = \frac{1}{2} \sum_{i=1}^{m} |d\phi(X_i)|^2,$$

where $\{X_1, ..., X_m\}$ is any local orthonormal frame for the tangent bundle $TM$ of $M$. For any compact subset $\Omega$ of $M$, the energy of $\phi$ on $\Omega$ is given by

$$E(\phi, \Omega) := \int_\Omega e_\phi \, d\sigma_M,$$

where $d\sigma_M$ is the volume element on $M$. A map $\phi : (M, g) \to (N, h)$ is called harmonic if it is a critical point of the energy functional $E(\ , \Omega)$ for every compact subset $\Omega$ of $M$. This means that for every compact subset $\Omega$ and 1-dimensional family of maps $\phi_t : (M, g) \to (N, h)$, such that $\phi = \phi_0$ and $\phi(x) = \phi_t(x)$ for all $x \in M - \Omega$ and all $t$, we have

$$\frac{d}{dt} E(\phi_t, \Omega)|_{t=0} = 0.$$

If $N = \mathbb{R}$, then a harmonic map $\phi : (M, g) \to \mathbb{R}$ is called a harmonic function on $M$. For a detailed account on harmonic maps we refer the reader to [Eel-Lem-1-2-3].

We now define the objects of central interest for this work, the harmonic morphisms.

Definition 1.1.1. A map $\pi : (M, g) \to (N, h)$ is called a harmonic morphism if for any harmonic function $f : U \to \mathbb{R}$, defined on an open subset $U$ of $N$ with $\pi^{-1}(U)$ non-empty, $f \circ \pi : \pi^{-1}(U) \to \mathbb{R}$ is a harmonic function.
Thus, harmonic morphisms are maps which pull back germs of harmonic functions on $N$ to germs of harmonic functions on $M$. An alternative description of non-constant harmonic morphisms $\pi : (M, g) \rightarrow (N, h)$ is that they map Brownian motions on $M$ to Brownian motions on $N$. For this see [Lévy], [Ber-Cam-Dav] and [Dar].

1.2. Basic Properties.

First of all we mention the following composition law for harmonic morphisms:

**Lemma 1.2.1.** (The First Composition Law) If $\pi_1 : (M, g) \rightarrow (\tilde{N}, \tilde{h})$ and $\pi_2 : (\tilde{N}, \tilde{h}) \rightarrow (N, h)$ are harmonic morphisms, so is the composition $\pi_2 \circ \pi_1 : (M, g) \rightarrow (N, h)$.

**Proof.** This follows directly from Definition 1.1.1. ◼

The next result was proved independently by Fuglede and Ishihara in [Fug-1] and [Ish].

**Proposition 1.2.2.** Let $\pi : (M^m, g) \rightarrow (N^n, h)$ be a harmonic morphism. If $m < n$, then $\pi$ is a constant map.

One can think of a Brownian motion on a Riemannian manifold $(M, g)$ as a mathematical description of a particle travelling at random along $M$. If one does, then the last proposition becomes intuitively clear, as follows: If $\beta$ was a Brownian motion on $M$, then a non-constant harmonic morphism $\pi : (M^m, g) \rightarrow (N^n, h)$ would map $\beta$ to a Brownian motion $\pi(\beta)$ on $N$. The corresponding particle would therefore stay inside the image $\pi(M)$ of $\pi$. If $m < n$, then $\pi(M)$ has measure zero in $N$, so $\pi(\beta)$ cannot describe a random motion on $N$.

Constant maps are trivially harmonic morphisms but not very interesting, so from now on we will assume that all our maps are non-constant. For our harmonic morphisms $\pi : (M^m, g) \rightarrow (N^n, h)$ this means that we will always have $m \geq n$. By the codimension of $\pi$ we mean the non-negative integer $\text{codim}(\pi) := m - n$.

The following result, proved independently in [Fug-1] and [Ish], serves as one of the most useful devices when dealing with the geometry of harmonic morphisms.

**Theorem 1.2.3.** A non-constant map $\pi : (M^m, g) \rightarrow (N^n, h)$ with $m \geq n$ is a harmonic morphism if and only if it is a harmonic map, and horizontally conformal.
Since the notion of horizontal conformality is still not a standard feature in the literature, we now give its definition.

**Definition 1.2.4.** For \( m \geq n \), a non-constant map \( \pi : (M^m, g) \rightarrow (N^n, h) \) and \( x \in M \) put \( \mathcal{V}_x := \text{Ker } d\pi_x \subset T_x M \) and \( \mathcal{H}_x := \mathcal{V}_x^\perp \subset T_x M \). If \( C_\pi := \{ x \in M \mid d\pi_x = 0 \} \) and \( \hat{M}^m := M - C_\pi \), then \( \pi : (M, g) \rightarrow (N, h) \) is said to be **horizontally (weakly) conformal** if there exists a function \( \lambda : \hat{M} \rightarrow \mathbb{R}^+ \) such that

\[
\lambda^2(x)g(X, Y) = h(d\pi(X), d\pi(Y)),
\]

for all \( X, Y \in \mathcal{H}_x \), and \( x \in \hat{M} \). The function \( \lambda \) is then extended to the whole of \( M \) by putting \( \lambda|_{C_\pi} \equiv 0 \). The extended function \( \lambda : M \rightarrow \mathbb{R}_0^+ \) is called the **dilation** of \( \pi \).

Note that it follows from the last definition that if \( \pi : (M^m, g) \rightarrow (N^n, h) \) is horizontally conformal, then \( d\pi_x : T_x M \rightarrow T_{\pi(x)} N \) is of rank \( n \) on \( \hat{M}^m \), and 0 on \( C_\pi \).

If \( M^2 \) and \( N^2 \) are connected Riemann surfaces, then \( \pi : M^2 \rightarrow N^2 \) is a harmonic morphism if and only if \( \pi \) is \( \pm \)-holomorphic. Harmonic morphisms can therefore be thought of as generalizing \( \pm \)-holomorphic maps between Riemann surfaces to higher dimensions.

It follows directly from Definition 1.2.4 that the function \( \lambda^2 : M \rightarrow \mathbb{R}_0^+ \) is smooth. By \( \text{grad}(\lambda^2) \) we shall denote the gradient of \( \lambda^2 \), which is a smooth section of \( TM \).

On \( (\hat{M}^m, g) \), \( \mathcal{V} := \{ \mathcal{V}_x \mid x \in \hat{M} \} \) and \( \mathcal{H} := \{ \mathcal{H}_x \mid x \in \hat{M} \} \) are smooth distributions or subbundles of \( T\hat{M} \), the tangent bundle of \( \hat{M} \). They are called the **vertical** and **horizontal distributions** defined by \( \pi \). By \( \mathcal{V} \) and \( \mathcal{H} \) we also denote the projections onto \( \mathcal{V}_x \) and \( \mathcal{H}_x \) at each point \( x \in \hat{M} \). On \( \hat{M} \) we have the unique orthogonal decomposition of the gradient of \( \lambda^2 \) into its vertical and horizontal parts, given by

\[
\text{grad}(\lambda^2) = \text{grad}_\mathcal{V}(\lambda^2) + \text{grad}_\mathcal{H}(\lambda^2).
\]

**Definition 1.2.5.** A non-constant map \( \pi : (M, g) \rightarrow (N, h) \) is said to be **horizontally homothetic** if it is horizontally conformal and \( \text{grad}_\mathcal{H}(\lambda^2) \equiv 0 \) on \( \hat{M} \).

For the horizontally homothetic case we have the following lemma due to Fuglede, see [Fug-2].
Lemma 1.2.6. If \( \pi : (M, g) \to (N, h) \) is a horizontally homothetic map, then \( \pi \) is a submersion, that is \( \mathcal{M} = M \).

The horizontal homothety is therefore equivalent to \( \lambda^2 : M \to \mathbb{R}^+_0 \) being constant along horizontal curves in \((M, g)\). Since the horizontal homothety will play a major role in our work, we now mention some of its geometric consequences. The following Theorem 1.2.7 is due to Baird and Eells and was first published in [Bai-Eel]. It turns out to be one of our main tools throughout this thesis.

Theorem 1.2.7. Let \( m > n \geq 2 \) and \( \pi : (M^m, g) \to (N^n, h) \) be a horizontally conformal submersion. If

(a) \( n = 2 \), then \( \pi \) is a harmonic map if and only if \( \pi \) has minimal fibres,
(b) \( n \geq 3 \), then two of the following conditions imply the other,

(1) \( \pi \) is a harmonic map,
(2) \( \pi \) has minimal fibres,
(3) \( \pi \) is horizontally homothetic.

For further understanding of the horizontal homothety we now draw attention to our observation of a duality between horizontally conformal submersions and weakly conformal immersions.

For \( m > n \geq 2 \), a non-constant map \( i : (N^n, h) \to (M^m, g) \) is called a weakly conformal immersion if there exists a function \( \mu : N \to \mathbb{R}^+_0 \), such that

\[
\mu^2(x)h(X, Y) = g(di(X), di(Y)),
\]
for all \( X, Y \in T_xN \) and \( x \in N \). The function \( \mu \) is called the conformal factor of \( i \). It follows immediately from the definition, that \( \mu^2 : N \to \mathbb{R}^+_0 \) is a smooth function. For weakly conformal immersions just as for the horizontally conformal submersions, the harmonicity is in fact a very geometric condition:

Theorem 1.2.8. Let \( m > n \geq 2 \) and \( i : (N^n, h) \to (M^m, g) \) be a non-constant weakly conformal immersion with conformal factor \( \mu : N \to \mathbb{R}^+_0 \). If

(a) \( n = 2 \), then \( i \) is a harmonic map if and only if \( i(N) \) is minimal in \((M, g)\),
(b) \( n \geq 3 \), then two of the following conditions imply the other,

(1) \( i \) is a harmonic map,
(2) \( i(N) \) is minimal in \((M, g)\),
(3) \( i \) is homothetic, that is, \( \mu \) is constant.
Proof. This is just a combination of Proposition page 119 of [Eel-Sam] and Example 3.3 of [Bai-Eel].

1.3. Examples.

For $m \geq 3$ we shall always denote by $S^m$, $\mathbb{R}^m$ and $H^m$ the $m$-dimensional simply connected space forms of constant sectional curvature $+1$, $0$ and $-1$, i.e. the sphere, the euclidean and hyperbolic spaces. The following examples are all maps between open subsets of these spaces. It is fairly easy to see that they are horizontally homothetic and have totally geodesic fibres, so by Theorem 1.2.7 they are all harmonic morphisms. For $\pi_6$ and $\pi_7$ the dilations are constant, in contrast to $\pi_1$ to $\pi_5$. The horizontal distribution $\mathcal{H}$ is integrable for $\pi_1$ to $\pi_6$ but for $\pi_7$ it is non-integrable.

Example $\pi_1$. Using the standard model for $S^m \subset \mathbb{R}^{m+1}$ we have $S^m - S^0 = \{(\cos(s),\sin(s)\cdot e) \in \mathbb{R} \times \mathbb{R}^m \mid s \in (0, \pi) \text{ and } e \in S^{m-1}\}$, $\langle, >_{\mathbb{R}^{m+1}} \rangle$. Let $\pi_1 : S^m - S^0 \to S^{m-1}$ be the projection along the longitudes onto the equatorial hypersphere, given by $\pi_1 : (\cos(s),\sin(s)\cdot e) \mapsto e$. For $e \in S^{m-1}$ the fibre of $\pi_1$ over $e$ is parametrized w.r.t. arclength by $\gamma_e(s) = (\cos(s),\sin(s)\cdot e)$, where $s \in (0, \pi)$. Along the fibres we have $\lambda_1^2(s) = 1/\sin^2(s)$. The level hypersurfaces of $\lambda_1$ are parallel small spheres $S^m_{\sin(s)}$ with constant sectional curvatures $K_{S^m_{\sin(s)}} = 1/\sin^2(s)$.

Example $\pi_2$. Let $\pi_2 : \mathbb{R}^m - \mathbb{R}^0 \to S^{m-1}$ be the radial projection, given by $\pi_2 : x \mapsto x/|x|$. For $e \in S^{m-1}$ the fibre of $\pi_2$ over $e$ is parametrized w.r.t. arclength by $\gamma_e(s) = s\cdot e$, where $s \in \mathbb{R}^+$. Along the fibres we have $\lambda_2^2(s) = 1/s^2$. The level hypersurfaces are parallel spheres $S^m_{s}$ with constant sectional curvatures $K_{S^m_{s}} = 1/s^2$.

Example $\pi_3$. Using the standard Poincaré model for $H^m$ we have $H^m - H^0 = (B_1^m(0) - \{0\}, \frac{4}{1-|x|^2} \cdot \langle, >_{\mathbb{R}^m} \rangle$, where $B_1^m(0) := \{x \in \mathbb{R}^m \mid |x| < 1\}$. Let $\pi_3 : H^m - H^0 \to S^{m-1}$ be the radial projection, given by $\pi_3 : x \mapsto x/|x|_{\mathbb{R}^m}$. For $e \in S^{m-1}$ the fibre of $\pi_3$ over $e$ is parametrized w.r.t. arclength by $\gamma_e(s) = \tanh(s/2)\cdot e$, where $s \in \mathbb{R}^+$, and along the fibres we have $\lambda_3^2(s) = 1/\sinh^2(s)$. The level hypersurfaces of $\lambda_3$ are parallel spheres $S^m_{\tanh(s/2)}$ with constant sectional curvatures $K_{S^m_{\tanh(s/2)}} = 1/\sinh^2(s)$. 

Example \( \pi_4 \). Using the standard upper half space model for \( H^m \) we have \( H^m = (\mathbb{R}^{m-1} \times \mathbb{R}^+, \frac{1}{x_m} <, >_{\mathbb{R}^m}) \). Let \( \pi_4 : H^m \to \mathbb{R}^{m-1} \) be the projection onto \( \mathbb{R}^{m-1} \) followed by a homothety, given by \( \pi_4 : (p, x) \mapsto \alpha p \), where \( \alpha \in \mathbb{R} - \{0\} \). For \( p \in \mathbb{R}^{m-1} \) the fibre of \( \pi_4 \) over \( \alpha p \) is parametrized w.r.t. arclength by \( \gamma_{\alpha}(s) = (p, e^s) \), where \( s \in \mathbb{R} \). Along the fibres we have \( \lambda_4^2(s) = \alpha^2 e^{2s} \). The level hypersurfaces of \( \lambda_4 \) are parallel affine subspaces \( \mathbb{R}^n_{e^s-1} := \{(p, e^s) \in \mathbb{R}^{m-1} \times \mathbb{R}^+ \mid p \in \mathbb{R}^{m-1}\} \) with constant sectional curvatures \( K_{\mathbb{R}^n_{e^s-1}} = 0 \).

Example \( \pi_5 \). For \( H^m = (\mathbb{R}^{m-2} \times \mathbb{R} \times \mathbb{R}^+, \frac{1}{x_m} <, >_{\mathbb{R}^m}) \) and \( H^{m-1} = (\mathbb{R}^{m-2} \times \{0\} \times \mathbb{R}^+, \frac{1}{x_m} <, >_{\mathbb{R}^{m-1}}) \) define \( \pi_5 : H^m \to H^{m-1} \) by \( \pi_5 : (p, x, y) \mapsto (p, 0, \sqrt{x^2 + y^2}) \). The fibre over \( (p, 0, r) \) is the semicircle in \( \mathbb{R}^{m-2} \times \mathbb{R} \times \mathbb{R}^+ \) with centre \((p, 0, 0)\), radius \( r \) and parallel to the coordinate plane \( \{(0, a, b) \mid a, b \in \mathbb{R}\} \). The fibre is parametrized w.r.t. arclength by \( \gamma_{(p,r)}(s) = (p, r \cdot \tanh(s), r / \cosh(s)) \). Geometrically this map is a projection along the geodesics of \( H^m \) orthogonal to \( H^{m-1} \). Along the fibre \( \lambda_5^2(s) = 1 / \cosh^2(s) \). The level hypersurfaces of \( \lambda_5 \) are \( H^m_{s-1} := \{(p, r \cdot \tanh(s), r / \cosh(s)) \in H^m \mid p \in \mathbb{R}^{m-2}, r \in \mathbb{R}^+\} = \mathbb{R}^{m-2} \times \text{span}_{\mathbb{R}}\{(0, \tanh(s), 1 / \cosh(s))\} \). They are parallel hyperbolic hyperplanes with constant sectional curvatures \( K_{H^m_{s-1}} = -1 / \cosh^2(s) \).

Example \( \pi_6 \). Let \( \pi_6 : \mathbb{R}^m \to \mathbb{R}^{m-1} \) be the orthogonal projection followed by a homothety, given by \( \pi_6 : (x_1, \ldots, x_m) \mapsto \alpha(x_1, \ldots, x_{m-1}) \), where \( \alpha \in \mathbb{R} - \{0\} \). For \( p \in \mathbb{R}^{m-1} \) the fibre of \( \pi_6 \) over \( p \) is parametrized w.r.t. arclength by \( \gamma_{\pi}(s) = (p, s) \), where \( s \in \mathbb{R} \). The dilation is constant \( \lambda_6^2(s) = \alpha^2 \). The horizontal distribution is obviously integrable, and its integral submanifolds are the parallel affine hyperplanes \( \mathbb{R}^m_{s-1} := \{(p, s) \in \mathbb{R}^m \mid p \in \mathbb{R}^{m-1}\} \).

Example \( \pi_7 \). Let \( F = \mathbb{C}, \mathbb{H} \) or \( C\mathbb{a} \), that is, the complex numbers, the quaternions or the Cayley numbers. Put \((m, n) := (2 \cdot \dim F - 1, \dim F) = (3, 2), (7, 4) \) or \((15, 8)\). Define \( \pi : F \times F = \mathbb{R}^{m+1} \to F \times \mathbb{R} = \mathbb{R}^{n+1} \) by \( \pi : (x, y) \mapsto (2xy, |x|^2 - |y|^2) \). The restrictions \( \pi_7 \) of \( \pi \) to \( S^m \subset \mathbb{R}^{m+1} \) are the well known Hopf maps. They are harmonic morphisms \( \pi_7 : S^m \to S^n \) with constant dilation \( \lambda_7 = 2 \). The fibres are totally geodesic and therefore isometric to \( S^{m-n} \subset S^m \). It is well known that the horizontal distributions are nowhere integrable.
CHAPTER 2. GEOMETRIC CONSTRAINTS.

In his very important paper [O’Ne], O’Neill studies the geometry of Riemannian submersions \( \pi : (M, g) \to (N, h) \). He derives equations which relate the sectional curvatures of the two manifolds involved. These give necessary conditions for the existence of such maps between \( M \) and \( N \).

Riemannian submersions are special cases of horizontally conformal maps, namely those with constant dilation \( \lambda \equiv 1 \). It is therefore very natural for us to generalize O’Neill’s work to the case of horizontal conformality.

2.1. Horizontal Conformality.

Let \( \pi : M \to N \) be a submersion. A vector field \( E \) on \( M \) is said to be projectable if there exists a vector field \( \hat{E} \) on \( N \), such that \( d\pi(E_x) = \hat{E}_{\pi(x)} \) for all \( x \in M \). In this case \( E \) and \( \hat{E} \) are called \( \pi \)-related. A horizontal vector field \( Y \) on \( (M, g) \) is called basic, if it is projectable. It is a well known fact, that if \( \hat{Z} \) is a vector field on \( N \), then there exists a unique basic vector field \( Z \) on \( M \), such that \( Z \) and \( \hat{Z} \) are \( \pi \)-related. The vector field \( Z \) is called the horizontal lift of \( \hat{Z} \).

The fundamental tensors of a submersion were introduced in [O’Ne]. They play a similar role to that of the second fundamental form of an immersion. O’Neill’s definition is as follows:

Definition 2.1.1. Let \( E \) and \( F \) be two vector fields on \( M \) and \( \pi : (M, g) \to N \) be a submersion. Then the fundamental tensors of \( \pi \) are given by:

\[
T_{EF} := \mathcal{H}\nabla_{\nabla_{V}E}F + \nabla_{\nabla_{V}E}\mathcal{H}F, \quad \text{and}
\]
\[
A_{EF} := \mathcal{H}\nabla_{\mathcal{H}E}V + \nabla_{\mathcal{H}E}\mathcal{H}F.
\]

It is easily seen that for \( x \in M \), \( X \in \mathcal{H}_x \) and \( V \in \mathcal{V}_x \) the linear operators \( T_V, A_X : T_xM \to T_xM \) are skew-symmetric, that is

\[-g(T_VE,F) = g(E,T_VF) \quad \text{and} \quad -g(A_XE,F) = g(E,A_XF)\]

for all \( E, F \in T_xM \). We also see that the restriction of \( T \) to the vertical distribution \( T|_{\mathcal{V}_x \times \mathcal{V}_x} \) is exactly the second fundamental form of the fibres of \( \pi \). Since \( T_V \) is skew-symmetric we get: \( \pi \) has totally geodesic fibres if and only if \( T \equiv 0 \).

For the special case when \( \pi \) is horizontally conformal we have the following.
Proposition 2.1.2. Let $\pi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with dilation $\lambda$ and $X, Y$ be horizontal vectors, then

$$AXY = \frac{1}{2} \{V[X, Y] - \lambda^2 g(X, Y) \text{grad}_V\left(\frac{1}{\lambda^2}\right)\}.$$

Proof. Extend $X, Y$ to basic vector fields, and let $\{V_{n+1}, \ldots, V_m\}$ be a local orthonormal frame for the vertical distribution. Then by using the well known formula for the Levi-Civita connection $\nabla$, we get

$$AXY = \nabla_X Y = \sum_{i=n+1}^m g(\nabla_X Y, V_i)V_i = \frac{1}{2} \sum_{i=n+1}^m \left\{X g(Y, V_i) + Y g(V_i, X) - V_i g(X, Y) - g([X, Y], V_i) + g(V_i, [X, Y])\right\}V_i = \frac{1}{2} \{V[X, Y] - \text{grad}_V(g(X, Y))\},$$

since $V_i, [X, V_i], [Y, V_i]$ are all vertical, because $X$ and $Y$ are basic. Now

$$\text{grad}_V(g(X, Y)) = \text{grad}_V\left(\frac{1}{\lambda^2} h(\tilde{X}, \tilde{Y})\right) = h(\tilde{X}, \tilde{Y}) \text{grad}_V\left(\frac{1}{\lambda^2}\right) = \lambda^2 g(X, Y) \text{grad}_V\left(\frac{1}{\lambda^2}\right),$$

and hence the result.

We see that the skew-symmetric part of $A|_{\mathcal{H} \times \mathcal{H}}$ measures the obstruction to integrability of the horizontal distribution $\mathcal{H}$, and is therefore of special interest. If $\mathcal{H}$ is integrable, then we have a horizontal foliation on $(M, g)$, which we denote by $\mathcal{F}_{\mathcal{H}}$.

We will now show that the horizontal conformality of $\pi$ has some very important geometric consequences for $\mathcal{F}_{\mathcal{H}}$. The next proposition was observed by Wood in [Woo].

Proposition 2.1.3. Let $\pi : (M, g) \to (N, h)$ be a horizontally conformal submersion with integrable horizontal distribution $\mathcal{H}$. Then the horizontal foliation $\mathcal{F}_{\mathcal{H}}$ is totally umbilic in $(M, g)$. 
Proof. Let $X, Y$ be two local horizontal vector fields. Since the horizontal distribution is integrable we have $\mathcal{V}[X,Y] = 0$, so from Proposition 2.1.2 we obtain

$$A_{X}Y = \frac{\lambda^2}{2} g(X,Y) \text{grad}_V(\frac{1}{\lambda^2}).$$

The second fundamental form $\Pi_L$ of a leaf $L \in \mathcal{F}_H$ is given by

$$\Pi_L(X,Y) = \mathcal{V} \nabla_X Y = A_X Y = \frac{\lambda^2}{2} g(X,Y) \text{grad}_V(\frac{1}{\lambda^2}),$$

so $L$ is an totally umbilic submanifold of $(M, g)$.

Definition 2.1.4. Let $\mathcal{F}$ be a totally umbilic foliation on a Riemannian manifold $(M, g)$. A leaf $L \in \mathcal{F}$ is called spherical if the mean curvature vector $H_L$ of $L$ is parallel in the normal bundle $\nu(L)$ of $L$ in $(M, g)$. The foliation $\mathcal{F}$ is called spherical if each leaf $L \in \mathcal{F}$ is spherical.

Lemma 2.1.5. Let $(M, g)$ be a Riemannian manifold and $f : M \to \mathbb{R}$ be a function on $M$. If $X, Y$ are vector fields on $M$, then

$$g(\nabla_X \text{grad}(f), Y) = g(\nabla_Y \text{grad}(f), X).$$

Proof.

$$g(\nabla_X \text{grad}(f), Y) - g(\nabla_Y \text{grad}(f), X)$$

$$= X(Y(f)) - g(\text{grad}(f), \nabla_X Y) - Y(X(f)) + g(\text{grad}(f), \nabla_Y X)$$

$$= X(Y(f)) - Y(X(f)) - [X,Y](f) = 0.$$

For harmonic morphisms we now get the following result:

Proposition 2.1.6. Let $m > n \geq 3$ and $\pi : (M^m, g) \to (N^n, h)$ be a non-constant harmonic morphism with totally geodesic fibres. If the horizontal distribution $\mathcal{H}$ is integrable, then the foliation $\mathcal{F}_H$ is spherical.

Proof. It follows from Theorem 1.2.7 that $\pi$ is horizontally homothetic, and then by Lemma 1.2.6 that $\pi$ is a submersion. If $L$ is a leaf in $\mathcal{F}_H$ then the corresponding mean curvature vector field $H_L$ is given by

$$H_L = -\frac{\lambda^2}{2} \text{grad}_V(\frac{1}{\lambda^2}) = -\frac{\lambda^2}{2} \text{grad}(\frac{1}{\lambda^2}).$$
Let $X \in \mathcal{H}$ and $V \in \mathcal{V}$ be two local vector fields. Then
\begin{align*}
g(\nabla_X H_L, V) &= -\frac{\lambda^2}{2} g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), V) \\
&= -\frac{\lambda^2}{2} g(\nabla_V \text{grad}(\frac{1}{\lambda^2}), X) \\
&= 0,
\end{align*}

since $\pi$ is horizontally homothetic with totally geodesic fibres. This means that $H_L$ is parallel in the normal bundle. \hfill \blacksquare

Remark 2.1.7. It follows from Theorem 1.2.7 that we could obtain the same result for $n = 2$ by assuming that $\pi$ is horizontally homothetic.

2.2. Necessary Curvature Conditions.

Before generalizing O'Neill’s fundamental curvature equations to the horizontally conformal case, we now state two standard results used in our proofs. Lemma 2.2.1 describes how the curvature tensor of $(M, g, \nabla, R)$ changes when the metric is changed by a conformal factor. This formula can for example be found on page 90 in [Gro-Kli-Mey].

**Lemma 2.2.1.** Let $m \geq 2$ and $(M^m, g, \nabla, R)$, $(\tilde{M}^m, \tilde{g}, \tilde{\nabla}, \tilde{R})$ be two Riemannian manifolds with their Levi-Civita connections and the corresponding curvature tensors. If $\tilde{g} = \lambda^2 g$, then
\begin{align*}
g(R(X, Y)Z, H) &= \frac{1}{\lambda^2} \tilde{g}(\tilde{R}(X, Y)Z, H) \\
&+ \frac{\lambda^2}{2} \left[ g(X, Z)g(\nabla_Y \text{grad}(\frac{1}{\lambda^2}), H) - g(Y, Z)g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), H) \\
&+ g(Y, H)g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), Z) - g(X, H)g(\nabla_Y \text{grad}(\frac{1}{\lambda^2}), Z) \right] \\
&+ \frac{\lambda^4}{4} \left[ (g(X, H)g(Y, Z) - g(Y, H)g(X, Z)) \cdot |\text{grad}(\frac{1}{\lambda^2})|^2 \\
&+ g(X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X, H(\frac{1}{\lambda^2})Z - Z(\frac{1}{\lambda^2})H) \right].
\end{align*}

The second fact needed is given in Lemma 2.2.2 and is simply one of O'Neill’s original equations. For this see Theorem 2.4 of [O’Ne].
Lemma 2.2.2. Let \( m > n \geq 2 \) and \((M^m, \bar{g}, \bar{\nabla}, \bar{R})\), \((N^n, h, \nabla^N, R^N)\) be two Riemannian manifolds with their Levi-Civita connections and the corresponding curvature tensors. If \( \bar{\pi} : (M^m, \bar{g}) \to (N^n, h) \) is a Riemannian submersion, then

\[
\bar{g}(\bar{R}(X,Y)Z,H) = h(R^N(\bar{X},\bar{Y})\bar{Z},\bar{H}) + \frac{1}{4} \left[ \bar{g}(\bar{V}[X,Z],\bar{V}[Y,H]) - \bar{g}(\bar{V}[Y,Z],\bar{V}[X,H]) + 2\bar{g}(\bar{V}[X,Y],\bar{V}[Z,H]) \right].
\]

We now state our promised fundamental curvature equations for horizontally conformal submersions.

Theorem 2.2.3. Let \( m > n \geq 2 \) and \((M^m, g, \nabla, R)\), \((N^n, h, \nabla^N, R^N)\) be two Riemannian manifolds with their Levi-Civita connections and the corresponding curvature tensors. Let \( \pi : (M, g) \to (N, h) \) be a horizontally conformal submersion, with dilation \( \lambda : M \to \mathbb{R}^+ \) and let \( R^V \) be the curvature tensor of the fibres of \( \pi \). If \( X, Y, Z, H \) are horizontal and \( U, V, W, F \) vertical vectors, then

\[
g(R(U, V)W, F) = g(R^V(U, V)W, F)
+ g(T_U W, T_V F) - g(T_V W, T_U F),
\]

(1)

\[
g(R(U, V)W, X) = g((\nabla_U T) V W, X) - g((\nabla_V T) U W, X)
\]

(2)

\[
g(R(U, X)Y, V) = g((\nabla_U A) X Y, V) + g(A_X U, A_Y V)
- g((\nabla_X T) U Y, V) - g(T_V Y, T_U X)
+ \lambda^2 g(A_X Y, U)g(V, \text{grad}_V(\frac{1}{\lambda^2}))
\]

(3)

\[
g(R(X, Y)Z, U) = g((\nabla_X A) Y Z, U) - g((\nabla_Y A) X Z, U)
- g(T_U Z, V[X,Y]),
\]

(4)

\[
g(R(X, Y)Z, H) = \frac{1}{\lambda^2} h(R^N(\bar{X},\bar{Y})\bar{Z},\bar{H}) + \frac{1}{4} \left[ g(V[X,Z],V[Y,H]) - g(V[Y,Z],V[X,H]) + 2g(V[X,Y],V[Z,H]) \right]
+ \frac{\lambda^2}{2} \left[ g(X,Z)g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), H) - g(Y,Z)g(\nabla_Y \text{grad}(\frac{1}{\lambda^2}), H) \right].
\]
\[ g(Y, H)g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), Z) - g(X, H)g(\nabla_Y \text{grad}(\frac{1}{\lambda^2}), Z) \]  
\[ + \frac{\lambda^4}{4} \left[ (g(X, H)g(Y, Z) - g(Y, H)g(X, Z))|\text{grad}(\frac{1}{\lambda^2})|^2 \right. \]
\[ + g(X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X, H(\frac{1}{\lambda^2})Z - Z(\frac{1}{\lambda^2})H) \]  
\[ = g(Y, H)g(\nabla_X \text{grad}(\frac{1}{\lambda^2}), Z) - g(X, H)g(\nabla_Y \text{grad}(\frac{1}{\lambda^2}), Z) \]
\[ + \frac{\lambda^4}{4} \left[ (g(X, H)g(Y, Z) - g(Y, H)g(X, Z))|\text{grad}(\frac{1}{\lambda^2})|^2 \right. \]
\[ + g(X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X, H(\frac{1}{\lambda^2})Z - Z(\frac{1}{\lambda^2})H) \].

**Proof.**

(1) This is simply the Gauss equation for the fibres.

(2) The proof is exactly the same as in [O'Ne].

(3) Extend \( X \) and \( Y \) to basic vector fields, then

\[ \forall R(U, X)Y = \forall \nabla_U \forall \nabla_X Y + \forall \nabla_U H \forall \nabla_X Y \]
\[ - \forall \nabla_X \forall \nabla_U Y - \forall \nabla_X H \forall \nabla_U Y - \forall \nabla_U [X, X]Y. \]

From this follows:

\[ g(R(U, X)Y, V) \]
\[ = g(\nabla_U (A_X Y), V) + g(T_U (H \forall \nabla_X Y), V) \]
\[ - g(\nabla_X (T_U Y), V) - g(A_X (H \forall \nabla_U Y), V) - g(T_V [U, X]Y, V) \]
\[ = g((\nabla_U A)_X Y, V) + g(A_X \nabla_X Y, V) \]
\[ - g((\nabla_X T)_U Y, V) - g(T_{V}(\nabla_U X)Y, V) \]
\[ = g((\nabla_U A)_X Y, V) - g(A_Y (H \forall \nabla_X U), V) \]
\[ - \lambda^2 g(Y, \nabla_X U)g(\text{grad}(\frac{1}{\lambda^2}), V) \]
\[ - g((\nabla_X T)_U Y, V) - g(T_V Y, T_U X), \]

which is the same as stated.

(4) Extend \( X, Y \) and \( Z \) to basic vector fields, then

\[ \forall R(X, Y)Z \]
\[ = \forall \nabla_X \forall \nabla_Y Z + \forall \nabla_X H \forall \nabla_Y Z - \forall \nabla_Y H \forall \nabla_X Z \]
\[ - \forall \nabla_Y H \forall \nabla_X Z - \forall \nabla_Y [X, Y]Z - \forall \nabla_Y [X, Y]Z, \] so

\[ g(R(X, Y)Z, U) \]
\[ = g(\nabla_X (A_Y Z), U) + g(A_X (H \forall \nabla_Y Z), U) - g(\nabla_X (A_X Z), U) \]
Conformal submersion, with dilation $\lambda$.

**Corollary 2.2.4.** Let $\pi : (M, g) \to (N, h)$ be a horizontally conformal submersion, with dilation $\lambda : M \to \mathbb{R}^+$ and $X, Y$ horizontal. Then

$$
g(R(X, Y)Y, X) = \frac{1}{\lambda^2} h(R^N(\hat{X}, \hat{Y})\hat{Y}, \hat{X}) - \frac{3}{4} |\nabla[X, Y]|^2 \\
+ \frac{\lambda^2}{2} \left[ g(Y, Y)g(\nabla_Y\text{grad}(\frac{1}{\lambda^2}), X) - g(Y, Y)g(\nabla_X\text{grad}(\frac{1}{\lambda^2}), X) \\
+ g(Y, Y)g(\nabla_X\text{grad}(\frac{1}{\lambda^2}), Y) - g(Y, Y)g(\nabla_Y\text{grad}(\frac{1}{\lambda^2}), Y) \right] \\
+ \frac{\lambda^4}{4} \left[ |X \wedge Y|^2 |\text{grad}(\frac{1}{\lambda^2})|^2 + |X(\frac{1}{\lambda^2})Y - Y(\frac{1}{\lambda^2})X|^2 \right].
$$

**Proof.** Put $Z = Y$ and $H = X$ into the equation in Theorem 2.2.3.(5), and the result follows immediately. 

This last formula seems to be too complicated to be of any use in the general case. But when assuming that $\pi : (M, g) \to (N, h)$ is horizontally homothetic, we get the following necessary curvature condition.

**Theorem 2.2.5.** Let $m > n \geq 2$ and $\pi : (M^m, g) \to (N^n, h)$ be a horizontally homothetic map, with dilation $\lambda : M \to \mathbb{R}^+$. If $X$ and $Y$ are horizontal vectors,
such that $|X| = |Y| = 1$ and $g(X, Y) = 0$, then

$$K_M(X \wedge Y) = \lambda^2 \cdot K_N(\tilde{X} \wedge \tilde{Y}) - {3 \over 4} |V[X, Y]|^2 - {\lambda^4 \over 4} |\text{grad}_V(1 \over \lambda^2)|^2.$$ 

**Proof.** Since $\pi : (M, g) \rightarrow (N, h)$ is horizontally homothetic, it follows from Lemma 1.2.6, that it is a submersion. The equation $\text{grad}_H(1 \over \lambda^2) \equiv 0$ implies that $X(1 \over \lambda^2) = Y(1 \over \lambda^2) = 0$ and that

$$g(\nabla_Z \text{grad}(1 \over \lambda^2), Z) = -g(\text{grad}_V(1 \over \lambda^2), \nabla_Z Z)$$

for $Z \in \{X, Y\}$. Then Corollary 2.2.4 and $g(X, Y) = 0$ give

$$K_M(X \wedge Y) = g(R(X, Y)Y, X)$$

$$= {1 \over \lambda^2} h(R^N(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X}) - {3 \over 4} |V[X, Y]|^2$$

$$+ {\lambda^2 \over 2} g(\text{grad}_V(1 \over \lambda^2), \nabla_X X + \nabla_Y Y) + {\lambda^4 \over 4} |\text{grad}_V(1 \over \lambda^2)|^2$$

$$= \lambda^2 \cdot K_N(\tilde{X} \wedge \tilde{Y}) - {3 \over 4} |V[X, Y]|^2 - {\lambda^4 \over 4} |\text{grad}_V(1 \over \lambda^2)|^2,$$

since by Proposition 2.1.2, $\nabla_Z Z = A_Z Z = -{\lambda^2 \over 2} g(Z, Z)\text{grad}_V(1 \over \lambda^2)$ for $Z \in \{X, Y\}$. ■

**Corollary 2.2.6.** Let $m > n \geq 2$ and $(M^m, g)$, $(N^n, h)$ be two Riemannian manifolds with $K_M(H) \geq 0$ and $K_N \leq 0$. If $\pi : M \rightarrow N$ is a horizontally homothetic map, then

1. $K_N \equiv 0$ and $K_M(H) \equiv 0$,
2. the dilation $\lambda : M \rightarrow \mathbb{R}^+$ is constant, and
3. the horizontal distribution $H$ is integrable.

**Proof.** It follows from Lemma 1.2.6 that if such a map $\pi$ exists, then it is a submersion. Let $X$ and $Y$ be horizontal vectors, such that $|X| = |Y| = 1$ and $g(X, Y) = 0$. For the above cases we then obtain from Theorem 2.2.5

$$0 \leq K_M(X \wedge Y) - \lambda^2 \cdot K_N(\tilde{X} \wedge \tilde{Y}) = -{3 \over 4} |V[X, Y]|^2 - {\lambda^4 \over 4} |\text{grad}_V(1 \over \lambda^2)|^2 \leq 0,$$

from which the result follows. ■

Another necessary curvature condition is given by the following:
Proposition 2.2.7. Let $m > n \geq 2$ and $\pi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion with constant dilation and totally geodesic fibres. Further let $X \in \mathcal{H}$ and $V \in \mathcal{V}$, such that $|X| = 1$ and $|V| = 1$, then

$$K_M(X \wedge V) = |A_X V|^2.$$ 

Proof. Since $\pi$ has totally geodesic fibres we have $T \equiv 0$. The dilation $\lambda$ of $\pi$ is constant, so it follows from Proposition 2.1.2 that the fundamental tensor $A$ is skew-symmetric. Hence $A_X X = 0$ and

$$\mathcal{V}((\nabla_V A) X \wedge X) = \mathcal{V}(\nabla_V A_X X - A_{\nabla_V X} X - A_X \nabla_V X) = 0.$$ 

The result now immediately follows from Theorem 2.2.3.(3).
CHAPTER 3. CONSTANT CURVATURE.

In this chapter we assume that \((M, g)\) and \((N, h)\) are simply connected space forms. We study non-constant harmonic morphisms \(\pi : U \to N\) from open connected subsets \(U\) of \(M\). The horizontal conformality of \(\pi\) is obviously independent of homothetic changes of the metrics concerned. The same is true for the harmonicity of \(\pi\). Since we are assuming that our manifolds have constant sectional curvatures, we can without loss of generality restrict our attention to the cases when \((M, g)\) and \((N, h)\) are the standard spheres, euclidean or hyperbolic spaces with \(K_M, K_N \in \{-1, 0, 1\}\).

3.1. Known Results.

First of all we remind the reader that for any \(m \geq 3\) we have the following examples of harmonic morphisms of codimension 1, given in section 1.3:

\[
\begin{align*}
\pi_1 &: S^m - S^0 \to S^{m-1}, \\
\pi_2 &: \mathbb{R}^m - \mathbb{R}^0 \to S^{m-1}, \\
\pi_3 &: H^m - H^0 \to S^{m-1}, \\
\pi_4 &: H^m \to \mathbb{R}^{m-1}, \\
\pi_5 &: H^m \to H^{m-1}, \\
\pi_6 &: \mathbb{R}^m \to \mathbb{R}^{m-1}, \\
\end{align*}
\]

and the harmonic morphism of codimension \(m - n\):

\[
\pi_7 : S^m \to S^n,
\]

with \((m, n) = (3, 2), (7, 4)\) or \((15, 8)\).

For harmonic morphisms from simply connected space forms there exist two very interesting classification results. The first one is due to Baird and Wood, see [Bai-Woo-1-2].

**Theorem 3.1.1.** Let \((M^3, g) = S^3, \mathbb{R}^3\) or \(H^3\) and \((N^2, h)\) be a surface. Further let \(U\) be an open and connected subset of \(M^3\).

(i) If \(\pi : M^3 \to N^2\) is a non-constant harmonic morphism, then up to isometries of \(M^3\), \(\pi\) is one of \(\pi_4, \pi_5, \pi_6,\) or \(\pi_7\), followed by a weakly conformal map.
(ii) If \( \pi : U \to N^2 \) is a harmonic morphism with an isolated singularity, then up to isometries of \( M^3 \), \( \pi \) is a restriction of one of \( \pi_1, \pi_2 \) or \( \pi_3 \), followed by a weakly conformal map.

In [Kas-Was], Kasue and Washio were able to generalize to higher dimensions in the case of \( M = \mathbb{R}^m \), when assuming that \( \pi \) has totally geodesic fibres:

**Theorem 3.1.2.** Let \( m > n \geq 3 \) and \( \pi : \mathbb{R}^m \to (\mathbb{R}^n, h) \) be a non-constant harmonic morphism with totally geodesic fibres. Then \( N = \mathbb{R}^n \) and \( \pi \) is an orthogonal projection, followed by a homothety.

The reader should note that both Theorem 3.1.1.(i) and Theorem 3.1.2 are of global nature, that is, \((M, g)\) is complete. In contrast, most of our results will be local, that is, concerning harmonic morphisms \( \pi : U \to N \) defined on an arbitrary open and connected subset \( U \) of \( M \).

### 3.2. Examples of Higher Codimension.

The examples \( \pi_1 \) to \( \pi_6 \) are all of codimension 1. The First Composition Law for harmonic morphisms given in Lemma 1.2.1 allows us to compose these maps and thereby construct examples of arbitrary codimension. In this way we define the following maps \( \hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_6 \), which we shall call the **standard harmonic morphisms**.

\[
\begin{align*}
\hat{\pi}_1 &: S^m \to S^{m-(n+1)} \xrightarrow{\pi_1} \ldots \xrightarrow{\pi_1} S^{n+1} \to S^0 \xrightarrow{\pi_1} S^m, \\
\hat{\pi}_2 &: \mathbb{R}^m \to \mathbb{R}^{m-(n+1)} \xrightarrow{\pi_6} \ldots \xrightarrow{\pi_6} \mathbb{R}^{n+1} \to \mathbb{R}^0 \xrightarrow{\pi_2} S^n, \\
\hat{\pi}_3 &: H^m \to H^{m-(n+1)} \xrightarrow{\pi_5} \ldots \xrightarrow{\pi_5} H^{n+1} \to H^0 \xrightarrow{\pi_3} S^n, \\
\hat{\pi}_4 &: H^m \to \ldots \xrightarrow{\pi_5} H^{n+1} \xrightarrow{\pi_4} \mathbb{R}^n, \\
\hat{\pi}_5 &: H^m \to \ldots \xrightarrow{\pi_5} H^n, \\
\hat{\pi}_6 &: \mathbb{R}^m \to \ldots \xrightarrow{\pi_6} \mathbb{R}^n.
\end{align*}
\]

We now study some special properties of these maps. To be able to do so we need the next lemma. For two basic vector fields \( X \) and \( Y \) on \((M, g)\) we denote by \( \hat{\nabla}_X Y \) the horizontal lift of \( \nabla^N_X Y \).
Lemma 3.2.1. If $\pi : (M^m, g) \to (N^n, h)$ is a horizontally conformal submersion and $X, Y$ are basic vector fields on $M$, then

$$\mathcal{H}\nabla_X Y = \nabla_X Y + \frac{\lambda^2}{2} \{ X(\frac{1}{\lambda^2})Y + Y(\frac{1}{\lambda^2})X - g(X, Y)\text{grad}\,\mathcal{H}(\frac{1}{\lambda^2}) \}.$$

Proof. We choose a local orthonormal frame $\{\tilde{Z}_i | i = 1, \ldots, n\}$ on $(N, h)$ and lift $\tilde{Z}_i$ horizontally to $Z_i$. Then $\{\lambda Z_i | i = 1, \ldots, n\}$ is a local orthonormal frame for the horizontal distribution of $(M, g)$, so we get

$$\mathcal{H}\nabla_X Y = \sum_{i=1}^{n} g(\nabla_X Y, \lambda Z_i) \lambda Z_i = \lambda^2 \sum_{i=1}^{n} g(\nabla_X Y, Z_i) Z_i$$

$$= \lambda^2 \sum_{i=1}^{n} \left[ X(g(Y, Z_i)) + Y(g(Z_i, X)) - Z_i(g(X, Y)) ight. $$

$$- g(X, [Y, Z_i]) + g(Y, [Z_i, X]) + g(Z_i, [X, Y]) \left] Z_i \right.$$}

$$= \lambda^2 \sum_{i=1}^{n} \left[ X(\frac{1}{\lambda^2})h(\tilde{Y}, \tilde{Z}_i) + \frac{1}{\lambda^2} \tilde{X}(h(\tilde{Y}, \tilde{Z}_i)) ight.$$

$$+ Y(\frac{1}{\lambda^2})h(\tilde{Z}_i, \tilde{X}) + \frac{1}{\lambda^2} \tilde{Y}(h(\tilde{Z}_i, \tilde{X})) $$

$$- Z_i(\frac{1}{\lambda^2})h(\tilde{X}, \tilde{Y}) - \frac{1}{\lambda^2} \tilde{Z}_i(h(\tilde{X}, \tilde{Y})) $$

$$+ \frac{1}{\lambda^2} \{-h(\tilde{X}, [\tilde{Y}, \tilde{Z}_i]) + h(\tilde{Y}, [\tilde{Z}_i, \tilde{X}]) + h(\tilde{Z}_i, [\tilde{X}, \tilde{Y}])\} \right] Z_i.$$}

We then apply the differential $d\pi$ and obtain

$$d\pi(\mathcal{H}\nabla_X Y)$$

$$= \sum_{i=1}^{n} h(\nabla^N_X \tilde{Y}, \tilde{Z}_i) \tilde{Z}_i + \frac{\lambda^2}{2} \{ X(\frac{1}{\lambda^2}) \sum_{i=1}^{n} h(\tilde{Y}, \tilde{Z}_i) \tilde{Z}_i $$

$$+ Y(\frac{1}{\lambda^2}) \sum_{i=1}^{n} h(\tilde{X}, \tilde{Z}_i) \tilde{Z}_i - h(\tilde{X}, \tilde{Y}) \sum_{i=1}^{n} Z_i(\frac{1}{\lambda^2}) \tilde{Z}_i \} $$

$$= \nabla^N_X \tilde{Y} + \frac{\lambda^2}{2} \{ X(\frac{1}{\lambda^2})\tilde{Y} + Y(\frac{1}{\lambda^2})\tilde{X} - g(\tilde{X}, \tilde{Y})d\pi(\text{grad}_\mathcal{H}(\frac{1}{\lambda^2})) \},$$

from which the result immediately follows. ■

As already mentioned in section 1.3, $\pi_1$ to $\pi_6$ are all horizontally homothetic with totally geodesic fibres. They also have integrable horizontal distributions. The following two lemmas show that this is also true for their compositions $\hat{\pi}_1$ to $\hat{\pi}_6$. 
Lemma 3.2.2. (The Second Composition Law) If $\pi_1 : (M, g) \to (\tilde{N}, \tilde{h})$ and $\pi_2 : (\tilde{N}, \tilde{h}) \to (N, h)$ are two horizontally homothetic maps, with totally geodesic fibres, so is the composition $\pi = \pi_2 \circ \pi_1 : (M, g) \to (N, h)$.

Proof. For $i = 1, 2$ let $\lambda_i$ denote the dilation of $\pi_i$ and $\lambda_i^* \lambda_2$ the pull-back of $\lambda_2$ via $\pi_1$, given by $(\lambda_i^* \lambda_2)(x) := \lambda_2 \circ \pi_1(x)$ for all $x \in M$. If $\lambda$ is the dilation of $\pi$, then $\lambda^2 = \lambda_1^2(\lambda_1^* \lambda_2)^2$. If $X$ is a horizontal vector, then

$$X(\lambda^2) = X((\lambda_1^*)^2(\lambda_2)^2) + \lambda_1^2 X((\lambda_1^*)^2(\lambda_2)^2)$$

$$= \lambda_1^2 d\pi_1(X)(\lambda_2^2) = 0,$$

so $\pi$ is horizontally homothetic.

The vertical distribution $V$ of $\pi$ splits naturally into two orthogonal parts, $V = V_1 \oplus W$, where $V_1 := \text{Ker } d\pi_1$ and $W := V_1^\perp \cap V$. It is easily seen, that $V_2 = \text{Ker } d\pi_2 = d\pi_1(W)$. If $x \in M$ and $V = V_1 + W \in V_x$ such that $V_1 \in (V_1)_x$ and $W \in W_x$, then we extend $V$ onto a small neighbourhood of $x$, such that $W$ is a basic vector field w.r.t $\pi_1$. Let $X \in \mathcal{H}_x$ be any horizontal vector w.r.t. $\pi$. Then

$$g(\nabla \mathcal{V} V, X) = g(\nabla V_1 V, X) + g(\nabla W + \nabla W V_1, X) + g(\nabla W W, X)$$

$$= 2 g(T V_1 W, X) + \frac{1}{\lambda^2} \tilde{h}(d\pi_1(\nabla W), d\pi_1(X))$$

$$= \frac{1}{\lambda^2} \tilde{h}(\nabla d\pi_1(W), d\pi_1(W), d\pi_1(X)) = 0$$

by Lemma 3.2.1 and the fact that $\pi_2$ has totally geodesic fibres.

Since the second fundamental form $\Pi_F$ of a fibre $F$ of $\pi$ is symmetric, it follows from

$$\Pi_F(U, V) = \frac{1}{4} \{ \Pi_F(U + V, U + V) - \Pi_F(U - V, U - V) \},$$

that $\pi$ has totally geodesic fibres. 

Lemma 3.2.3. (Third Composition Law) Let $\pi_1 : (M, g) \to (\tilde{N}, \tilde{h})$ and $\pi_2 : (\tilde{N}, \tilde{h}) \to (N, h)$ be two submersions. If $\pi_1$ and $\pi_2$ have integrable horizontal distribution, so has the composition $\pi = \pi_2 \circ \pi_1 : (M, g) \to (N, h)$.

Proof. For $i = 1, 2$ let $\mathcal{V}_i$ and $\mathcal{H}_i$ denote the vertical and horizontal distributions of $\pi_i$, respectively. Let $\tilde{X}, \tilde{Y}$ be two local vector fields on $N$, and $X, Y$ their
horizontal lifts via $\pi_2$. Further let $\tilde{X}, \tilde{Y} \in \mathcal{H} \subset \mathcal{H}_1$ be the horizontal lifts of $X$, $Y$ via $\pi_1$. The integrability of $\mathcal{H}_1$ implies $\mathcal{V}_1[\tilde{X}, \tilde{Y}] = 0$, so $\mathcal{V}[\tilde{X}, \tilde{Y}] \subset \mathcal{H}_1$. But now $d\pi_1(\mathcal{V}[\tilde{X}, \tilde{Y}]) = \mathcal{V}_2[d\pi_1(\tilde{X}), d\pi_1(\tilde{Y})] = \mathcal{V}_2[X, Y] = 0$, since $\mathcal{H}_2$ is integrable, so $\mathcal{V}[\tilde{X}, \tilde{Y}] = 0$.

3.3. A Higher Dimensional Classification.

First of all we have a non-existence result. It explains why the following cases are missing in the list of examples, given in the last section 3.2.

**Theorem 3.3.1.** Let $m > n \geq 2$ and $(M, N) = (S^m, \mathbb{R}^n)$, $(S^m, H^n)$ or $(\mathbb{R}^m, H^n)$. If $U$ is an open subset of $M$, then there exists no horizontally homothetic map $\pi : U \rightarrow N$.

**Proof.** It follows from Lemma 1.2.6 that if such a map $\pi$ exists, then it is a submersion. Let $X$ and $Y$ be horizontal vectors, such that $|X| = |Y| = 1$ and $g(X, Y) = 0$. For the above cases we then obtain from Theorem 2.2.5

$$0 < K_M(X \wedge Y) - \lambda^2 \cdot K_N(\tilde{X} \wedge \tilde{Y}) = -\frac{3}{4} \cdot |\mathcal{V}[X, Y]|^2 - \frac{\lambda^4}{4} \cdot (\nabla_{\lambda} (\frac{1}{\lambda^2}))^2 \leq 0.$$  

This contradicts the existence of $\pi$. 

It was noted in (5.6) of [Bai-Eel] that a non-constant dilation $\lambda : M \rightarrow \mathbb{R}^+$ of a horizontally homothetic harmonic morphism $\pi : (M^m, g) \rightarrow (N^{m-1}, h)$ is an isoparametric function, or equivalently, $\mathcal{F}_\mathcal{H}$ is an isoparametric foliation of codimension 1. For basic facts on isoparametric systems, see Appendix A.

**Proposition 3.3.2.** Let $(M, g)$ be a Riemannian manifold of constant sectional curvature, and $\mathcal{F}_\mathcal{H}$ be a totally umbilic foliation on $M$. Then the following are equivalent:

1. $\mathcal{F}_\mathcal{H}$ is spherical, and
2. $\mathcal{F}_\mathcal{H}$ is isoparametric.

**Proof.** Let $\mathcal{V}$ be the full distribution orthogonal to $\mathcal{F}_\mathcal{H}$, $L$ be a leaf in $\mathcal{F}_\mathcal{H}$ and $H_L$ be the mean curvature vector field of $L$. Then $\Pi_L(X, Y) = g(X, Y)H_L$, and consequently the shape operator $S_V$ satisfies $S_V = -g(V, H_L)\text{id}_{TM}$ for any normal field $V \in \mathcal{V}$. 

Suppose that (1) holds. To show that $L$ is isoparametric we must prove that
its normal bundle $\nu(L) = \mathcal{V}|_L$ is flat and that the principal curvatures of $L$ in the
direction of any parallel normal field are constant.

Let $V$ and $\bar{V}$ be two local normal fields along $L$. Then the two shape operators
$S_V$ and $S_{\bar{V}}$ are both multiples of the identity so they commute, i.e.

$$[S_V, S_{\bar{V}}] = S_V \cdot S_{\bar{V}} - S_{\bar{V}} \cdot S_V = 0.$$ 

It then follows from Proposition 2.1.2 of [Pal-Ter] that $\nu(L)$, the normal bundle of
$L$ in $(M, g)$, is flat.

Let $X$ be a local horizontal vector field and $W$ be a parallel normal field along $L$, then

$$Xg(W, H_L) = g(\nabla_X W, H_L) + g(W, \nabla_X H_L) = 0,$$

since both $W$ and $H_L$ are parallel in the normal bundle. This means that $L$ has
constant principal curvatures in the direction of any parallel normal field, so $\mathcal{F}_H$ is
isoparametric.

Conversely, suppose that (2) holds. Let $V$ be a local parallel normal field along $L$. Then

$$g(\nabla_X H_L, V) = Xg(H_L, V) - g(H_L, \nabla_X V) = 0,$$

since $V$ is parallel and $-g(V, H_L)$ is constant along $L$ because $L$ is isoparametric.
This means that $H_L$ is parallel in the normal bundle, so $\mathcal{F}_H$ is spherical. ■

**Corollary 3.3.3.** Let $m > n \geq 2$, $(M^m, g)$, $(N^n, h)$ be simply connected space
forms and $U$ be an open and connected subset of $M$. Let $\pi : U \to N$ be a horizontally
homothetic harmonic morphism with totally geodesic fibres and integrable horizontal
distribution. Then $\mathcal{F}_H$ is a totally umbilic isoparametric foliation on $U$.

**Proof.** This follows directly from Proposition 2.1.6, Remark 2.1.7 and Lemma 3.3.2. ■

**Corollary 3.3.4.** Let $m > n \geq 3$ and $(M, N) = (S^m, S^n)$, $(\mathbb{R}^m, S^n)$ or $(H^m, S^n)$.
Then there exists no harmonic morphism $\pi : M \to N$ with totally geodesic fibres
and integrable horizontal distribution.

**Proof.** Suppose that such a map $\pi : M \to N$ exists. Then by Theorem 1.2.7 and
Lemma 1.2.6 $\pi$ is a submersion so by Corollary 3.3.3 it would define a totally umbilic
isoparametric foliation \( F \) on the whole of \( M \) without focal varieties. Let \( L \) be a leaf in \( F \), then the restriction \( \pi|_L : L \to S^n \) is a homothety, so \( L \) has constant positive curvature. Such a foliation must have focal varieties, which contradicts the hypothesis.

**Theorem 3.3.5.** Let \( m > n \geq 2 \), \((M, N) = (S^m, S^n), (\mathbb{R}^m, S^n), (H^m, S^n), (H^m, \mathbb{R}^n), (H^m, H^n) \) or \((\mathbb{R}^m, \mathbb{R}^n)\), and \( \hat{\pi}_i : \hat{U}_i \to N \) be the corresponding standard harmonic morphism. Further let \( U \) be an open and connected subset of \( M \) and \( \pi : U \to N \) be a horizontally homothetic harmonic morphism. If \( \pi \) has totally geodesic fibres and integrable horizontal distribution, then up to isometries of \( M \) and \( N \), \( \pi = \hat{\pi}_i|_{\hat{U}_i} \).

**Proof.** It follows from Corollary 3.3.3 that \( \pi \) determines a totally umbilic isoparametric foliation \( F_U \) on \( U \) without singularities. By Appendix A this foliation is up to isometries of \( M \) uniquely determined. This means that there exists a foliation preserving isometric embedding \( \sigma : (U, F_U) \to (\hat{U}_i, F_{\hat{U}_i}) \), where \( F_{\hat{U}_i} \) is the foliation given by \( \hat{\pi}_i : \hat{U}_i \to N \). Since \( \sigma \) is foliation preserving, there exists a map \( \hat{\sigma} : N \to N \), such that the following diagram commutes.

\[
\begin{array}{ccc}
U & \xrightarrow{\sigma} & \hat{U}_i \\
\downarrow{\pi} & & \downarrow{\hat{\pi}_i} \\
N & \xrightarrow{\hat{\sigma}} & N
\end{array}
\]

The maps \( \pi \) and \( \hat{\pi}_i \) are horizontally conformal and \( \sigma \) is an isometry. This implies, that \( \hat{\sigma} \) is conformal. If \( \lambda^2_\sigma \) is the corresponding conformal factor, then \( \pi^*\lambda^2_\sigma : U \to \mathbb{R}^+ \), the pullback of \( \lambda^2_\sigma \) via \( \pi \), satisfies

\[
\lambda^2 \cdot \pi^*\lambda^2_\sigma = \sigma^*\lambda^2_\sigma.
\]

Now \( \sigma^*\lambda^2_\sigma \) and \( \lambda^2 \) are horizontally constant, so \( \pi^*\lambda^2_\sigma \) is. But being a pull-back \( \pi^*\lambda^2_\sigma \) is also vertically constant. This means that \( \lambda^2_\sigma \) is constant, so \( \hat{\sigma} \) is a homothety. If \( N = S^n \) or \( H^n \) this means that \( \lambda^2_\sigma = 1 \), but when \( N = \mathbb{R}^n \), \( \lambda_\sigma \) can be any element of \( \mathbb{R}^+ \).

For \( m > n \geq 2 \) Theorems 3.3.1 and 3.3.5 give a classification for horizontally homothetic harmonic morphisms \( \pi : U \subset (M^m, g) \to (N^n, h) \) with totally geodesic fibres and integrable horizontal distribution between open and connected subsets of simply connected space forms. For \( m > n \geq 3 \) we have the following:
Corollary 3.3.6. Let $m > n \geq 3$ and $(M, N) = (S^m, S^n), (\mathbb{R}^m, \mathbb{S}^n), (H^m, S^n), (H^m, \mathbb{R}^n), (H^m, H^n)$ or $(\mathbb{R}^m, \mathbb{R}^n)$, and $\hat{\pi}_i : \hat{U}_i \to N$ be the corresponding standard harmonic morphism. Further let $U$ be an open and connected subset of $M$ and $\pi : U \to N$ be a harmonic morphism with totally geodesic fibres. If $\pi$ has integrable horizontal distribution, then up to isometries of $M$ and $N$, $\pi = \hat{\pi}_i|_{\hat{U}_i}$.

Proof. It follows from Theorem 1.2.7 that the map $\pi : U \to N$ is horizontally homothetic. The result then follows from Theorem 3.3.5. ■

To see that the condition of integrability is in general necessary, consider the following harmonic morphisms.

$$\hat{\pi}_8 : \mathbb{R}^4 - \mathbb{R}^1 \xrightarrow{\pi_6} \mathbb{R}^3 - \mathbb{R}^0 \xrightarrow{\pi_7} S^2,$$

$$\hat{\pi}_9 : \mathbb{R}^4 - \mathbb{R}^1 \xrightarrow{\pi_2} S^3 - S^0 \xrightarrow{\pi_7} S^2.$$

They are both horizontally homothetic with totally geodesic fibres. The corresponding vertical foliations are fundamentally different, so the maps must be different. It is the non-integrability of the horizontal distribution of $\hat{\pi}_9$ that makes this possible.

In the special case of $(M, N) = (\mathbb{R}^m, \mathbb{R}^n)$ the condition of integrability of the horizontal distribution is automatically satisfied, so we get:

Corollary 3.3.7. Let $m > n \geq 3$, $U$ be an open and connected subset of $\mathbb{R}^m$ and $\pi : U \to \mathbb{R}^n$ be a harmonic morphism, with totally geodesic fibres. Then $\pi$ is an orthogonal projection, followed by a homothety.

Proof. Since $\pi$ is a harmonic morphism with totally geodesic fibres and $n \geq 3$, it follows from Theorem 1.2.7, that it is horizontally homothetic. It then follows directly from Theorem 2.2.5, that its horizontal distribution $\mathcal{H}$ is integrable and the dilation is constant. The result then follows from Theorem 3.3.5. ■

The reader should compare Corollary 3.3.7 with the result of Kasue and Washio, given in Theorem 3.1.2.

If $\pi$ has codimension 1, then its fibres are automatically totally geodesic, which implies:

Corollary 3.3.8. Let $m \geq 3$, $(M, N) = (S^m, S^{m-1}), (\mathbb{R}^m, S^{m-1}), (H^m, S^{m-1}), (H^m, \mathbb{R}^{m-1}), (\mathbb{R}^m, \mathbb{R}^{m-1})$, or $(H^m, H^{m-1})$ and $U$ be an open and connected subset
of $M$. If $\pi : U \to N$ is a horizontally homothetic harmonic morphism, then $\pi$ has constant dilation, or up to isometries of $M$ and $N$, $\pi$ is one of the standard harmonic morphisms.

**Proof.** If the dilation $\lambda$ is not constant, there exists a point $x \in U$, such that $\text{grad}(\lambda^2) \neq 0$ on an open neighbourhood $W \subset U$ of $x$. It follows from $\text{grad}H(\lambda^2) = 0$ that $H$ is integrable on $W$ and its integral manifolds are the level hypersurfaces of $\lambda$. Since $\pi$ is of codimension 1 it follows from Theorem 1.2.7, that its fibres are totally geodesic. We can therefore apply Theorem 3.3.5 on $W$. Hence up to isometries $\pi|_W$ is the restriction of one of the standard maps $\tilde{\pi}_i$. Then by the unique continuation principle of Sampson (see [Sam]) for harmonic maps this is in fact the case on the whole of $U$. $\blacksquare$

When $\pi$ has constant dilation and totally geodesic fibres, we get the following:

**Theorem 3.3.9.** Let $m > n \geq 2$, $(M^m, g)$, $(N^n, h)$ be simply connected space forms, and $U$ an open and connected subset of $M$. If $\pi : U \to N$ is a harmonic morphism with constant dilation and totally geodesic fibres, then $(M, N) = (S^m, S^n)$ or $(\mathbb{R}^m, \mathbb{R}^n)$.

**Proof.** The cases $(S^m, \mathbb{R}^n)$, $(S^m, H^n)$ and $(\mathbb{R}^m, H^n)$ are excluded by Theorem 3.3.1. Let $X, Y \in \mathcal{H}$ and $V \in \mathcal{V}$ be local vector fields, such that $|X| = |Y| = |V| = 1$ and $g(X, Y) = 0$. By Theorem 2.2.5 and Proposition 2.2.7 we get:

1. $K_M(X \wedge V) = |A_XV|^2$, and
2. $K_M(X \wedge Y) = \lambda^2 \cdot K_N(\overline{X} \wedge \overline{Y}) - \frac{3}{4}|V[X, Y]|^2$.

Equation (1) excludes the cases $(H^m, S^n)$, $(H^m, \mathbb{R}^n)$ and $(H^m, H^n)$. If $M = \mathbb{R}^m$, then follows from (1) that $A_XV = 0$. The linear operator $A_X$ is skew-symmetric, so $A_XY = \frac{1}{2}V[X, Y] = 0$, which is equivalent to the horizontal distribution being integrable. This makes the case $(\mathbb{R}^m, S^n)$ impossible by (2). $\blacksquare$

In Corollary 3.3.7 we have already dealt with $(M, N) = (\mathbb{R}^m, \mathbb{R}^n)$. If $(M, N) = (S^m, S^n)$ we have the following theorem:

**Theorem 3.3.10.** Let $m > n \geq 2$ and $\pi : S^m \to S^n$ be a harmonic morphism with constant dilation. If $\pi$ has totally geodesic fibres, then up to isometries of $S^m$ and $S^n$ it is one of the Hopf maps.
Proof. The map $\pi$ has constant dilation so, up to a homothety of $S^n$, it is a Riemannian submersion with totally geodesic fibres. The result then follows from Escobales classification of such maps between spheres, see [Esc].

For the local case we conjecture the following:

Conjecture 3.3.11. Let $m > n \geq 2$, $U$ be an open and connected subset of $S^m$ and $\pi : U \to S^n$ be a harmonic morphism with constant dilation. If $\pi$ has totally geodesic fibres, then up to isometries of $S^m$ and $S^n$, $\pi$ is a restriction of one of the Hopf maps.
CHAPTER 4. MINIMAL SUBMANIFOLDS.

Throughout this chapter we shall assume that \( m > n \geq 2 \) and that \( \pi : (M^m, g) \rightarrow (N^n, h) \) is a horizontally conformal submersion. We are mainly interested in the following question: Under what extra conditions on \( \pi \) are the inverse images of minimal submanifolds of \( N \) minimal in \( M \)?


As before we denote by \( \mathcal{H} \) and \( \mathcal{V} \) the horizontal and vertical distributions of \( \pi : (M, g) \rightarrow (N, h) \). Let \( L \) be a submanifold of \( N \). Since \( \pi \) is a submersion \( K := \pi^{-1}(L) \) is a submanifold of \( M \). For \( x \in K \) we define

\[
(\mathcal{H}_1)_x := T_x K \cap \mathcal{H}_x \quad \text{and} \quad (\mathcal{H}_2)_x := T_x K^\perp.
\]

This means that along \( K \) we get the following orthogonal decompositions

\[
TK = \mathcal{V} \oplus \mathcal{H}_1, \quad \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \quad \text{and} \quad TM = \mathcal{V} \oplus \mathcal{H} = TK \oplus \mathcal{H}_2.
\]

By \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) we shall also denote the orthogonal projections onto the corresponding subbundles of \( TM \) along \( K \).

By \( \Pi_K \) we denote the second fundamental form \( \Pi_K : TK \times TK \rightarrow \mathcal{H}_2 \) of \( K \) in \( M \), and similarly by \( \Pi_L \) the second fundamental form \( \Pi_L : TL \times TL \rightarrow \nu(L) \) of \( L \) in \( N \). The corresponding mean curvature vector fields are denoted by \( H_K \) and \( H_L \).

Definition 4.1.1. Let \( \pi : (M, g) \rightarrow (N, h) \) be a horizontally conformal submersion, \( L \) be a submanifold of \( N \) and \( K = \pi^{-1}(L) \subset M \). The map \( \pi \) is called normally homothetic along \( K \) if \( \text{grad}_{\mathcal{H}_2}(\frac{1}{\lambda^2}) = 0 \) on \( K \).

Note that if \( \pi \) is horizontally homothetic, i.e. \( \text{grad}_{\mathcal{H}}(\frac{1}{\lambda^2}) = 0 \), then \( \pi \) is automatically normally homothetic along \( K \) for any submanifold \( L \) of \( N \).

Lemma 4.1.2. Let \( \pi : (M^m, g) \rightarrow (N^n, h) \) be a horizontally conformal submersion with minimal fibres and let \( L \) be a submanifold of \( N \). Then the following conditions are equivalent:

1. \( K = \pi^{-1}(L) \) is minimal in \( M \), and
2. \( H_L = \frac{1}{2} d\pi(\text{grad}_{\mathcal{H}_2}(\frac{1}{\lambda^2})) \).
Proof. Choose a local orthonormal frame \( \{ \hat{X}_i \mid i = 1, \ldots, l \} \) for TL, then the normalized horizontal lifts \( \{ \lambda X_i \mid i = 1, \ldots, l \} \) form a local orthonormal frame for \( \mathcal{H}_1 \). Further choose a local orthonormal frame \( \{ V_r \mid r = n+1, \ldots, m \} \) for \( V \). For the mean curvature vector \( H_K \) of \( K \) in \( M \) we now get

\[
(m-(n-l))H_K = \sum_{i=1}^{l} \Pi_K(\lambda X_i, \lambda X_i) + \sum_{r=n+1}^{m} \Pi_K(V_r, V_r)
\]

\[
= \lambda^2 \sum_{i=1}^{l} \mathcal{H}_2(\nabla X_i X_i),
\]

and from Lemma 3.2.1

\[
= \lambda^2 \sum_{i=1}^{l} \{ \mathcal{H}_2(\nabla X_i X_i) - \frac{\lambda^2}{2} g(X_i, X_i) \text{grad}_{\mathcal{H}_2}(\frac{1}{\lambda^2}) \}.
\]

If we now apply the differential \( d\pi \) then

\[
(m-(n-l))d\pi(H_K) = \lambda^2 \sum_{i=1}^{l} \Pi_L(X_i, X_i) - \frac{\lambda^2}{2} \sum_{i=1}^{l} h(X_i, X_i) d\pi(\text{grad}_{\mathcal{H}_2}(\frac{1}{\lambda^2}))
\]

\[
= \lambda^2 l \bar{H}_L - \frac{l\lambda^2}{2} d\pi(\text{grad}_{\mathcal{H}_2}(\frac{1}{\lambda^2})),
\]

from which the the statement immediately follows.

Corollary 4.1.3. Let \( \pi : (M, g) \to (N, h) \) be a horizontally conformal submersion with minimal fibres and \( L \) be a submanifold of \( N \). If \( \pi \) is normally homothetic along \( \pi^{-1}(L) \), then the two following conditions are equivalent:

1. \( \pi^{-1}(L) \) is minimal in \( M \), and
2. \( L \) is minimal in \( N \).

Proof. This follows directly from Proposition 4.1.2.

For harmonic morphisms we get the following version of Corollary 4.1.3.

Theorem 4.1.4. Let \( m > n \geq 2 \), \( \pi : (M^m, g) \to (N^n, h) \) be a horizontally homothetic harmonic morphism and \( L \) be a submanifold of \( N \). Then the following conditions are equivalent.

1. \( L \) is minimal in \( N \), and
2. \( \pi^{-1}(L) \) is minimal in \( M \).
Proof. The map $\pi$ is horizontally homothetic, so by Lemma 1.2.6 it is a horizontally conformal submersion. It then follows from Theorem 1.2.7 that $\pi$ has minimal fibres, so Corollary 4.1.3 applies.

Note that Theorem 4.1.4 can be applied to any of the standard harmonic morphisms $\hat{\pi}_1, \ldots, \hat{\pi}_6$ defined in section 3.2 and to the Hopf maps $\pi_7$. They could therefore be used to construct minimal submanifolds in space forms.

4.2. Examples.

In this section we find horizontally conformal submersions $\pi : (\mathbb{C}^*)^m \to \mathbb{C}^*$ which are in general not horizontally homothetic. They have minimal, non-totally geodesic fibres. We then use these maps to construct minimal foliations $\hat{\mathcal{F}} := \{F^{-1}(\{e^t \alpha| t \in \mathbb{R}\})| \alpha \in S^1\}$ of real hypersurfaces of $(\mathbb{C}^*)^m$.

Let $\mathcal{F}$ be the minimal foliation of $\mathbb{C}^*$ given by

$$\mathcal{F} := \{ l_\alpha = \{e^t \alpha| t \in \mathbb{R}\} | \alpha \in S^1 \subset \mathbb{C}\}.$$

For a holomorphic function $f : U \to \mathbb{C}^*$ from an open subset $U$ of $\mathbb{C}^m$ we define $\alpha_f : U \to \mathbb{C}$, by

$$\alpha_f := 2 \frac{\langle \text{grad}_\mathbb{C}(f), \text{grad}_\mathbb{C}(\|\text{grad}_\mathbb{C}(f)\|^2) \rangle_{\mathbb{C}}}{f},$$

where $\text{grad}_\mathbb{C}(f) = (\partial f/\partial z_1, \ldots, \partial f/\partial z_n)$.

Lemma 4.2.1. Let $f : U \to \mathbb{C}$ be a holomorphic submersion on an open subset $U$ of $\mathbb{C}^m$. If $\alpha_f(U) \subset \mathbb{R} - \{0\}$, then the foliation $\hat{\mathcal{F}}_f = \{f^{-1}(l_\alpha)|\alpha \in S^1\}$ of real hypersurfaces in $U$ is minimal.

Proof. The function $f = u + iv : U \to \mathbb{C}$ is holomorphic, so by [Fug-1] it is horizontally conformal with dilation $\lambda$, satisfying $\lambda = \frac{1}{\lambda^2} = \|\text{grad}_\mathbb{C}(f)\|^2$. Then one easily checks that

$$\text{grad} \left( \frac{1}{\lambda^2} \right) = 2 \text{grad}_\mathbb{C} \left( \frac{1}{\lambda^2} \right) = 2 \text{grad}_\mathbb{C}(\|\text{grad}_\mathbb{C}(f)\|^2).$$

The horizontal space $\mathcal{H}_x$ at $x \in U$ is given by $\mathcal{H}_x = \text{span}_\mathbb{R}\{\text{grad}(u), \text{grad}(v)\} = \text{span}_\mathbb{C}\{\text{grad}_\mathbb{C}(f)\}$, so

$$\text{grad}_\mathcal{H} \left( \frac{1}{\lambda^2} \right) = \frac{\langle \text{grad} \left( \frac{1}{\lambda^2} \right), \text{grad}_\mathbb{C}(f) \rangle_{\mathbb{C}}}{\|\text{grad}_\mathbb{C}(f)\|^2} \text{grad}_\mathbb{C}(f).$$
Then applying the differential $df$ gives
\[
df(\text{grad}_H(\frac{1}{\lambda^2})) = \frac{<\text{grad}(\frac{1}{\lambda^2}), \text{grad}_C(f)>_C}{|\text{grad}_C(f)|^2} [\text{grad}_C(f)]^t \\
= <\text{grad}(\frac{1}{\lambda^2}), \text{grad}_C(f)>_C \\
= 2 <\text{grad}_C(f), \text{grad}(\frac{1}{\lambda^2})> = \alpha_f \cdot f.
\]

It then follows from $\alpha_f(U) \subset \mathbb{R} - \{0\}$, that
\[
df(\text{grad}_H(\frac{1}{\lambda^2})/\alpha_f) = f,
\]
so the vector field $X := \text{grad}_H(\frac{1}{\lambda^2})/\alpha_f$ is basic. It then follows that the vector field
\[
\dot{X} := df(\text{grad}_H(\frac{1}{\lambda^2})/\alpha_f)
\]
satisfies $\dot{X}(p) = p$, so the elements of $\mathcal{F}$ are the integral curves of $\dot{X}$. Hence $\text{grad}_H(\frac{1}{\lambda^2})$ is tangential to $f^{-1}(l_\alpha)$ for all $\alpha \in S^1$, so $f$ is normally homothetic along $f^{-1}(l_\alpha)$ for all $\alpha \in S^1$. It then follows from Corollary 4.1.2 that $\mathcal{F}_f$ is a minimal foliation of $U \subset (\mathbb{C}^*)^m$. \hfill \blacksquare

For our examples we need the following lemma.

**Lemma 4.2.2.** Let $f : U \subset \mathbb{C}^m \rightarrow \mathbb{C}^*$, and $g : V \subset \mathbb{C}^n \rightarrow \mathbb{C}^*$ be holomorphic functions, such that $\alpha_f(U) \subset \mathbb{R}^+$ and $\alpha_g(V) \subset \mathbb{R}^+$. If $h : U \times V \rightarrow \mathbb{C}^*$ is given by $h(z, w) := f(z)g(w)$, then $\alpha_h(U \times V) \subset \mathbb{R}^+$.

**Proof.** This follows directly from the fact that
\[
\alpha_h = |g|^2 \alpha_f + 2|\text{grad}_C(f)|^2 |\text{grad}_C(g)|^2 + |f|^2 \alpha_g.
\]

\hfill \blacksquare

**Examples 4.2.3.** Define $F : (\mathbb{C}^*)^m \rightarrow \mathbb{C}^*$ by $F(z) := a_0 \prod_{i=1}^m z_i^{k_i}$, where $a_0 \in \mathbb{C}^*$ and $k_i > 1$. Further let $\mathcal{F}_F := \{F^{-1}(l_\alpha) | \alpha \in S^1\}$. Then the foliation $\mathcal{F}_F$ of real hypersurfaces in $(\mathbb{C}^*)^m$ is minimal.

**Proof.** If $f : \mathbb{C}^* \rightarrow \mathbb{C}^*$ is given by $f : z \mapsto a_0 z^k$ then $\alpha_f(z) = 2k^3(k-1)|a_0|^2 |z|^{2k-4} \in \mathbb{R}^+$ for all $z \in \mathbb{C}^*$. The result then follows from Lemmas 4.2.1 and 4.2.2. \hfill \blacksquare
CHAPTER 5. MULTIVALUED HARMONIC MORPHISMS.

5.1. Locally Defined Harmonic Morphisms.

In [Bai-Woo-1-2], Baird and Wood classify locally defined harmonic morphisms from 3-dimensional simply connected space forms \((M^3, g_M)\) to Riemann surfaces \(N^2\), in terms of meromorphic functions defined on the surface. This description implies that globally there exist only very few such examples, see Theorem 3.1.1.

For later use we describe this local classification in the case that \(M^3 = \mathbb{R}^3\):

Let \(g, h : N^2 \to \mathbb{C} \cup \{\infty\}\) be two meromorphic functions satisfying the following conditions:

1. If \(g(z_0)\) is finite, then \(h(z_0)\) is finite, and
2. If \(g(z_0) = \infty\), then \(\lim_{z \to z_0} h(z)/g^2(z)\) is finite.

For such meromorphic functions \(g, h : N^2 \to \mathbb{C} \cup \{\infty\}\), \(z \in N^2\) and \(x \in \mathbb{R}^3\) we have the following equation:

\[
\hat{G}(z, x) := (1 - g^2(z))x_1 + i(1 + g^2(z))x_2 + 2g(z)x_3 - 2h(z) = 0.
\]  

(5.1.3)

To make sense of this equation at a pole \(z_0 \in N^2\) of \(g\) we must divide through by \(g^2(z)\) and treat it as a limit.

By a local solution of this equation we mean a map \(\phi : U \to N^2\) defined on an open subset \(U\) of \(\mathbb{R}^3\), satisfying \(\hat{G}(\phi(x), x) = 0\). The above mentioned classification says: Every local solution to equation (5.1.3) is a harmonic morphism, and every locally defined harmonic morphism is a local solution to (5.1.3) for some meromorphic pair \((g, h)\) as above.

For the cases of \((M^3, g_M) = S^3\) or \(H^3\) there exists a similar description for locally defined harmonic morphisms. For \(M^3 = S^3\) see section 5.4.
5.2. The Covering Construction.

In complex analysis the analytic continuation or “gluing together” of locally defined holomorphic functions leads to the construction of Riemann surfaces. In this section we generalize this to harmonic morphisms from an arbitrary Riemannian manifold \((M^m, g_M)\) to a surface \((N^2, g_N)\). Proposition 5.2.2 and Theorem 5.2.5 resulted from joint work with J.C.Wood, see also [Gud-Woo].

Let the product manifold \(N^n \times M^m\) be equipped with the product metric \(\hat{g} = g_N \times g_M\).

**Definition 5.2.1.** Let \(G : (N^n \times M^m, \hat{g}) \rightarrow (P^p, g_P)\) be a map. For \(z \in N\) and \(x \in M\) we denote by \(G_z : M \rightarrow P\) and \(G_x : N \rightarrow P\) the maps given by \(G_z : y \mapsto G(z, y)\) and \(G_x : w \mapsto G(w, x)\), respectively. \(G\) is said to be a harmonic morphism in each variable separately if \(G_z : (M, g_M) \rightarrow (P, g_P)\) and \(G_x : (N, g_N) \rightarrow (P, g_P)\) are harmonic morphisms for every \(z \in N\) and \(x \in M\).

If \(f : U \subset P^p \rightarrow \mathbb{R}\) is a function and \(G : N^n \times M^m \rightarrow P^p\) is a smooth map, then it is easily verified that

\[
\Delta_{N \times M}(f \circ G)(z, x) = \Delta_N(f \circ G_x)(z) + \Delta_M(f \circ G_z)(x)
\]

for all \((z, x) \in N \times M\). Thus if \(G\) is a harmonic morphism in each variable separately, then it is a harmonic morphism as a map from the product manifold.

**Proposition 5.2.2.** Let \((M^m, g_M)\), \((N^2, g_N)\) and \((P^2, g_P)\) be Riemannian manifolds with \(\dim N^2 = \dim P^2 = 2\), and let \(G : N^2 \times M^m \rightarrow P^2\) be a harmonic morphism in each variable separately. If \(w\) is a fixed point on \(P^2\) and \(dG \neq 0\) on \(G^{-1}(w)\), then any smooth local solution \(\phi : U \rightarrow N^2, z = \phi(x)\) to the equation

\[
G(z, x) = w, \quad (5.2.1)
\]

defined on an open subset \(U\) of \(M^m\), is a harmonic morphism.

**Proof.** Let \(z = \phi(x)\) be a solution to (5.2.1) on \(U \subset M^m\) through \(x_0\) and put \(z_0 = \phi(x_0)\). To prove that \(\phi\) is a harmonic morphism we show that it is horizontally conformal and harmonic.

Since \(G_x\) is conformal and non-constant in a neighbourhood of \((z_0, x_0)\) and our problem is local, we can without loss of generality assume that \(N^2\) and \(P^2\) are
connected and oriented and that $G_x$ is holomorphic. Then we may choose local complex coordinates $z$ and $\hat{w}$ on $N^2$ and $P^2$ in neighbourhoods of $z_0$ and $w$ respectively, and normal coordinates $x = (x_1, ..., x_n)$ centred at the point $x_0 \in M^m$. In a neighbourhood of $x_0$ a local solution $φ : x \mapsto z$ satisfies

$$G(z(x), x) = w.$$  

Differentiating with respect to $x_i$ gives:

$$\frac{\partial G}{\partial z} \frac{\partial z}{\partial x_i} + \frac{\partial G}{\partial x_i} = 0. \quad (5.2.2)$$

Note that $dG \neq 0$ on $G^{-1}(w)$, so equation (5.2.2) implies that $dG_x = \frac{\partial G}{\partial z} \neq 0$, for if $\frac{\partial G}{\partial z} = 0$ then by (5.2.2) $\frac{\partial G}{\partial x_i} = 0$ for all $i$, so $dG = 0$, contradicting the hypothesis. Hence

$$\frac{\partial z}{\partial x_i} = -\left(\frac{\partial G}{\partial z}\right)^{-1} \frac{\partial G}{\partial x_i}.$$  

Since $G_z$ is a harmonic morphism it is horizontally conformal, that is $\sum_{i=1}^m (\frac{\partial G}{\partial x_i})^2 \neq 0$ at $x_0$, thus at that point,

$$\sum_{i=1}^m (\frac{\partial z}{\partial x_i})^2 = 0. \quad (5.2.3)$$

Differentiating (5.2.2) with respect to $x_i$ gives:

$$\frac{\partial^2 G}{\partial z^2} \left(\frac{\partial z}{\partial x_i}\right)^2 + \frac{\partial G}{\partial z} \frac{\partial^2 z}{\partial x_i^2} + \frac{\partial^2 G}{\partial x_i^2} = 0.$$  

Summing using (5.2.3), we have at $x_0$,

$$\frac{\partial G}{\partial z} \sum_{i=1}^m \frac{\partial^2 z}{\partial x_i^2} + \sum_{i=1}^m \frac{\partial^2 G}{\partial x_i^2} = 0.$$  

Since $\frac{\partial G}{\partial z} \neq 0$ and the last term vanishes we conclude that

$$\sum_{i=1}^m \frac{\partial^2 z}{\partial x_i^2} = 0. \quad (5.2.4)$$

At the point $x_0$ equations (5.2.3) and (5.2.4) are the conditions for horizontal conformality and harmonicity respectively. This implies that $φ$ is a harmonic morphism.

In general there is not a unique solution to equation (5.2.1); we need the following concept:
Definition 5.2.3. Let $C(N^n)$ be the set of all closed subsets of $N^n$. By a multivalued harmonic morphism $\Phi$ from $(M^m, g_M)$ to $(N^n, g_N)$ we shall mean a mapping $\Phi : M^m \to C(N^n)$, such that any smooth map $\phi : U \to N^m$ defined on an open subset $U$ of $M^m$ satisfying $\phi(x) \in \Phi(x)$ for all $x \in U$ is a harmonic morphism. Such a map $\phi$ will be called a branch of $\Phi$.

Note that Proposition 5.2.2 says that the set-valued mapping $\Phi : M^m \to C(N^2)$ with $\Phi(x) = \{z \in N^2 \mid G(z, x) = w\}$ is a multivalued harmonic morphism.

Definition 5.2.4. We call $\Phi : M^m \to C(N^2)$ given by $\Phi(x) := \{z \in N^2 \mid G(z, x) = w\}$ the multivalued harmonic morphism defined by equation (5.2.1).

For a map $G : N^2 \times M^m \to P^2$ and a point $w \in P^2$, define

$$\tilde{M}^m = \tilde{M}^m_w := \{(z, x) \in N \times M \mid G(z, x) = w\},$$

and let $\pi = \pi_1|_\tilde{M} : \tilde{M} \to N$ and $\psi = \pi_2|_{\tilde{M}} : \tilde{M} \to M$ be the restrictions of the natural projections $\pi_1 : N \times M \to N$ and $\pi_2 : N \times M \to M$ to $\tilde{M}$. We call the closed subset $\tilde{E} := \{(z, x) \in \tilde{M}^m \mid dG_z(z, x) = 0\}$ of $\tilde{M}^m$ the envelope of $G$ and its image $E = \psi(\tilde{E})$ in $M^m$ its geometric envelope. Further let $\tilde{F} := \{(z, x) \in \tilde{M}^m \mid dG_z(z, x) = 0\}$.

The next theorem explains how $\tilde{M}^m$ is the “Riemannian covering manifold” of the multivalued harmonic morphism defined by equation (5.2.1).

Theorem 5.2.5. Let $G : (N^2 \times M^m, \hat{g}) \to (P^2, g_P)$ be a harmonic morphism in each variable separately. Suppose that for some $w \in P^2$, $dG \neq 0$ along $\tilde{M}^m := G^{-1}(w)$. Then, with notations as above,

1. $\tilde{M}^m$ is an $m$-dimensional minimal submanifold of $(N^2 \times M^m, \hat{g})$,
2. $\psi : \tilde{M}^m \to M^m$ is a local diffeomorphism except on $\tilde{E}$,
3. $\pi : (\tilde{M}^m, \hat{g}) \to (N^2, g_N)$ is a harmonic morphism with critical set $\tilde{F}$,
4. any local solution $\phi : U \subset M^m \to N^2$ of equation (5.2.1), necessarily a harmonic morphism by Proposition 5.2.2, satisfies $\pi = \phi \circ \psi$ on $\psi^{-1}(U)$.

Proof. First note that the tangent space of $\tilde{M}$ at $(z, x)$ is given by

$$T_{(z, x)}\tilde{M} = \{(Z, X) \in T_zN^2 \times T_xM^m \mid dG(Z, X) = 0\}$$

$$= \{(Z, X) \in T_zN^2 \times T_xM^m \mid dG_z(X) + dG_x(Z) = 0\}.$$
(1) Since $G$ is a harmonic morphism, it is horizontally conformal, so the fact that $dG \neq 0$ on $\tilde{M}^m$ implies that $dG$ is surjective. It follows from the implicit function theorem that $\tilde{M}^m$ is an $m$-dimensional submanifold of $N^2 \times M^m$. That $\tilde{M}^m$ is minimal is a consequence of Theorem 1.2.7.

(2) Since $M^m$ and $\tilde{M}^m$ have the same dimension, we only have to show that $d\psi$ is surjective outside $\tilde{E}$. Let $(z, x) \in \tilde{M}^m \setminus \tilde{E}$ and let $X \in T_x M^m$ be non-zero. Since $G_x$ is a harmonic morphism, $dG_x \neq 0$ means that $dG_x$ is non-singular. Let $Z := -(dG_x)^{-1} \circ dG_{z}(X) \in T_z N^2$, then $dG_{z}(X) + dG_{z}(Z) = 0$, so $(Z, X) \in T_{(z, x)} \tilde{M}^m$ and $d\psi(Z, X) = X$. Thus $d\psi$ is surjective.

(3) We look first at points $(z, x) \in \tilde{F}$. There the tangent space of $\tilde{M}^m$ is $\{(0, X) \in T_z N^2 \times T_x M^m\}$, so clearly $d\pi = 0$ on $\tilde{F}$.

On the other hand let $(z, x) \in \tilde{M}^m \setminus \tilde{F}$, then we have $\pi^{-1}(z) = \{(z, x) \in \tilde{M}^m \mid G(z, x) = w\}$, so that the vertical space $V_{(z, x)}^\pi \mid \tilde{M}^m = \tilde{M}^m$ with respect to $\pi$ is given by $V_{(z, x)}^\pi = \{(0, X) \in T_{(z, x)} \tilde{M}^m \mid X \in V_x^G\}$. Hence $H_{(z, x)}^\pi = \{(Z, X) \in T_{(z, x)} \tilde{M}^m \mid X \in H_x^G\}$. Given $Z \in T_z N^2$, there exists exactly one $X \in H_x^G$ such that $dG_{z}(X) + dG_{z}(Z) = 0$, namely $X = -(dG_{z}|_{H_x^G})^{-1} \circ dG_{z}(Z)$. From this it is clear that $d\pi|_{H_{(z, x)}^\pi} : H_{(z, x)}^\pi \rightarrow T_z N^2$ is given by

$$
(Z, X) = (Z, -(dG_{z}|_{H_x^G})^{-1} \circ dG_{z}(Z)) \mapsto Z.
$$

Since $dG_{z}|_{H_x^G}$ and $dG_{z}$ are conformal, this shows that $\pi$ is horizontally conformal.

As regards the harmonicity of $\pi$, note that $\pi$ is the composition of the inclusion map $i$ and the projection $\pi_2$,

$$
\pi : \tilde{M}^m \xrightarrow{i} M^m \times N^2 \xrightarrow{\pi_2} N^2.
$$

Now the composition law for the the tension field (see [Eel-Sam]) is:

$$
\tau(\pi) = \text{trace } \nabla d\pi_2 (di, di) + d\pi_2 (\tau(i)).
$$

The first term is zero since $\pi_2$ is totally geodesic; also $\tilde{M}^m$ is minimal in $N^2 \times M^m$ so that $\tau(i) = 0$. Hence $\pi$ is harmonic, and since horizontally conformal, a harmonic morphism.

(4) To say that $\phi$ is a local solution on $U \subset M^m$, means that $G(\phi(x), x) = w$ for all $x \in U$. Thus $(\phi(x), x) \in \tilde{M}^m$ and $\pi(\phi(x), x) = \phi(x)$, i.e. $\pi = \phi \circ \psi$ on $\psi^{-1}(U)$. ■
By Theorem 5.2.5 the local solutions $\phi : U \to N^2$ defined by equation (5.2.1) have been “glued together” to get a globally defined harmonic morphism $\pi : (\tilde{M}^m, \hat{g}) \to (N^2, g_N)$.

5.3. Multivalued Harmonic Morphisms from $\mathbb{R}^3$.

As already mentioned in section 5.1, any locally defined harmonic morphism from an open subset $U$ of $\mathbb{R}^3$ to a Riemann surface $N^2$, is given by two meromorphic functions $g, h : N^2 \to \mathbb{C} \cup \{\infty\}$. For such a pair $(g, h)$ satisfying conditions (5.1.1) and (5.1.2) we define a meromorphic map $\xi : N^2 \to (\mathbb{C} \cup \{\infty\})^3$ by

$$\xi := \frac{1}{2h}(1 - g^2, i(1 + g^2), 2g).$$

Then $G : (N^2 \times \mathbb{R}^3, \hat{g}) \to \mathbb{C} \cup \{\infty\}$ with $G(z, x) := \langle \xi(z), x >_\mathbb{C}$ is a harmonic morphism in each variable separately. Theorem 5.2.5 therefore allows us to construct the Riemannian covering manifold

$$\tilde{\mathbb{R}}^3 := \{ (z, x) \in N^2 \times \mathbb{R}^3 | G(z, x) = 1 \},$$

with suitable interpretation at the poles of $\xi$. Furthermore it gives a globally defined harmonic morphism $\pi : \tilde{\mathbb{R}}^3 \to N^2$ by $\pi : (z, x) \to z$. Note, that $G(z, x) = 1$ is equivalent to equation (5.1.3).

**Proposition 5.3.1.** The harmonic morphism $\pi : (\tilde{\mathbb{R}}^3, \hat{g}) \to (N^2, g_N)$ defined above is a bundle map for a trivial $\mathbb{R}$-bundle over $N^2$.

**Proof.** For each $z \in N^2$ we have $\sum_{i=1}^3 \xi_i^2(z) = 0$ or equivalently

$$|\text{Re } \xi(z)|^2 = |\text{Im } \xi(z)|^2 \quad \text{and} \quad < \text{Re } \xi(z), \text{Im } \xi(z) >= 0.$$

The fibre $\pi^{-1}(z)$ of $\pi$ is given by $< \xi(z), x >_\mathbb{C} = 1$ or equivalently

$$< \text{Re } \xi(z), x >= 1 \quad \text{and} \quad < \text{Im } \xi(z), x >= 0.$$

Re $\xi$ and Im $\xi$ are linearly independent, so $L_z := \psi(\pi^{-1}(z))$ is an affine line in $\mathbb{R}^3$. If $c, \gamma : N^2 \to \mathbb{R}^3$ are given by

$$c := \frac{\text{Re } \xi}{|\text{Re } \xi|^2} \quad \text{and} \quad \gamma := \frac{\text{Re } \xi \times \text{Im } \xi}{|\text{Re } \xi \times \text{Im } \xi|},$$
then $L_z$ is parametrized w.r.t. arclength by $t \mapsto c(z) + t \cdot \gamma(z)$. The claim then follows from the fact that $l : N^2 \times \mathbb{R} \to \mathbb{R}^3$ given by $l : (z, t) \mapsto (z, c(z) + t \cdot \gamma(z))$ is a global bundle homeomorphism. \hfill \blacksquare

For each $z \in N^2$, $c(z)$ is the point of the oriented affine line $L_z$ nearest to the origin. The point $\gamma(z) \in S^2 \subset \mathbb{R}^3$ is the direction of $L_z$. Obviously $c(z) \perp \gamma(z)$ for every $z \in N^2$. This means that the map $s : N^2 \to \mathbb{R}^3 \times \mathbb{R}^3$ given by $s : z \mapsto (\gamma(z), c(z))$ has values in $TS^2$, the tangent bundle of $S^2$. Identify $S^2$ with $\mathbb{C} \cup \{\infty\}$ via the stereographic projection $\sigma$ from the south pole $(0, 0, -1)$, whose inverse $\sigma^{-1}$ is given by $\sigma^{-1} : z \mapsto (\frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2})$.

It was shown in [Bai-Woo-1] that the meromorphic pair $(g, h)$ represents $(\gamma, c)$ via $\sigma$, i.e.

$$\sigma(\gamma) = g \quad \text{and} \quad d\sigma_\gamma(c) = h.$$ 

Using this it can be shown that

$$\gamma = \sigma^{-1}(g) = (\frac{2g}{1 + |g|^2}, \frac{1 - |g|^2}{1 + |g|^2}) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$$

and

$$c = 2 \cdot (\frac{h - \overline{h} \cdot g^2}{(1 + |g|^2)^2}, \frac{-\overline{h} \cdot g + h \cdot \overline{g}}{(1 + |g|^2)^2}) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3.$$

**Example 5.3.2.** (Orthogonal Projection) Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C}$ and $g, h : \mathbb{C} \to \mathbb{C}$ be given by $g : z \mapsto 0$ and $h : z \mapsto z/2$. Then equation (5.1.3) becomes $x_1 + ix_2 - z = 0$. This has a global solution the single-valued harmonic morphism $z = x_1 + ix_2$ which is an orthogonal projection. This means that the covering manifold $\tilde{\mathbb{R}}^3 = \{(z, (x_1, x_2, x_3)) \in \mathbb{C} \times \mathbb{R}^3 \mid x_1 + ix_2 = z\}$ is a 3-dimensional subspace of $\mathbb{R}^5$. The envelope $\tilde{E} \subset \tilde{\mathbb{R}}^3$ is the empty set. The harmonic morphism $\pi : \tilde{\mathbb{R}}^3 \to \mathbb{C}$ can be thought of as an orthogonal projection $\mathbb{R}^3 \to \mathbb{R}^2$, where the metric on the horizontal spaces has been multiplied by the constant factor 2.

**Example 5.3.3.** (Radial Projection) Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and $g, h : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g : z \mapsto z$ and $h : z \mapsto 0$. Then equation (5.1.3) becomes

$$(x_1 - ix_2)z^2 + 2x_3z - (x_1 + ix_2) = 0. \quad (5.3.1)$$
Solving this equation gives two local solutions $\sigma^{-1} \circ z^\pm : \mathbb{R}^3 - \{0\} \to S^2$ given by $x \mapsto \pm x/|x|$ which are radial projection and its negative, well-known harmonic morphisms. We can think of equation (5.3.1) as defining a multivalued harmonic morphism $z(x)$, 2-valued away from 0, with these local solutions as branches.

It is easily seen that the covering manifold is

$$\tilde{\mathbb{R}}^3 = \{(v, x) \in S^2 \times \mathbb{R}^3 \mid x = tv \text{ for some } t \in \mathbb{R}\},$$

that is, the tautological bundle over $S^2$; this has an explicit trivialization $S^2 \times \mathbb{R} \to \tilde{\mathbb{R}}^3$ given by $(v, t) \mapsto (v, tv)$. The harmonic morphism $\pi : \tilde{\mathbb{R}}^3 \to S^2$ is the projection map of this bundle. The projection $\psi : \tilde{\mathbb{R}}^3 \to \mathbb{R}^3$ is a double cover with $\psi(v, x) = \psi(-v, x)$ except on the envelope $\tilde{E} = S^2 \times \{0\}$. This means that the geometric envelope $E$ is the single point $\{0\}$. Thus on passing from $\mathbb{R}^3$ to $\tilde{\mathbb{R}}^3$ the origin $0 \in \mathbb{R}^3$ is “blown up” to an $S^2$.

**Example 5.3.4.** (The Outer Disk Family) Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and $g, h : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g : z \mapsto z$ and $h : z \mapsto r z$ for some $r \in \mathbb{R}^+$. Equation (5.1.3) is now:

$$(1 - z^2) x_1 + i(1 + z^2) x_2 - 2 z x_3 - 2 i r z = 0. \quad (5.3.2)$$

We can think of this as defining a multivalued harmonic morphism $z(x)$ which is 2-valued except on the zero set of the discriminant of this quadratic equation in $z$ where the two values coincide. This set gives the geometric envelope $E = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = r^2 \text{ and } x_3 = 0\}$. It is the circle of radius $r$ in the $(x_1, x_2)$-plane, whose centre is the origin $0 \in \mathbb{R}^3$. In this case $\tilde{E} = \psi^{-1}(E)$, and $\tilde{E}$ is also a circle. Outside $E$ equation (5.3.2) has the two solutions:

$$z^\pm_r = \frac{-(x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2 i r x_3}}{x_1 - i x_2}. \quad (5.3.3)$$

If we choose $\sqrt{\cdot}$ to be the principal square root on $\mathbb{C} - \mathbb{R}_0^-$ ($\sqrt{\rho \ e^{i \theta}} := \sqrt{\rho} \ e^{i \theta/2}$, where $\theta \in (-\pi, \pi)$), then we obtain two harmonic morphisms $z^+_r$ and $z^-_r$ defined on $\mathbb{R}^3 - D_r$, where $D_r := \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 \leq r^2 \text{ and } x_3 = 0\}$ is the disk in the $(x_1, x_2)$-plane with $E$ as its boundary. For every $(x_1, x_2, x_3) \in \mathbb{R}^3 - D_r$ we have

$$z^+_r (x_1, x_2, x_3) = z^-_r (-x_1, x_2, -x_3)$$
so $z^+_r$ and $z^-_r$ are, up to isometries of $\mathbb{R}^3 - D_r$ and $S^2$, the same map. We call
\[
\{z^+_r : \mathbb{R}^3 - D_r \to S^2 \mid r \in \mathbb{R}^+\}
\]
the outer disk family.

The map $(\gamma, c) : S^2 \to TS^2 \subset (\mathbb{C} \times \mathbb{R})^2$ is given by
\[
(\gamma, c) : \sigma^{-1}(z) \mapsto \left(\left(\frac{2z}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2}\right), \left(\frac{2irz}{1 + |z|^2}, 0\right)\right).
\]
In spherical polar coordinates $(\theta, t) \in [0, 2\pi] \times [-\pi/2, \pi/2] \mapsto (e^{i\theta} \cos t, \sin t) \in S^2$, this reads
\[
(\gamma, c) : (\theta, t) \mapsto ((e^{i\theta} \cos t, \sin t), (ire^{i\theta} \cos t, 0)).
\]
The covering manifold $\tilde{\mathbb{R}}^3$ is therefore parametrized by
\[
(\theta, t, s) \mapsto ((e^{i\theta} \cos t, \sin t), (ire^{i\theta} \cos t, 0) + s(e^{i\theta} \cos t, \sin t))
\]
where $\theta \in [0, 2\pi]$, $t \in [-\pi/2, \pi/2]$ and $s \in \mathbb{R}$. The fibre over the point $z^+_r$ at $(e^{i\theta} \cos t, \sin t) \in S^2$ is given by
\[
s \mapsto L^+(\theta, t, s) := (ire^{i\theta} \cos t, 0) + s(e^{i\theta} \cos t, \sin t), \quad s \in \mathbb{R}^+,
\]
and the fibres of $z^-_r$ are given by the same formula with $s \in \mathbb{R}^-$. Note that $z^+_r$ and $z^-_r$ map a point $(ire^{i\theta} \cos t, 0) + s(e^{i\theta} \cos t, \sin t) \in \mathbb{R}^3 - D_r$ to $(e^{i\theta} \cos t, \sin t) \in S^2$ so they are both surjective.

Note that as $r \to 0$, $L(\theta, t, s) \to s(e^{i\theta} \cos t, \sin t)$, which shows that as $r \to 0$ the foliation of half-lines given by the fibres of $z^+_r$ approaches the corresponding foliation for the radial projection.

Let $(x, 0)$ be a point in $\mathbb{C} \times \mathbb{R}$, such that $|x| \leq r$. Then setting $t = \cos^{-1}(|x|/r)$, we see that the fibres $f^u$ and $f^l$ of $z^+_r$ in the upper and lower half spaces, with boundaries $\partial f^u = \partial f^l = (x, 0)$ are orthogonal to the radius from $(0, 0)$ to $(x, 0)$ and make an angle $t$ with the $(x_1, x_2)$-plane. As $|x|$ increases from 0 to $r$, $t$ decreases from $\pi/2$ to 0. The fibres through a point $(x, 0)$ with $|x| > r$ lie in the $(x_1, x_2)$-plane and are tangent to the envelope $E$. Note that the direction of the fibres changes discontinuously as we cross the disk $D_r$.

Keeping $x$ fixed and letting $r \to \infty$, we have that $t \to \pi/2$. In this sense the foliation of $z^+_r$ approaches the corresponding one for the orthogonal projection. Thus the outer disk family “interpolates” between the radial and orthogonal projections.
It follows from Proposition 5.3.1 that the covering manifold $\tilde{\mathbb{R}}^3$ is homeomorphic to $S^2 \times \mathbb{R}$. This can also be seen directly as follows: The map $L^+ : S^2 \times \mathbb{R}^+ \to \mathbb{R}^3 - D_r$ given by (5.3.4) is a homeomorphism, and so is $L^- : S^2 \times \mathbb{R}^- \to \mathbb{R}^3 - D_r$ given by the same formula. Extending these maps continuously to $S^2 \times \{0\}$ and glueing them together along this manifold gives a homeomorphism $L : S^2 \times \mathbb{R} \to \tilde{\mathbb{R}}^3$. We can thus think of $\tilde{\mathbb{R}}^3$ as being obtained by glueing the two copies of $\mathbb{R}^3 - D_r$ across $D_r$ in an analogous way to that used in Riemann surface theory, the disk $D_r$ playing the role of a cut joining two branch points. Indeed $\partial D_r = E$ and $D_r$ is a Seifert surface in the sense of [Rol] where this procedure of glueing across a Seifert surface to obtain a branched covering is discussed. Note finally that the globally defined harmonic morphism $\pi : \tilde{\mathbb{R}}^3 \to S^2$ is, via the homeomorphism $L$, just natural projection $S^2 \times \mathbb{R} \to S^2$.

Note that the solution $z_r^+ : \mathbb{R}^3 - D_r \to S^2$ of (5.3.2) satisfies the following properties:

(1) it is surjective,

(2) it has connected fibres, \hspace{1cm} (5.3.5)

(3) no two fibres are parallel as oriented line segments.

This is also true for the radial projections of Example 5.3.3. The following result shows that, up to equivalence, any harmonic morphism satisfying these conditions is one of these two examples.

**Theorem 5.3.5.** Let $\phi : U \to N^2$ be a harmonic morphism from an open subset of $\mathbb{R}^3$ to a closed Riemann surface, satisfying conditions (5.3.5). Then $N^2$ is conformally equivalent to $S^2$ and, up to isometries of $\mathbb{R}^3$ and conformal transformations of $S^2$, $\phi$ is a restriction of radial projection or a solution to (5.3.2).

**Proof.** Condition (2) implies that $\phi$ is submersive, otherwise by [Bai-Woo-2], it would be locally of the form $\rho \circ \tilde{\phi}$ where $\tilde{\phi}$ is submersive and $\rho$ of the form $z \mapsto z^k$. Such a composition clearly does not have connected fibres. From the locally defined harmonic morphism $\phi : U \to N^2$ we obtain two meromorphic functions $g, h : N^2 \to \mathbb{C} \cup \{\infty\}$. From the interpretation of $g$ as giving, via stereographic projection $\sigma$, the direction of the fibres, condition (3) tell us that $\sigma^{-1} \circ g : N^2 \to S^2$ is injective and holomorphic, so bijective. Thus, up to composition with a conformal transformation, $N^2 = S^2$ and $g$ is the identity map $g(z) = z$. Since $h$ then represents
the holomorphic vector field $c$ on $S^2$, $h(z)$ must be given by a quadratic polynomial in $z$. This means that $h$ can be written in the form

$$h(z) = <v(z), \bar{p}>_\mathbb{C},$$

where $v(z) := (1 - z^2, i(1 + z^2), 2z)$ for some $p$ of $\mathbb{C}^3$ and $<, >_\mathbb{C}$ denote the inner product on $\mathbb{C}^3$ given by $<z, w> := z_1 \bar{w}_1 + z_2 \bar{w}_2 + z_3 \bar{w}_3$. Then equation (5.1.3) can be written in the neat form

$$<v(z), x - \bar{p}>_\mathbb{C} = 0.$$  \hspace{1cm} (5.3.6)

Choose $r \in \mathbb{R}_0^+$, $\alpha \in [0, 2\pi]$ and $\beta \in [-\pi/2, \pi/2]$ such that $\text{Im } p = r(\cos \alpha \cos \beta, \sin \alpha \cos \beta, \sin \beta)$ and let $A, B \in \text{SO}(3)$ be given by

$$A = \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} \sin \beta & 0 & -\cos \beta \\ 0 & 1 & 0 \\ \cos \beta & 0 & \sin \beta \end{pmatrix}.$$

Then equation (5.3.6) is equivalent to

$$< B \cdot A \cdot v(z), B \cdot A(x - \text{Re}(p)) + iB \cdot A(\text{Im}(p)) >_\mathbb{C} = 0.$$  \hspace{1cm} (5.3.7)

If we make the following isometric changes of coordinates on $\mathbb{R}^3$ and $S^2$:

$$y := B \cdot A(x - \text{Re}(p)) \text{ and } w := \frac{ae^{i\alpha}z - b}{be^{i\alpha}z + a},$$

where $a := \cos((2\beta - \pi)/4)$ and $b := \sin((2\beta - \pi)/4)$ then it is not difficult to see that equation (5.3.7) is equivalent to

$$< v(w), y + (0, 0, ir) >_\mathbb{C} = 0,$$

which is equation (5.3.2) for $r > 0$, or (5.3.1) for $r = 0$. In the latter case it is clear that $\phi$ must be a restriction of radial projection or its negative. \hspace{1cm}  \blacksquare

Whether a solution to equation (5.3.2) on a given domain $U$ satisfies conditions (1),(2) and (3) of (5.3.5) depends on the domain. For example if we choose a different square root in Example 5.3.4, for instance the one defined on $\mathbb{C} - \mathbb{R}_0^+$, then we obtain different harmonic morphisms

$$\hat{z}^+_r, \hat{z}^-_r : \mathbb{R}^3 - C_r \rightarrow S^2$$
where $C_r := \{x \in \mathbb{R}^3 | r^2 \leq x_1^2 + x_2^2 \text{ and } x_3 = 0\}$. We call $\{\hat{z}_r^+: \mathbb{R}^3 - C_r \to S^2 | r \in \mathbb{R}^+\}$ the *inner disk family*. The map $\hat{z}_r^+: \mathbb{R}^3 - C_1 \to S^2$ is the map described in Example 2.9 of [Ber-Cam-Dav], and in [Bai-1], [Bai-Woo-1] as the disk example. Note that, in contrast to the outer disk family, the image $\hat{z}_r^+(\mathbb{R}^3 - C_r) \subset S^2$ of $\hat{z}_r^+$ is the upper hemisphere, so $\hat{z}_r^+$ is not surjective. The direction of the fibres of $\hat{z}_r^+$ changes continuously when crossing the disk $D_r$ but discontinuously when crossing $C_r$, this being another possible Seifert surface for the branched covering $\psi: \tilde{\mathbb{R}}^3 \to \mathbb{R}^3$. More generally, we can find solutions to (5.3.5) on $\mathbb{R}^3 - S_r$ for any Seifert surface $S_r$, i.e. any surface with boundary the envelope $E$, the resulting harmonic morphisms having very different properties according as $S_r$ is bounded or not.

On the way we have shown that, if $N^2 = S^2$, $g(z) = z$ and $h(z)$ is a quadratic polynomial in $z$, then (5.1.3) has solutions satisfying conditions (5.3.5). This is not true, for example if $N^2 = S^2$, $g(z) = z$ and $h(z) = z^k$, where $k \geq 3$. In this case (5.1.3) can have no solution which is a surjective harmonic morphism to $S^2$, since $h$ cannot represent a vector field at $z = \infty$; indeed only quadratic polynomials in $z$ define vector fields globally on $S^2$.

*Example 5.3.6*. Let $M^3 = \mathbb{R}^3$, $N^2 = \mathbb{C}$ and $g, h: \mathbb{C} \to \mathbb{C}$ be given by $g: z \mapsto z$ and $h: z \mapsto z^k/2$ for some $k \geq 3$. Then $\hat{G}: \mathbb{C} \times \mathbb{R}^3 \to \mathbb{C}$ is given by $\hat{G}: (z, x) \mapsto z^k + (x_1 - ix_2)z^2 + 2x_3z - (x_1 + ix_2)$. The projection $\pi$ is therefore a $k$-fold covering of $\mathbb{R}^3$ except on the envelope $\tilde{E}$. As before one could use the map $(\gamma, c): \mathbb{C} \to TS^2$ to get a parametrization of $\tilde{\mathbb{R}}^3$ which we know by Proposition 5.3.1 is homeomorphic to $\mathbb{C} \times \mathbb{R} = \mathbb{R}^3$.

We are mainly interested in the geometric envelope $E \subset \mathbb{R}^3$, for which we give a complete parametrization for any $k \geq 3$. As before the envelope $\tilde{E}$ is given by solving two simultaneous equations:

\[ G(z, x) \equiv -z^k - (x_1 - ix_2)z^2 - 2x_3z + (x_1 + ix_2) = 0 \quad (5.3.8) \]

and

\[ \frac{\partial G}{\partial z}(z, x) \equiv -kz^{k-1} - 2(x_1 - ix_2)z - 2x_3 = 0. \quad (5.3.9) \]

By eliminating $z^k$ and $z^{k-1}$ between the two equations we get

\[ (k - 2)\bar{w} \cdot z^2 + 2x(k - 1) \cdot z - kw = 0. \quad (5.3.10) \]
It immediately follows, that if \( w = 0 \) then both \( x = 0 \) and \( z = 0 \), so \( E \) has only one point in common with the \( x \)-axis and that is \( (0, 0) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3 \). From now on we assume that \( w \neq 0 \). Solving (5.3.10) gives

\[
z = \frac{-x(k - 1) \pm \sqrt{x^2(k - 1)^2 + k(k - 2)|w|^2}}{(k - 2) \cdot \bar{w}}. \tag{5.3.11}
\]

It then follows from (5.3.9) that \( z^{k-1} = -(2x + 2\bar{w} \cdot z)/k \in \mathbb{R} \) since \( \bar{w} \cdot z \in \mathbb{R} \). Hence there exist \( n \in \{1, 2, ..., k - 1\} \) and \( r \in \mathbb{R} - \{0\} \), such that \( w = re^{i\pi(n/k)} \).

This means that \( E \subset \bigcup_{n=1}^{k-1} P_n \) where \( P_n \) is the 2-plane in \( \mathbb{C} \times \mathbb{R} \) given by \( P_n := \text{span}_\mathbb{R}\{(0, 1), (e^{i\pi(n/k)}, 0)\} \). For \( n \in \{1, 2, ..., 2(k - 1)\} \) define \( \tau_n : \mathbb{C} \times \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{C} \times \mathbb{R} \) by

\[
\tau_n : (z, w, x) \mapsto (e^{i\pi(n/k)} \cdot z, (-1)^n \cdot e^{i\pi(n/k)} \cdot w, (-1)^n \cdot x),
\]

then \( \{\tau_n| \ n \in \{1, 2, ..., 2(k - 1)\}\} \) is an isometry subgroup of \( \mathbb{C} \times \mathbb{C} \times \mathbb{R} \) and an easy calculation shows that a point \( p \in \tilde{E} \) if and only if \( \tau_n(p) \in \tilde{E} \) for all \( n \in \{1, 2, ..., 2(k - 1)\} \). This means that if one knows the part of the geometric envelope in the plane \( P_{k-1} = \text{span}_\mathbb{R}\{(0, 1), (1, 0)\} \subset \mathbb{C} \times \mathbb{R} \), then one obtains the rest by applying the maps \( \tau_n \) to \( E \cap P_{k-1} \). In \( P_{k-1} \) both \( x \) and \( w \) are real, so the corresponding \( z \) given by (5.3.11) is also real. For such points let \( t := z \) and \( y := w \).

Equations (5.3.8) and (5.3.9) now become

\[
\begin{bmatrix}
2t & t^2 - 1 \\
2 & 2t
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= 
\begin{bmatrix}
-t^k \\
-k t^{k-1}
\end{bmatrix},
\]

or equivalently

\[
\begin{bmatrix}
x \\
y
\end{bmatrix}(t) = \frac{1}{2(1 + t^2)} \begin{bmatrix}
(k - 2)t^{k+1} - kt^{k-1} \\
2(1 - k)t^k
\end{bmatrix},
\]

which parametrizes \( E \cap P_{k-1} \) completely. Thus \( E \) consists of \( 2(k - 1) \) such curves meeting at the origin, and so is not a manifold, not even topologically.

### 5.4. Multivalued Harmonic Morphisms from \( S^3 \).

In [Bai-Woo-1] harmonic morphisms defined locally on \( S^3 \) to a Riemann surface are classified in terms of two meromorphic functions on the surface. We now describe this classification, which can also be found in [Bai-3].
Let \( f, g : N^2 \to \mathbb{C} \) be meromorphic functions and \( \xi : N^2 \to \mathbb{C}^4 - \{0\} \) be the meromorphic map, given by
\[
\xi := (1 + fg, i(1 - fg), f - g, -i(f + g)).
\]

Then every local solution \( \phi : U \subset S^3 \to N^2 \) to the equation
\[
< \xi(z), x >_{\mathbb{C}} = 0,
\]
where \( x \in S^3 \subset \mathbb{R}^4 \) and \( z \in N^2 \), is a harmonic morphism. To make sense of this last equation at the poles of \( f \) and \( g \), we must divide it by \( f(z) \) and/or \( g(z) \) and treat it as a limit.

Conversely, every locally defined harmonic morphism \( \phi : U \to N^2 \) from an open and connected subset \( U \) of \( S^3 \), is a solution of equation (5.4.1) for some pair \((f, g)\) of meromorphic functions defined locally on the Riemann surface, as above.

As in the \( \mathbb{R}^3 \)-case equation (5.4.1) defines a Riemannian covering manifold
\[
\tilde{S}^3 := \{(z, x) \in N^2 \times S^3 | < \xi(z), x >_{\mathbb{C}} = 0\},
\]
with suitable interpretation at the poles of \( \xi \). Furthermore it gives a globally defined harmonic morphism \( \pi : \tilde{S}^3 \to N^2 \) with \( \pi : (z, x) \to z \) and a map \( \psi : \tilde{S}^3 \to S^3 \) given by \( \psi : (z, x) \to x \).

**Proposition 5.4.1.** For any meromorphic functions \( f, g \) on a Riemann surface \( N^2 \), the harmonic morphism \( \pi : \tilde{S}^3 \to N^2 \) given by \( \pi : (z, x) \mapsto z \) is a principal \( S^1 \)-bundle over \( N^2 \).

**Proof.** The fibres of \( \pi \) are closed geodesics of \( \tilde{S}^3 \) all of length \( 2\pi \), indeed for a fixed \( z \in N^2 \), \( \pi^{-1}(z) = \{z\} \times C_z \) where \( C_z \) is the great circle of \( S^3 \) given by equation (5.4.1). Such fibres have a natural orientation, so that, together with \( \text{Re} \ \xi(z) \) and \( \text{Im} \ \xi(z) \), they give an oriented basis for \( \{z\} \times S^3 \). We define the action of \( e^{i\theta} \in S^1 \) on each fibre of \( \tilde{S}^3 \) as rotation through \( +\theta \). Thus \( \psi : \tilde{S}^3 \to N^2 \) is a principal \( S^1 \)-bundle.

Regarding the degree of the bundle \( \pi : \tilde{S}^3 \to N^2 \), let us form the associated \( \mathbb{R}^2 \)-bundle, \( \tilde{\pi} : \mathbb{R}^4 \to N^2 \), given as for \( \tilde{S}^3 \) by equation (5.4.1) but with \( x \in \mathbb{R}^4 \); this is naturally oriented and \( \pi : \tilde{S}^3 \to N^2 \) is its unit circle bundle.
Lemma 5.4.2. For two meromorphic functions $f, g : N^2 \to \mathbb{C}$ let $\pi : \hat{S}^3 \to N^2$ and $\hat{\pi} : \mathbb{R}^3 \to N^2$ be the corresponding $S^1$-bundle and its associated $\mathbb{R}^2$-bundle over $N^2$. Further define a map $\xi : N^2 \to \mathbb{C}^4 \setminus \{0\}$, by

$$\xi := \left(1 - \frac{f}{g}, i(1 + \frac{f}{g}), f + \frac{1}{g}, -i(f - \frac{1}{g})\right).$$

Then the two sections $\text{Re} \, \xi$ and $\text{Im} \, \xi$ form an oriented basis for $\hat{\pi} : \mathbb{R}^4 \to N^2$, away from poles of $\xi$.

Proof. First note that from equation (5.4.1) it follows that $\text{Re} \, \xi$ and $\text{Im} \, \xi$ form an oriented basis for the horizontal space $H^\#_z$ of $\hat{\pi}$ at $z$. Then one easily checks that $\langle \xi, \xi \rangle = \langle \text{Re} \, \xi, \text{Re} \, \xi \rangle = \langle \text{Im} \, \xi, \text{Im} \, \xi \rangle = 0$, so that $\{\text{Re} \, \xi, \text{Im} \, \xi\}$ is an orthogonal basis for the vertical space $V^\#_z$ of $\hat{\pi} : \mathbb{R}^4 \to N^2$.

Finally, a lengthy calculation (done using REDUCE) shows that the determinant of the matrix, made up of the four vectors $\text{Re} \, \xi$, $\text{Im} \, \xi$, $\text{Re} \, \xi$ and $\text{Im} \, \xi$ is positive, so that $\text{Re} \, \xi$ and $\text{Im} \, \xi$ is an oriented basis for the vertical space $V^\#_z$ of $\hat{\pi} : \mathbb{R}^4 \to N^2$ as claimed. □

It follows from Lemma 5.4.2, that $\text{Re} \, \xi / |\text{Re} \, \xi|$ defines a section of $\pi : \hat{S}^3 \to S^2$ away from poles of $\xi$. We may calculate the degree of the bundle $\pi : \hat{S}^3 \to S^2$ as the obstruction to extending $\text{Re} \, \xi / |\text{Re} \, \xi|$ over those poles, see [Ste].

In the following examples it is convenient to write $w_1 := x_1 + ix_2$, $w_2 := x_3 + ix_4$, so that equation (5.4.1) reads:

$$w_1 + f(z)g(z)\bar{w}_1 - g(z)w_2 + f(z)\bar{w}_2 = 0. \quad (5.4.2)$$

Example 5.4.3. Let $M^3 = S^3$, $N^2 = \mathbb{C} \cup \{\infty\} = S^2$ and let $f, g : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ be given by $g : z \mapsto z^k$ ($k \in \mathbb{N}^+$) and $f : z \mapsto 0$. Then

$$\xi = (1, i, e^{i\theta}, \frac{ie^{i\theta}}{r^k}),$$

where $z = re^{i\theta}$. Further $v_1 := \text{Re} \, \xi / |\text{Re} \, \xi|$ and $v_2 := \text{Im} \, \xi / |\text{Im} \, \xi|$ are sections of $\pi : \hat{S}^3 \to N^2$ except at $z = 0$ and we have

$$v_1(0, \theta) := \lim_{r \to 0} \frac{\text{Re} \, \xi}{|\text{Re} \, \xi|} = (0, 0, \cos k\theta, -\sin k\theta),$$

and

$$v_2(0, \theta) := \lim_{r \to 0} \frac{\text{Im} \, \xi}{|\text{Im} \, \xi|} = (0, 0, \sin k\theta, \cos k\theta).$$
The ordered pair \((\alpha, \beta) = (v_1(0,0), v_2(0,0))\) = \(((0,0,1,0), (0,0,0,1))\) forms a positively oriented basis for the fibre \(\hat{\pi}^{-1}(0)\). The index of \(\text{Re} \xi^\perp/|\text{Re} \xi^\perp|\) at \(z = 0\) is \(-k\), so the degree of \(\pi : \tilde{S}^3 \to \tilde{S}^2\) is \(d = -k\). If \(k = 1\), then equation (5.4.2) is \(w_1 - zw_2 = 0\), and has the global solution \(\phi : (w_1, w_2) \mapsto w_1/w_2\), which simply is the Hopf-map \(\pi_7 : S^3 \to S^2\).

**Example 5.4.4.** Let \(M^3 = S^3\), \(N^2 = \mathbb{C}\cup\{\infty\} = S^2\) and let \(f, g : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}\) be given by \(f : z \mapsto 1/z^k\) \((k \in \mathbb{N}^+)\) and \(g : z \mapsto \infty\). Then

\[
\xi^\perp = (1, i, e^{-i\theta}r^k, -ie^{-i\theta}r^k),
\]

where \(z = re^{i\theta}\). As in Example 5.4.3 we get sections \(\text{Re} \xi^\perp/|\text{Re} \xi^\perp|\) and \(\text{Im} \xi^\perp/|\text{Im} \xi^\perp|\) of \(\pi : \tilde{S}^3 \to N^2\) except at \(z = 0\). Simple calculation yield

\[
v_1(0, \theta) := \lim_{r \to 0} \frac{\text{Re} \xi^\perp}{|\text{Re} \xi^\perp|} = (0, 0, \cos k\theta, -\sin k\theta), \text{ and}
\]

\[
v_2(0, \theta) := \lim_{r \to 0} \frac{\text{Im} \xi^\perp}{|\text{Im} \xi^\perp|} = (0, 0, -\sin k\theta, -\cos k\theta).
\]

Then one easily sees that the degree of \(\pi : \tilde{S}^3 \to S^2\) is \(d = k\).

**Example 5.4.5.** Let \(M^3 = S^3\), \(N^2 = \mathbb{C}\cup\{\infty\} = S^2\) and let \(f, g : \mathbb{C}\cup\{\infty\} \to \mathbb{C}\cup\{\infty\}\) be given by \(f : z \mapsto -z\), and \(g : z \mapsto z\). Then a calculation of the degree of \(\pi : \tilde{S}^3 \to S^2\) gives \(d = 0\), so \(\tilde{S}^3\) is homeomorphic to the product \(S^2 \times S^1\) and \(\pi\) is simply the natural projection \(\pi : S^2 \times S^1 \to S^2\), with \(\pi(x, y) \mapsto x\). Equation (5.4.2) becomes

\[
\bar{w}_1 z^2 + (w_2 + \bar{w}_2)z - w_1 = 0.
\]

Solving this equation gives two local solutions \(\phi^\pm : S^3 - S^0 \to S^2\), with \(\phi^\pm(x) = \sigma^{-1} \circ z^\pm(x) = \pm(x_1, x_2, x_3)/\sqrt{x_1^2 + x_2^2 + x_3^2}\). The map \(\phi^+\) is simply \(\pi_1 : S^3 - S^0 \to S^2\) given in section 1.3.

Note that Examples 5.4.3-5.4.5 give explicit examples of how any topological \(S^1\)-bundle over \(S^2\) can be given a metric such that its projection map \(\pi : \tilde{S}^3 \to S^2\) is a harmonic morphism. It is a general result of [Bai-Woo-3] that this is possible for any \(S^1\)-bundle over a surface.
CHAPTER 6. HOMOGENEOUS HARMONIC MORPHISMS.

In this chapter we use a result of L.Bérard Bergery to give new examples of harmonic morphisms between reductive homogeneous spaces. We then show how some already well known examples from 3-manifolds to surfaces can be obtained in this way.


Let $G$ be a Lie group and $K$ be a closed subgroup, then the homogeneous space $G/K = \{aK \mid a \in G\}$ of left cosets has a structure of an analytic manifold. The group $G$ acts on $G/K$ transitively in the natural way by $G \times G/K \to G/K$ with $(a, bK) \mapsto abK$. By $\mathfrak{k}$ and $\mathfrak{g}$ we denote the Lie algebras of $K$ and $G$ respectively. The homogeneous space $G/K$ is called reductive if the Lie algebra $\mathfrak{g}$ may be decomposed into a direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is an $\text{Ad}(K)$-invariant subspace i.e. $\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$. Since $\text{ad} = d(\text{Ad})_e$ it follows that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, which is equivalent to $\text{Ad}(K)\mathfrak{m} \subset \mathfrak{m}$ if $K$ is connected. A reductive homogeneous space $G/K$ is said to be symmetric if furthermore $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$.

If $G/K$ is a reductive homogeneous space with the $\text{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, then every $\text{Ad}(K)$-invariant positive definite scalar product $\langle, \rangle$ on $\mathfrak{m}$ induces a $G$-invariant Riemannian metric $g$ on $G/K$. The condition of $K$ being compact is sufficient for the existence of an $\text{Ad}(K)$-invariant decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and an $\text{Ad}(K)$-invariant scalar product $\langle, \rangle$ on $\mathfrak{m}$, but as we will see Example 6.3.4 then it is not necessary. If $G$ is compact then we can choose $\langle, \rangle := -B|_{\mathfrak{m} \times \mathfrak{m}}$, where $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is the Killing form on $\mathfrak{g}$.

The classical Lie groups will here be denoted by: $\text{GL}_n(\mathbb{R})$, $\text{SL}_n(\mathbb{R})$, $\text{O}(n)$, $\text{SO}(n)$, $\text{GL}_n(\mathbb{C})$, $\text{SL}_n(\mathbb{C})$, $\text{U}(n)$, $\text{SU}(n)$, and $\text{Sp}(n)$. Their tangent bundles are of course trivial and simply given by

$$TG := \{(x, xA) \mid x \in G, A \in T_xG \cong \mathfrak{g}\}.$$ 

For symmetric spaces there exists a well known classification, see for example Helgason’s book [Hel]. Amongst the irreducible ones we have: $\text{SL}_n(\mathbb{R})/\text{SO}(n)$, $\text{SU}(n)/\text{SO}(n)$, $\text{SU}(2n)/\text{Sp}(n)$, $\text{SU}(m+n)/\text{S(U}(n) \times \text{U}(m))$, $\text{SO}(m+n)/\text{SO}(m) \times $
$\text{SO}(n)$, $\text{SO}(2n)/\text{U}(n)$, $\text{Sp}(n)/\text{U}(n)$ and $\text{Sp}(m+n)/\text{Sp}(m) \times \text{Sp}(m)$. We now give two examples of reductive homogeneous spaces which are in general not symmetric.

**Example 6.1.1.** For $n, n_1, ..., n_q \in \mathbb{N}^+$ such that $n = \sum_{k=1}^{q} n_k$ put $G := \text{SL}_n(\mathbb{R})$ and $K := \text{SO}(n_1) \times \cdots \times \text{SO}(n_q)$. Then the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ are given by

$$\mathfrak{g} = \text{sl}_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} | \text{trace } A = 0 \}, \quad \text{and} \quad \mathfrak{k} = \text{so}(n_1) \times \cdots \times \text{so}(n_q)$$

$$= \{ A = \begin{pmatrix} A_1 & 0 \\ \vdots & \ddots \\ 0 & A_q \end{pmatrix} | A_k \in \mathbb{R}^{n_k \times n_k}, A^t_k = -A_k \}.$$

Let $\mathfrak{m} \subset \mathfrak{g}$ be given by

$$\mathfrak{m} = \{ B = \begin{pmatrix} B_{11} & \cdots & B_{1q} \\ \vdots & \ddots & \vdots \\ B_{q1} & \cdots & B_{qq} \end{pmatrix} | B_{ij} \in \mathbb{R}^{n_i \times n_j}, B^t_{kk} = B_{kk}, \text{trace } B = 0 \}.$$

Then clearly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and for $A \in \mathfrak{k}$ and $B \in \mathfrak{m}$ we have

$$[A, B]_{kk} = B^t_{kk} A^t_k - A^t_k B^t_{kk} = [A, B]_{kk}.$$

This means that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, so $\mathfrak{m}$ is an $\text{Ad}(K)$-invariant subspace of $\mathfrak{g}$, since $K$ is connected.

**Example 6.1.2.** For $n, n_1, ..., n_q \in \mathbb{N}^+$ such that $n = \sum_{k=1}^{q} n_k$ put $G := \text{SO}(n)$ and $K := \text{SO}(n_1) \times \cdots \times \text{SO}(n_q)$. Then the corresponding Lie algebras $\mathfrak{g}$ and $\mathfrak{k}$ are given by

$$\mathfrak{g} = \text{so}(n) = \{ A \in \mathbb{R}^{n \times n} | A^t = -A \}, \quad \text{and} \quad \mathfrak{k} = \text{so}(n_1) \times \cdots \times \text{so}(n_q)$$

$$= \{ A = \begin{pmatrix} A_1 & 0 \\ \vdots & \ddots \\ 0 & A_q \end{pmatrix} | A_k \in \mathbb{R}^{n_k \times n_k}, A^t_k = -A_k \}.$$

Let $\mathfrak{m} \subset \mathfrak{g}$ be given by

$$\mathfrak{m} = \{ B = \begin{pmatrix} 0 & B_{12} & \cdots & B_{1q} \\ -B^t_{12} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & B_{q-1q} \\ -B^t_{1q} & \cdots & -B^t_{q-1q} & 0 \end{pmatrix} | B_{ij} \in \mathbb{R}^{n_i \times n_j}, \text{for } i < j \}.$$

Then clearly $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, and it follows from the calculations above that $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$, so $\mathfrak{m}$ is an $\text{Ad}(K)$-invariant subspace of $\mathfrak{g}$. 
6.2. Existence for Reductive Homogeneous Spaces.

Let $G$ be a Lie group and $K \subset H$ be two closed subgroups, and $\mathfrak{k} \subset \mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebras. Then we have the natural fibration $\pi : G/K \to G/H$ given by $\pi : aK \mapsto aH$. Further let $\mathfrak{m}$ be an $\text{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ and $\mathfrak{p}$ an $\text{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{h}$. Then $\mathfrak{p} \oplus \mathfrak{m}$ is an $\text{Ad}(K)$-invariant complement of $\mathfrak{k}$ in $\mathfrak{g}$. As described above, then an $\text{Ad}(H)$-invariant scalar product $\langle , \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ defines a $G$-invariant Riemannian metric $\bar{g}$ on $G/H$ and an $\text{Ad}(K)$-invariant scalar product $\langle , \rangle_{\mathfrak{p}}$ on $\mathfrak{p}$ defines a $H$-invariant Riemannian metric $\bar{g}$ on $H/K$. The orthogonal direct sum $\langle , \rangle := \langle , \rangle_{\mathfrak{m}} \oplus \langle , \rangle_{\mathfrak{p}}$ on $\mathfrak{m} \oplus \mathfrak{p}$ defines a $G$-invariant Riemannian metric $g$ on $G/K$.

For the above situation we have the following result due to L. Bérard Bergery, see [Bér] or [Bes].

**Theorem 6.2.1.** If $K \subset H$ are compact subgroups of $G$, then the map $\pi : (G/K, g) \to (G/H, \bar{g})$, given by $\pi : aK \mapsto aH$ is a Riemannian submersion with totally geodesic fibres.

**Proof.** see [Bes].

In Theorem 6.2.1 the condition of $K$ and $H$ being compact ensures the existence of the $G$-invariant metrics $g$ and $\bar{g}$ on $G/K$ and $G/H$ respectively. The proof of the theorem as in [Bes] works however if one assumes that the two metrics $g$ and $\bar{g}$ exist but that $K$ and $H$ are not necessarily compact.

As a direct consequence for harmonic morphisms we have:

**Proposition 6.2.2.** Let $(G/K, g)$ and $(G/H, \bar{g})$ be two Riemannian homogeneous spaces, with $G$-invariant metrics as above. Then the map $\pi : (G/K, g) \to (G/H, \bar{g})$, given by $\pi : aK \mapsto aH$ is a harmonic morphism with totally geodesic fibres and constant dilation $\lambda \equiv 1$.

6.3. Applications.

Here we shall always equip the classical groups $G \subset \text{GL}_n(\mathbb{C})$ with the left-invariant Riemannian metric given by the positive definite scalar product $\langle A, B \rangle := \text{Re trace } A^t \cdot B$ on $\mathfrak{g}$. For the classical groups the map $\text{Ad}(\alpha) : \mathfrak{g} \to \mathfrak{g}$ is simply
given by \( \text{Ad}(\alpha) : X \mapsto \alpha X \alpha^{-1} \), so if \( \alpha \in \mathbf{U}(n) \), then

\[
< \text{Ad}(\alpha)X, \text{Ad}(\alpha)Y > = \text{Re} \text{trace}(\alpha X \alpha^{-1}) \cdot (\alpha Y \alpha^{-1}) = < X, Y > .
\]

This means that the scalar product \(<,>\) is \( \text{Ad}(K) \)-invariant for any subgroup \( K \) of \( \mathbf{U}(n) \).

The above construction can now be used to produce many new examples of harmonic morphisms between reductive homogeneous spaces, for instance we have:

\[
\pi : \text{SL}_{n_1 + \cdots + n_q}(\mathbb{R})/\mathbf{SO}(n_1) \times \cdots \times \mathbf{SO}(n_q) \to \text{SL}_{n_1 + \cdots + n_q}(\mathbb{R})/\mathbf{SO}(n_1 + \cdots + n_{k_1}) \times \cdots \times \mathbf{SO}(n_{k_q-1} + 1 + \cdots + n_q),
\]

\[
\pi : \mathbf{SO}(n_1 + \cdots + n_q)/\mathbf{SO}(n_1) \times \cdots \times \mathbf{SO}(n_q) \to \mathbf{SO}(n_1 + \cdots + n_q)/\mathbf{SO}(n_1 + \cdots + n_{k_1}) \times \cdots \times \mathbf{SO}(n_{k_q-1} + 1 + \cdots + n_q).
\]

Some already well known examples of harmonic morphisms from 3-dimensional manifolds to surfaces do arise in this way:

**Example 6.3.1.** Let \( \hat{\pi} \) be the homogeneous map

\[
\hat{\pi} : \mathbf{SO}(2) \times \mathbb{R}^2 \to \mathbf{SO}(2) \times \mathbb{R}^2/\mathbf{SO}(2) = \mathbb{R}^2
\]

and \( \sigma : \mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2 \xrightarrow{\mathbb{Z}:1} \mathbf{SO}(2) \times \mathbb{R}^2 \) be the universal cover. Then \( \pi := \hat{\pi} \circ \sigma : \mathbb{R}^3 \to \mathbb{R}^2 \) is a harmonic morphism.

**Example 6.3.2.** Let \( \hat{\pi} \) be the homogeneous harmonic morphism

\[
\hat{\pi} : \mathbf{SO}(3) \to \mathbf{SO}(3)/\mathbf{SO}(2) = S^2
\]

and \( \sigma : S^3 \xrightarrow{2:1} \mathbf{SO}(3) \) be the universal cover. Then \( \pi := \hat{\pi} \circ \sigma : S^3 \to S^2 \) is a harmonic morphism.

**Example 6.3.3.** Let \( \hat{\pi} \) be the homogeneous harmonic morphism

\[
\hat{\pi} : \text{SL}_2(\mathbb{R})/\{\pm I\} \to \text{SL}_2(\mathbb{R})/\mathbf{SO}(2) = H^2.
\]

Further let \( \sigma : \text{SL}_2(\mathbb{R}) \xrightarrow{2:1} \text{SL}_2(\mathbb{R})/\{\pm I\} \) and \( \hat{\sigma} : \text{SL}_2(\mathbb{R}) \xrightarrow{\mathbb{Z}:1} \text{SL}_2(\mathbb{R}) \) be the cover and universal cover, respectively. Then \( \pi := \hat{\pi} \circ \sigma \circ \hat{\sigma} : \text{SL}_2(\mathbb{R}) \to H^2 \) is a harmonic morphism.
Example 6.3.4. Let $G$ be the 3-dimensional Heisenberg group
\[ G = \{ \begin{pmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \}, \]
and $K$ be the closed non-compact subgroup of $G$ given by
\[ K = \{ \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \}. \]

A basis for the Lie algebra $\mathfrak{g}$ is given by $\{X, Y, Z\}$, where
\[ X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \]

The Lie algebra $\mathfrak{k}$ of $K$ is given by $\mathfrak{k} = \text{span}_{\mathbb{R}}\{X\}$ and one easily checks that $[X, Y] = 0$, $[X, Z] = 0$ and $[Y, Z] = X$, so if $\mathfrak{m} := \text{span}_{\mathbb{R}}\{Y, Z\}$ then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is an $\text{Ad}(K)$-invariant decomposition. If $a \in K$, then $\text{Ad}(a) = \text{Id}_{\mathfrak{g}}$, so any scalar product on $\mathfrak{g}$ is $\text{Ad}(K)$-invariant and therefore induces a $G$-invariant metric on the symmetric space $G/K$. We choose $g$ to be the metric on $\mathfrak{g}$ that makes $\{X, Y, Z\}$ into an orthonormal basis for $\mathfrak{g}$. Then $G/K$ is simply $\mathbb{R}^2$, and the projection $\pi : G \to G/K$ is a harmonic morphism with totally geodesic fibres. Define a map $\phi : \mathbb{R}^3 \to G$ by
\[ \phi : (x, y, z) \mapsto \begin{pmatrix} 1 & y & x \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}. \]

Then the pull-back metric $ds^2 := \phi^*g$ on $\mathbb{R}^3$ is given by
\[ ds^2 = dy^2 + dz^2 + (dx - ydz)^2. \]

This means that $(\mathbb{R}^3, ds^2)$ is the space Nil discussed in [Bai-3], and the map $\pi \circ \phi : \text{Nil} \to \mathbb{R}^2$ is a harmonic morphism.

It is trivial that the harmonic morphisms in example 6.3.1 to 6.3.4 are all Riemannian submersions, so the uniqueness results of [Bai-Woo-3] tell us exactly what they are, namely Example 6.3.1 is a orthogonal projection, 6.3.2 the Hopf map and 6.3.3, 6.3.4 the maps discussed on p. 179 of [Bai-Woo-3]. Proposition 6.2.2 can be seen as a unifying framework for the above already well known harmonic morphisms. It is our hope that it will be a useful tool for further studies of harmonic morphisms between homogeneous spaces.
APPENDIX A.

In this appendix we mention a few basic facts about isoparametric systems, needed in chapter 3.

The concept of an isoparametric map generalizes that of an isoparametric function, first studied by Cartan in [Car-1-2]. The Definition A.1 used here was first given by Terng in [Ter] after Carter and West had studied the case of codimension 2 in [Car-Wes-1-2].

Definition A.1. If \((M, g) = S^m, \mathbb{R}^m\) or \(H^m\), then a map \(f = (f_1, \ldots, f_{m-n}) : M \to \mathbb{R}^{m-n}\) is called isoparametric if

1. \(f\) has a regular value,
2. the functions \(g(\text{grad}(f_i), \text{grad}(f_j))\) and \(\Delta f_k\) are constant along the inverse images of \(f\) for all \(i, j, k\), and
3. the vector field \([\text{grad}(f_i), \text{grad}(f_j)]\) is a linear combination of \(\text{grad}(f_1), \ldots, \text{grad}(f_{m-n})\) whose coefficients are constant along the inverse images of \(f\), for all \(i\) and \(j\).

The isoparametric foliation \(\mathcal{F}_f\) associated to \(f\) is the decomposition of the manifold \(M\) into the inverse images of \(f\). The foliation \(\mathcal{F}_f\) is \(n\)-dimensional with possible singularities, so called focal varieties. The inverse image of a regular value of an isoparametric map is called an isoparametric submanifold of \(M\). Isoparametric submanifolds of space forms have been characterised as follows:

Characterization A.2. Let \(L\) be a submanifold of a simply connected space form \((M, g)\). Then \(L\) is isoparametric if and only if

1. the normal bundle \(\nu(L)\) of \(L\) in \((M, g)\) is flat, and
2. the principal curvatures of \(L\) in the direction of any parallel normal field are constant.

Given one isoparametric submanifold \(L\) of \((M, g)\) the corresponding isoparametric foliation \(\mathcal{F}_f\) on \((M, g)\) is uniquely determined. Its regular leaves are simply the submanifolds of \((M, g)\), which are parallel to \(L\).

In the light of Proposition 3.3.2 we are mainly interested in the simplest possible isoparametric foliations, namely those which are totally umbilic. For totally umbilic submanifolds \(L^n\) of simply connected space forms \(S^m, \mathbb{R}^m\) or \(H^m\) there exists a
well known classification, see for example Spivak’s book [Spi]: $L^n$ has constant sectional curvature $K_L$ and is either totally geodesic or lies in an $(n+1)$-dimensional totally geodesic submanifold $\tilde{M}^{n+1}$ of $(M^m, g)$. Totally umbilic submanifolds $L^n$ of $(M^m, g)$ exist exactly in the cases when $(M^m, K_L) = (S^m, \geq 1), (\mathbb{R}^m, > 0), (H^m, > 0), (H^m, = 0), (H^m, \in [-1, 0])$ or $(\mathbb{R}^m, = 0)$. This means that up to isometries on $(\tilde{M}^{n+1}, g), L^n$ is an integral manifold of the horizontal distribution of exactly one of the standard maps $\pi_i : U_i \subset (\tilde{M}^{n+1}, g) \to (N^n, h), i = 1, 2, ..., 6$, given in section 1.3. Now choose the appropriate map $\hat{\pi}_i : \hat{U}_i \subset (M^m, g) \to (\tilde{M}^{n+1}, g)$ from

$$
\hat{\pi}_1 : S^m - S^m-(n+1) \xrightarrow{\pi_1} \cdots \xrightarrow{\pi_1} S^{n+1} - S^0,
$$

$$
\hat{\pi}_6 : \mathbb{R}^m \xrightarrow{\pi_6} \cdots \xrightarrow{\pi_6} \mathbb{R}^{n+1},
$$

$$
\hat{\pi}_5 : H^m \xrightarrow{\pi_5} \cdots \xrightarrow{\pi_5} H^{n+1},
$$

and put $\pi := \pi_i \circ \hat{\pi}_i : \hat{U}_i \subset (M^m, g) \to (N^n, h)$. Lifting $TN$ horizontally via $\pi$, gives an $n$-dimensional integrable distribution $\mathcal{H} := \pi^*TN$ on $\hat{U}_i$. The leaves of the corresponding foliation $\mathcal{F}_\mathcal{H}$ are exactly the $n$-dimensional submanifolds of $(M^m, g)$ parallel to $L^n$.

From what has just been said it is clear, that given $(M^m, g)$ and the sign of $K_L$, the constant sectional curvatures of the leaves, the totally umbilic isoparametric foliation $\mathcal{F}_\mathcal{H}$ is uniquely determined up to isometries of $(M^m, g)$.
LIST OF REFERENCES.


