

Lecture Notes in Mathematics

An Introduction to Gaussian Geometry

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Preface

These lecture notes grew out of a course on elementary differential geometry which I have given at Lund University for a number of years.

Their purpose is to introduce the beautiful theory of Gaussian geometry i.e. the theory of curves and surfaces in three dimensional Euclidean space. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work.

The text is written for students with a good understanding of linear algebra, real analysis of several variables and basic knowledge of the classical theory of ordinary differential equations and some topology.

I am grateful to my enthusiastic students who have contributed to the text by finding numerous typing errors and giving many helpful comments on the presentation.

Norra Nöbbelev the 1st of January 2017.

Sigmundur Gudmundsson

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CHAPTER 1

Introduction

Around 300 BC Euclid wrote "The Thirteen Books of the Elements". These were used as the basic text on geometry throughout the Western world for about 2000 years. Euclidean geometry is the theory one yields when assuming Euclid's five axioms, including the parallel postulate.

Gaussian geometry is the study of curves and surfaces in three dimensional Euclidean space. This theory was initiated by the ingenious Carl Friedrich Gauss (1777-1855) in his famous *Disquisitiones generales circa superficies curvas* from 1827.

The work of Gauss, János Bolyai (1802-1860) and Nikolai Ivanovich Lobachevsky (1792-1856) lead to their independent discovery of non-Euclidean geometry. This solved the best known mathematical problem ever and proved that the parallel postulate is indeed independent of the other four axioms that Euclid used for his theory.

Curves in the Euclidean Plane \mathbb{R}^2

In this chapter we study regular curves in the two dimensional Euclidean plane. We define their curvature and show that this determines the curves up to orientation preserving Euclidean motions. We then prove the isoperimetric inequality for plane curves.

Let the n -dimensional real vector space \mathbb{R}^n be equipped with its standard Euclidean **scalar product** $\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n$$

and induces the norm $|\cdot| : \mathbb{R}^n \rightarrow \mathbb{R}_0^+$ on \mathbb{R}^n with

$$|x| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Definition 2.1. A map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is said to be a **Euclidean motion** of \mathbb{R}^n if it is given by $\Phi : x \mapsto A \cdot x + b$ where $b \in \mathbb{R}^n$ and

$$A \in \mathbf{O}(n) = \{X \in \mathbb{R}^{n \times n} \mid X^t \cdot X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(n) = \{X \in \mathbf{O}(n) \mid \det X = 1\}.$$

Definition 2.2. A differentiable **parametrised curve** in \mathbb{R}^n is a C^1 -map $\gamma : I \rightarrow \mathbb{R}^n$ from an open interval I on the real line \mathbb{R} . The image $\gamma(I)$ in \mathbb{R}^n is the corresponding **geometric curve**. We say that the map $\gamma : I \rightarrow \mathbb{R}^n$ parametrises $\gamma(I)$. The derivative $\gamma'(t)$ is called the **tangent** of γ at the point $\gamma(t)$ and

$$L(\gamma) = \int_I |\gamma'(t)| dt \leq \infty$$

is the **arclength** of γ . The differentiable curve γ is said to be **regular** if $\gamma'(t) \neq 0$ for all $t \in I$.

Example 2.3. If p and q are two distinct points in \mathbb{R}^2 then the differentiable curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ with

$$\gamma : t \mapsto (1-t) \cdot p + t \cdot q$$

parametrises the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 2.4. If $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^2$ then the differentiable curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ with

$$\gamma : t \mapsto p + r \cdot (\cos t, \sin t)$$

parametrises the **circle** with **center** p and **radius** r . The arclength of the curve $\gamma|_{(0,2\pi)}$ satisfies

$$L(\gamma|_{(0,2\pi)}) = \int_0^{2\pi} |\gamma'(t)| dt = 2\pi r.$$

Definition 2.5. A regular curve $\gamma : I \rightarrow \mathbb{R}^n$ is said to **parametrise** $\gamma(I)$ **by arclength** if $|\dot{\gamma}(s)| = 1$ for all $s \in I$ i.e. the tangents $\dot{\gamma}(s)$ are elements of the **unit sphere** S^{n-1} in \mathbb{R}^n .

Theorem 2.6. Let $\gamma : I = (a, b) \rightarrow \mathbb{R}^n$ be a regular curve in \mathbb{R}^n . Then the image $\gamma(I)$ of γ can be parametrised by arclength.

PROOF. Define the arclength function $\sigma : (a, b) \rightarrow \mathbb{R}^+$ by

$$\sigma(t) = \int_a^t |\gamma'(u)| du.$$

Then $\sigma'(t) = |\gamma'(t)| > 0$ so σ is strictly increasing and

$$\sigma((a, b)) = (0, L(\gamma)).$$

Let $\tau : (0, L(\gamma)) \rightarrow (a, b)$ be the inverse of σ such that $\sigma(\tau(s)) = s$ for all $s \in (0, L(\gamma))$. By differentiating we get

$$\frac{d}{ds}(\sigma(\tau(s))) = \sigma'(\tau(s)) \cdot \dot{\tau}(s) = 1.$$

If we define the curve $\alpha : (0, L(\gamma)) \rightarrow \mathbb{R}^n$ by $\alpha = \gamma \circ \tau$ then the chain rule gives $\dot{\alpha}(s) = \gamma'(\tau(s)) \cdot \dot{\tau}(s)$. Hence

$$\begin{aligned} |\dot{\alpha}(s)| &= |\gamma'(\tau(s))| \cdot \dot{\tau}(s) \\ &= \sigma'(\tau(s)) \cdot \dot{\tau}(s) \\ &= 1. \end{aligned}$$

The function τ is bijective so α parametrises $\gamma(I)$ by arclength. \square

For a regular planar curve $\gamma : I \rightarrow \mathbb{R}^2$, parametrised by arclength, we define its **tangent** $T : I \rightarrow S^1$ along γ by

$$T(s) = \dot{\gamma}(s)$$

and its **normal** $N : I \rightarrow S^1$ with

$$N(s) = R \circ T(s).$$

Here $R : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear rotation by the angle $+\pi/2$ satisfying

$$R : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}.$$

It follows that for each $s \in I$ the set $\{T(s), N(s)\}$ is an orthonormal basis for \mathbb{R}^2 . It is called the **Frenet frame** along the curve.

Definition 2.7. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 -curve parametrised by arclength. Then we define its **curvature** $\kappa : I \rightarrow \mathbb{R}$ by

$$\kappa(s) = \langle \dot{T}(s), N(s) \rangle.$$

Note that the curvature is a measure of how fast the unit tangent $T(s) = \dot{\gamma}(s)$ is bending in the direction of the normal $N(s)$, or equivalently, out of the line generated by $T(s)$.

Theorem 2.8. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a C^2 -curve parametrised by arclength. Then the Frenet frame satisfies the following system of ordinary differential equations.

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) \\ -\kappa(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \end{bmatrix}.$$

PROOF. The curve $\gamma : I \rightarrow \mathbb{R}^2$ is parametrised by arclength so

$$2\langle \dot{T}(s), T(s) \rangle = \frac{d}{ds}(\langle T(s), T(s) \rangle) = 0$$

and

$$2\langle \dot{N}(s), N(s) \rangle = \frac{d}{ds}(\langle N(s), N(s) \rangle) = 0.$$

As a direct consequence we have

$$\dot{T}(s) = \langle \dot{T}(s), N(s) \rangle N(s) = \kappa(s)N(s)$$

$$\dot{N}(s) = \langle \dot{N}(s), T(s) \rangle T(s) = -\kappa(s)T(s),$$

since

$$\langle \dot{T}(s), N(s) \rangle + \langle T(s), \dot{N}(s) \rangle = \frac{d}{ds}(\langle T(s), N(s) \rangle) = 0.$$

□

Theorem 2.9. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a C^2 -curve parametrised by arclength. Then its curvature $\kappa : I \rightarrow \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.

PROOF. It follows from Theorem 2.8 that the curvature $\kappa(s)$ vanishes identically if and only if the tangent is constant i.e. there exist a unit vector $Z \in S^1$ and a point $p \in \mathbb{R}^2$ such that

$$\gamma(s) = p + s \cdot Z.$$

□

The following result tells us that a planar curve is, up to orientation preserving Euclidean motions, completely determined by its curvature.

Theorem 2.10. *Let $\kappa : I \rightarrow \mathbb{R}$ be a continuous function. Then there exists a C^2 -curve $\gamma : I \rightarrow \mathbb{R}^2$ parametrised by arclength with curvature κ . If $\tilde{\gamma} : I \rightarrow \mathbb{R}^2$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(2)$ and an element $b \in \mathbb{R}^2$ such that*

$$\tilde{\gamma}(s) = A \cdot \gamma(s) + b.$$

PROOF. See the proof of Theorem 3.11.

□

In differential geometry we are interested in properties of geometric objects which are independent of how these objects are parametrised. The curvature of a geometric curve should therefore not depend on its parametrisation.

Definition 2.11. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 -curve in \mathbb{R}^2 not necessarily parametrised by arclength. Let $t : J \rightarrow I$ be a strictly increasing C^2 -function such that the composition $\alpha = \gamma \circ t : J \rightarrow \mathbb{R}^2$ is a curve parametrised by arclength. Then we define the **curvature** $\kappa : I \rightarrow \mathbb{R}$ of $\gamma : I \rightarrow \mathbb{R}^2$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \rightarrow \mathbb{R}$ is the curvature of α .

Proposition 2.12. *Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 -curve in \mathbb{R}^2 . Then its curvature κ satisfies*

$$\kappa(t) = \frac{\det[\gamma'(t), \gamma''(t)]}{|\gamma'(t)|^3}.$$

PROOF. See Exercise 2.5.

□

Corollary 2.13. *Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^2 -curve in \mathbb{R}^2 . Then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 2.9 and Proposition 2.12.

□

We complete this chapter by proving the isoperimetric inequality. Let us first remind ourselves of the following topological facts.

Definition 2.14. A continuous map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ is said to parametrise a **closed curve** if it is periodic with period $L \in \mathbb{R}^+$. The image $\gamma(I)$ is said to be **simple** if the restriction

$$\gamma|_{[0,L)} : [0, L) \rightarrow \mathbb{R}^2$$

is injective.

The following result is called the **Jordan curve theorem**.

Fact 2.15. *Let the continuous map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ parametrise a simple closed curve. Then the subset $\mathbb{R}^2 \setminus \gamma(\mathbb{R})$ of the plane has exactly two connected components. The interior $\text{Int}(\gamma)$ of γ is bounded and the exterior $\text{Ext}(\gamma)$ is unbounded.*

Definition 2.16. A regular differentiable map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$, parametrising a simple closed curve, is said to be **positively oriented** if its normal

$$N(t) = R \cdot \gamma'(t) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \cdot \gamma'(t)$$

is an inner normal to the interior $\text{Int}(\gamma)$ for all $t \in \mathbb{R}$. It is said to be **negatively oriented** otherwise.

We are now ready for the **isoperimetric inequality**.

Theorem 2.17. *Let C be a regular simple closed curve in the plane with arclength L and let A be the area of the region enclosed by C . Then*

$$4\pi \cdot A \leq L^2,$$

with equality if and only if C is a circle.

PROOF. Let l_1 and l_2 be two parallel lines touching the curve C such that C is contained in the strip between them. Introduce a coordinate system in the plane such that l_1 and l_2 are orthogonal to the x -axis and given by

$$l_1 = \{(x, y) \in \mathbb{R}^2 \mid x = -r\} \quad \text{and} \quad l_2 = \{(x, y) \in \mathbb{R}^2 \mid x = r\}.$$

Let $\gamma = (x, y) : \mathbb{R} \rightarrow \mathbb{R}^2$ be a positively oriented curve parameterising C by arclength, such that $x(0) = r$ and $x(s_1) = -r$ for some $s_1 \in (0, L)$. Define the curve $\alpha : \mathbb{R} \rightarrow \mathbb{R}^2$ by $\alpha(s) = (x(s), \tilde{y}(s))$ where

$$\tilde{y}(s) = \begin{cases} +\sqrt{r^2 - x^2(s)} & \text{if } s \in [0, s_1), \\ -\sqrt{r^2 - x^2(s)} & \text{if } s \in [s_1, L). \end{cases}$$

Then this new curve parameterises the circle given by $x^2 + \tilde{y}^2 = r^2$.

As an immediate consequence of Lemma 2.18 we have

$$A = \int_0^L x(s) \cdot y'(s) ds \quad \text{and} \quad \pi \cdot r^2 = - \int_0^L \tilde{y}(s) \cdot x'(s) ds.$$

Employing the Cauchy-Schwartz inequality we then get

$$\begin{aligned} A + \pi \cdot r^2 &= \int_0^L (x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s)) ds \\ &\leq \int_0^L \sqrt{(x(s) \cdot y'(s) - \tilde{y}(s) \cdot x'(s))^2} ds \\ &\leq \int_0^L \sqrt{(x(s)^2 + \tilde{y}(s)^2) \cdot ((x'(s))^2 + (y'(s))^2)} ds \\ &= L \cdot r. \end{aligned}$$

The inequality

$$0 \leq (\sqrt{A} - r\sqrt{\pi})^2 = A - 2r\sqrt{A}\sqrt{\pi} + \pi r^2$$

implies that

$$2r\sqrt{A}\sqrt{\pi} \leq A + \pi r^2 \leq Lr$$

so

$$4A\pi r^2 \leq L^2 r^2$$

or equivalently

$$4\pi A \leq L^2.$$

It follows from our construction above that the positive real number r depends on the direction of the two parallel lines l_1 and l_2 chosen. In the case of equality $4\pi A = L^2$ we get $A = \pi r^2$. Since A is independent of the direction of the two lines, we see that so is r . This implies that in that case the curve C must be a circle. \square

The following result is a direct consequence of Green's theorem in the plane.

Lemma 2.18. *Let the regular, positively oriented map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ parameterise a simple closed curve in the plane. If A is the area of the interior $\text{Int}(\gamma)$ of γ then*

$$\begin{aligned} A &= \frac{1}{2} \int_{\gamma(\mathbb{R})} (x(t)y'(t) - y(t)x'(t)) dt \\ &= \int_{\gamma(\mathbb{R})} x(t)y'(t) dt \\ &= - \int_{\gamma(\mathbb{R})} x'(t)y(t) dt. \end{aligned}$$

Exercises

Exercise 2.1. A **cycloid** is a planar curve parametrised by a map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ of the form

$$\gamma(t) = r(t, 1) - r(\sin(-t), \cos(-t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.2. An **astroid** is a planar curve parametrised by a map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ of the form

$$\gamma(t) = (4r \cos^3 t, 4r \sin^3 t) = 3r(\cos t, \sin t) + r(\cos(-3t), \sin(-3t)),$$

where $r \in \mathbb{R}^+$. Describe the curve geometrically and calculate the arclength

$$\sigma(2\pi) = \int_0^{2\pi} |\gamma'(t)| dt.$$

Is the curve regular ?

Exercise 2.3. For $a \in \mathbb{R}^+$ let the curves $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^2$ be given by

$$\gamma_1(t) = r(\cos(at), \sin(at)) \quad \text{and} \quad \gamma_2(t) = r(\cos(-at), \sin(-at)).$$

Calculate the curvatures κ_1, κ_2 of γ_1 and γ_2 , respectively. Find a Euclidean motion $\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ of \mathbb{R}^2 such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 2.4. Let $\gamma : I \rightarrow \mathbb{R}^2$ be a regular C^3 -curve, parametrised by arclength, with Frenet frame $\{T(s), N(s)\}$. For $r \in \mathbb{R}$ we define the parallel curves $\gamma_r : I \rightarrow \mathbb{R}^2$ by

$$\gamma_r(t) = \gamma(t) + r \cdot N(t).$$

Calculate the curvature κ_r of those curves γ_r which are regular.

Exercise 2.5. Prove the curvature formula in Proposition 2.12.

Exercise 2.6. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ be the parametrised curve in \mathbb{R}^2 given by $\gamma(t) = (\sin t, \sin 2t)$. Is γ regular, closed and simple ?

Exercise 2.7. Let the positively oriented $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ parametrise a simple closed C^2 -curve by arclength. Show that if the period of γ is $L \in \mathbb{R}^+$ then the total curvature satisfies

$$\int_0^L \kappa(s) ds = 2\pi.$$

Curves in the Euclidean Space \mathbb{R}^3

In this chapter we study regular curves in the three dimensional Euclidean space. We define their curvature and torsion and show that these determine the curves up to orientation preserving Euclidean motions.

Let the 3-dimensional real vector space \mathbb{R}^3 be equipped with its standard Euclidean **scalar product** $\langle \cdot, \cdot \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}$. This is given by

$$\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3$$

and induces the norm $|\cdot| : \mathbb{R}^3 \rightarrow \mathbb{R}_0^+$ on \mathbb{R}^3 with

$$|x| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Further we equip the three dimensional real vector space \mathbb{R}^3 with the standard **cross product** $\times : \mathbb{R}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfying

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1).$$

Definition 3.1. A map $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is said to be a **Euclidean motion** of \mathbb{R}^3 if it is given by $\Phi : x \mapsto A \cdot x + b$ where $b \in \mathbb{R}^3$ and

$$A \in \mathbf{O}(3) = \{X \in \mathbb{R}^{3 \times 3} \mid X^t \cdot X = I\}.$$

A Euclidean motion Φ is said to be **rigid** or **orientation preserving** if

$$A \in \mathbf{SO}(3) = \{X \in \mathbf{O}(3) \mid \det X = 1\}.$$

Example 3.2. If p and q are two distinct points in \mathbb{R}^3 then the differentiable map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$\gamma : t \mapsto (1-t) \cdot p + t \cdot q$$

parametrises the **straight line** through $p = \gamma(0)$ and $q = \gamma(1)$.

Example 3.3. Let $\{Z, W\}$ be an orthonormal basis for a two dimensional subspace V of \mathbb{R}^3 , $r \in \mathbb{R}^+$ and $p \in \mathbb{R}^3$. Then then the differentiable map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$\gamma : t \mapsto p + r \cdot (\cos t \cdot Z + \sin t \cdot W)$$

parametrises a **circle** in the affine 2-plane $p + V$ with **center** p and **radius** r .

Example 3.4. If $r, b \in \mathbb{R}^+$ then the differentiable map $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$\gamma = (x, y, z) : t \mapsto (r \cdot \cos(t), r \cdot \sin(t), bt)$$

parametrises a **helix**. It is easy to see that $x^2 + y^2 = r^2$ so the image $\gamma(\mathbb{R})$ is contained in the **circular cylinder**

$$\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = r^2\}$$

of radius r .

Definition 3.5. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^2 -curve parametrised by arclength. Then the **curvature** $\kappa : I \rightarrow \mathbb{R}_0^+$ of γ is defined by

$$\kappa(s) = |\ddot{\gamma}(s)|.$$

Theorem 3.6. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^2 -curve parametrised by arclength. Then its curvature $\kappa : I \rightarrow \mathbb{R}_0^+$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a line.

PROOF. The curvature $\kappa(s) = |\ddot{\gamma}(s)|$ vanishes identically if and only if there exist a unit vector $Z \in S^2$ and a point $p \in \mathbb{R}^3$ such that

$$\gamma(s) = p + s \cdot Z$$

i.e. the geometric curve $\gamma(I)$ is contained in a straight line. \square

Definition 3.7. A regular C^3 -curve $\gamma : I \rightarrow \mathbb{R}^3$, parametrised by arclength, is said to be a **Frenet curve** if its curvature κ is non-vanishing i.e. $\kappa(s) \neq 0$ for all $s \in I$.

For a Frenet curve $\gamma : I \rightarrow \mathbb{R}^3$ we define its **tangent** $T : I \rightarrow S^2$ along γ by

$$T(s) = \dot{\gamma}(s),$$

the **principal normal** $N : I \rightarrow S^2$ with

$$N(s) = \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|} = \frac{\ddot{\gamma}(s)}{\kappa(s)}$$

and its **binormal** $B : I \rightarrow S^2$ as the cross product

$$B(s) = T(s) \times N(s).$$

The Frenet curve $\gamma : I \rightarrow \mathbb{R}^3$ is parametrised by arclength so

$$0 = \frac{d}{ds} \langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle = 2 \langle \ddot{\gamma}(s), \dot{\gamma}(s) \rangle.$$

This means that for each $s \in I$ the set $\{T(s), N(s), B(s)\}$ is an orthonormal basis for \mathbb{R}^3 . It is called the **Frenet frame** along the curve.

Definition 3.8. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a Frenet curve. Then we define the **torsion** $\tau : I \rightarrow \mathbb{R}$ of γ by

$$\tau(s) = \langle \dot{N}(s), B(s) \rangle.$$

Note that the torsion is a measure of how fast the principal normal $N(s) = \ddot{\gamma}(s)/|\ddot{\gamma}(s)|$ is bending in the direction of the binormal $B(s)$, or equivalently, out of the plane generated by $T(s)$ and $N(s)$.

Theorem 3.9. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a Frenet curve. Then its Frenet frame satisfies the following system of ordinary differential equations

$$\begin{bmatrix} \dot{T}(s) \\ \dot{N}(s) \\ \dot{B}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \cdot \begin{bmatrix} T(s) \\ N(s) \\ B(s) \end{bmatrix}.$$

PROOF. The first equation is a direct consequence of the definition of the curvature

$$\dot{T}(s) = \ddot{\gamma}(s) = |\ddot{\gamma}(s)| \cdot N = \kappa(s) \cdot N(s).$$

We get the second equation from

$$\begin{aligned} \langle \dot{N}(s), T(s) \rangle &= \frac{d}{ds} \langle N(s), T(s) \rangle - \langle N(s), \dot{T}(s) \rangle \\ &= -\left\langle \frac{\ddot{\gamma}(s)}{|\ddot{\gamma}(s)|}, \ddot{\gamma}(s) \right\rangle \\ &= -\kappa(s), \end{aligned}$$

$$2\langle \dot{N}(s), N(s) \rangle = \frac{d}{ds} \langle N(s), N(s) \rangle = 0$$

and

$$\langle \dot{N}(s), B(s) \rangle = \frac{d}{ds} \langle N(s), B(s) \rangle - \langle N(s), \dot{B}(s) \rangle = \tau(s).$$

When differentiating $B(s) = T(s) \times N(s)$ we yield

$$\begin{aligned} \dot{B}(s) &= \dot{T}(s) \times N(s) + T(s) \times \dot{N}(s) \\ &= \kappa(s) \cdot N(s) \times N(s) + T(s) \times \dot{N}(s) \\ &= T(s) \times \dot{N}(s), \end{aligned}$$

hence $\langle \dot{B}(s), T(s) \rangle = 0$. The definition of the torsion

$$\langle \dot{B}(s), N(s) \rangle = -\langle B(s), \dot{N}(s) \rangle = -\tau(s)$$

and the fact

$$2\langle \dot{B}(s), B(s) \rangle = \frac{d}{ds} \langle B(s), B(s) \rangle = 0$$

provide us the third and last equation. \square

Theorem 3.10. *Let $\gamma : I \rightarrow \mathbb{R}^3$ be a Frenet curve. Then its torsion $\tau : I \rightarrow \mathbb{R}$ vanishes identically if and only if the geometric curve $\gamma(I)$ is contained in a plane.*

PROOF. It follows from the third Frenet equation that if the torsion vanishes identically then

$$\frac{d}{ds} \langle \gamma(s) - \gamma(0), B(s) \rangle = \langle T(s), B(s) \rangle = 0.$$

Because $\langle \gamma(0) - \gamma(0), B(0) \rangle = 0$ it follows that $\langle \gamma(s) - \gamma(0), B(s) \rangle = 0$ for all $s \in I$. This means that $\gamma(s)$ lies in a plane containing $\gamma(0)$ with constant normal $B(s)$.

Let us now assume that the geometric curve $\gamma(I)$ is contained in a plane i.e. there exists a point $p \in \mathbb{R}^3$ and a normal $n \in \mathbb{R}^3 \setminus \{0\}$ to the plane such that

$$\langle \gamma(s) - p, n \rangle = 0$$

for all $s \in I$. When differentiating we get

$$\langle T(s), n \rangle = \langle \dot{\gamma}(s), n \rangle = 0$$

and

$$\langle \ddot{\gamma}(s), n \rangle = 0$$

so $\langle N(s), n \rangle = 0$. This means that $B(s)$ is a constant multiple of n , so $\dot{B}(s) = 0$ and hence $\tau \equiv 0$. \square

The next result is called the **fundamental theorem of curve theory**. It tells us that a Frenet curve is, up to orientation preserving Euclidean motions, completely determined by its curvature and torsion.

Theorem 3.11. *Let $\kappa : I \rightarrow \mathbb{R}^+$ and $\tau : I \rightarrow \mathbb{R}$ be two continuous functions. Then there exists a Frenet curve $\gamma : I \rightarrow \mathbb{R}^3$ with curvature κ and torsion τ . If $\tilde{\gamma} : I \rightarrow \mathbb{R}^3$ is another such curve, then there exists a matrix $A \in \mathbf{SO}(3)$ and an element $b \in \mathbb{R}^3$ such that*

$$\tilde{\gamma}(s) = A \cdot \gamma(s) + b.$$

PROOF. The proof is based on Theorem 3.9 and a well-known result of **Picard-Lindelöf** formulated here as Fact 3.12, see Exercise 3.6. \square

Fact 3.12. *Let $f : U \rightarrow \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, x) - f(t, y)| \leq L \cdot |x - y|$$

for all $(t, x), (t, y) \in U$. If $(t_0, x_0) \in U$ then there exists a unique local solution $x : I \rightarrow \mathbb{R}^n$ to the following initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

In differential geometry we are interested in properties of geometric objects which are independent of how these objects are parametrised. The curvature and the torsion of a geometric curve should therefore not depend on its parametrisation.

Definition 3.13. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^2 -curve in \mathbb{R}^3 not necessarily parametrised by arclength. Let $t : J \rightarrow I$ be a strictly increasing C^2 -function such that the composition $\alpha = \gamma \circ t : J \rightarrow \mathbb{R}^3$ is a curve parametrised by arclength. Then we define the **curvature** $\kappa : I \rightarrow \mathbb{R}^+$ of $\gamma : I \rightarrow \mathbb{R}^3$ by

$$\kappa(t(s)) = \tilde{\kappa}(s),$$

where $\tilde{\kappa} : J \rightarrow \mathbb{R}^+$ is the curvature of α . If further $\gamma : I \rightarrow \mathbb{R}^3$ is a regular C^3 -curve with non-vanishing curvature and $t : J \rightarrow I$ is C^3 , then we define the **torsion** $\tau : I \rightarrow \mathbb{R}$ of γ by

$$\tau(t(s)) = \tilde{\tau}(s),$$

where $\tilde{\tau} : J \rightarrow \mathbb{R}$ is the torsion of α .

We are now interested in deriving formulae for the curvature κ and the torsion τ in terms of γ , under the above mentioned conditions.

Proposition 3.14. *If $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^2 -curve in \mathbb{R}^3 then its curvature satisfies*

$$\kappa(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}.$$

PROOF. By differentiating $\gamma(t) = \alpha(s(t))$ we get

$$\gamma'(t) = \dot{\alpha}(s(t)) \cdot s'(t),$$

$$\langle \gamma'(t), \gamma'(t) \rangle = s'(t)^2 \langle \dot{\alpha}(s(t)), \dot{\alpha}(s(t)) \rangle = s'(t)^2$$

and

$$2 \langle \gamma''(t), \gamma'(t) \rangle = \frac{d}{dt}(s'(t)^2) = 2 \cdot s'(t) \cdot s''(t).$$

When differentiating once more we get

$$s'(t) \cdot \ddot{\alpha}(s(t)) = \frac{s'(t) \cdot \gamma''(t) - s''(t) \cdot \gamma'(t)}{s'(t)^2}$$

and

$$\begin{aligned} \ddot{\alpha}(s(t)) &= \frac{s'(t)^2 \cdot \gamma''(t) - s'(t) \cdot s''(t) \cdot \gamma'(t)}{s'(t)^4} \\ &= \frac{\gamma''(t) \langle \gamma'(t), \gamma'(t) \rangle - \gamma'(t) \langle \gamma''(t), \gamma'(t) \rangle}{|\gamma'(t)|^4} \end{aligned}$$

$$= \frac{\gamma'(t) \times (\gamma''(t) \times \gamma'(t))}{|\gamma'(t)|^4}.$$

Finally we get a formula for the curvature of $\gamma : I \rightarrow \mathbb{R}^3$ by

$$\begin{aligned} \kappa(t) &= \tilde{\kappa}(s(t)) \\ &= |\ddot{\alpha}(s(t))| \\ &= \frac{|\gamma'(t)| \cdot |\gamma''(t) \times \gamma'(t)|}{|\gamma'(t)|^4} \\ &= \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}. \end{aligned}$$

□

Corollary 3.15. *If $\gamma : I \rightarrow \mathbb{R}^3$ is a regular C^2 -curve in \mathbb{R}^3 then the geometric curve $\gamma(I)$ is contained in a line if and only if $\gamma'(t)$ and $\gamma''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.6 and Proposition 3.14. □

Proposition 3.16. *Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^3 -curve with non-vanishing curvature. Then its torsion τ satisfies*

$$\tau(t) = \frac{\det[\gamma'(t), \gamma''(t), \gamma'''(t)]}{|\gamma'(t) \times \gamma''(t)|^2}.$$

PROOF. See Exercise 3.5. □

Corollary 3.17. *Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular C^3 -curve with non-vanishing curvature. Then the geometric curve $\gamma(I)$ is contained in a plane if and only if $\gamma'(t)$, $\gamma''(t)$ and $\gamma'''(t)$ are linearly dependent for all $t \in I$.*

PROOF. The statement is a direct consequence of Theorem 3.10 and Proposition 3.16. □

Exercises

Exercise 3.1. For $r, a, b \in \mathbb{R}^+$ parametrise the helices $\gamma_1, \gamma_2 : \mathbb{R} \rightarrow \mathbb{R}^3$ by

$$\begin{aligned}\gamma_1 : t &\mapsto (r \cdot \cos(at), r \cdot \sin(at), abt), \\ \gamma_2 : t &\mapsto (r \cdot \cos(-at), r \cdot \sin(-at), abt).\end{aligned}$$

Calculate their curvatures κ_1, κ_2 and torsions τ_1, τ_2 , respectively. Find a Euclidean motion $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of \mathbb{R}^3 such that $\gamma_2 = \Phi \circ \gamma_1$. Is Φ orientation preserving ?

Exercise 3.2. For any $\kappa \in \mathbb{R}^+$ and $\tau \in \mathbb{R}$ construct a Frenet curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ with constant curvature κ and constant torsion τ .

Exercise 3.3. Prove that the curve $\gamma : (-\pi/2, \pi/2) \rightarrow \mathbb{R}^3$ with

$$\gamma : t \mapsto (2 \cos^2 t - 3, \sin t - 8, 3 \sin^2 t + 4)$$

is regular. Determine whether the image of γ is contained in

- ii) a straight line in \mathbb{R}^3 or not,
- i) a plane in \mathbb{R}^3 or not.

Exercise 3.4. Show that the curve $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ given by

$$\gamma(t) = (t^3 + t^2 + 3, t^3 - t + 1, t^2 + t + 1)$$

is regular. Determine whether the image of γ is contained in

- ii) a straight line in \mathbb{R}^3 or not,
- i) a plane in \mathbb{R}^3 or not.

Exercise 3.5. Prove the torsion formula in Proposition 3.16.

Exercise 3.6. Use your local library to find a proof of Theorem 3.11.

Exercise 3.7. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular C^2 -map parametrising a closed curve in \mathbb{R}^3 by arclength. Use your local library to find a proof of Fenchel's theorem i.e.

$$L(\dot{\gamma}) = \int_0^P \kappa(s) ds \geq 2\pi,$$

where P is the period of γ .

Surfaces in the Euclidean Space \mathbb{R}^3

In this chapter we introduce the notion of a regular surface in the three dimensional Euclidean space. We give several examples of regular surfaces and study differentiable maps between them. We then define the tangent space at a point and show that this is a two dimensional vector space. Further we introduce the first fundamental form which enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.1. A non-empty subset M of \mathbb{R}^3 is said to be a **regular surface** if for each point $p \in M$ there exist open, connected and simply connected neighbourhoods U in \mathbb{R}^2 , V in \mathbb{R}^3 with $p \in V$ and a bijective C^1 -map $X : U \rightarrow V \cap M$, such that X is a homeomorphism and

$$X_u(q) \times X_v(q) \neq 0$$

for all $q \in U$. The map $X : U \rightarrow V \cap M$ is said to be a **local parametrisation** of M and the inverse $X^{-1} : V \cap M \rightarrow U$ a **local chart** or **local coordinates** on M . An **atlas** on M is a collection

$$\mathcal{A} = \{(V_\alpha \cap M, X_\alpha^{-1}) \mid \alpha \in I\}$$

of local charts on M such that \mathcal{A} covers the whole of M i.e.

$$M = \bigcup_{\alpha} (V_\alpha \cap M).$$

Example 4.2. Let $f : U \rightarrow \mathbb{R}$ be a C^1 -function from an open subset U of \mathbb{R}^2 . Then $X : U \rightarrow M$ with

$$X : (u, v) \mapsto (u, v, f(u, v))$$

is a local parametrisation of the graph

$$M = \{(u, v, f(u, v)) \mid (u, v) \in U\}$$

of f . The corresponding local chart $X^{-1} : M \rightarrow U$ is given by

$$X^{-1} : (x, y, z) \mapsto (x, y).$$

Example 4.3. Let S^2 denote the unit sphere in \mathbb{R}^3 given by

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

Let $N = (0, 0, 1)$ be the north pole and $S = (0, 0, -1)$ be the south pole of S^2 , respectively. Put $U_N = S^2 \setminus \{N\}$, $U_S = S^2 \setminus \{S\}$ and define the stereographic projection from the north pole $\sigma_N : U_N \rightarrow \mathbb{R}^2$ by

$$\sigma_N : (x, y, z) \mapsto \frac{1}{1-z}(x, y)$$

and the stereographic projection from the south pole $\sigma_S : U_S \rightarrow \mathbb{R}^2$

$$\sigma_S : (x, y, z) \mapsto \frac{1}{1+z}(x, y).$$

Then $\mathcal{A} = \{(U_N, \sigma_N), (U_S, \sigma_S)\}$ is an atlas on S^2 . The inverses

$$X_N = \sigma_N^{-1} : \mathbb{R}^2 \rightarrow U_N \quad \text{and} \quad X_S = \sigma_S^{-1} : \mathbb{R}^2 \rightarrow U_S$$

are local parametrisations of the sphere S^2 given by

$$X_N : (u, v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, u^2+v^2-1),$$

$$X_S : (u, v) \mapsto \frac{1}{1+u^2+v^2}(2u, 2v, 1-u^2-v^2).$$

Our next step is to prove the implicit function theorem which is a useful tool for constructing surfaces in \mathbb{R}^3 . For this we use the classical inverse mapping theorem stated below. Remember that if $F : U \rightarrow \mathbb{R}^m$ is a C^1 -map defined on an open subset U of \mathbb{R}^n then its differential

$$dF(p) : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

at a point $p \in U$ is a linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1 / \partial x_1(p) & \dots & \partial F_1 / \partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m / \partial x_1(p) & \dots & \partial F_m / \partial x_n(p) \end{bmatrix}.$$

The **classical inverse mapping theorem** can be formulated as follows.

Theorem 4.4. *Let r be a positive integer, U be an open subset of \mathbb{R}^n and $F : U \rightarrow \mathbb{R}^n$ be a C^r -map. If $p \in U$ and the differential*

$$dF(p) : \mathbb{R}^n \rightarrow \mathbb{R}^n$$

of F at p is invertible then there exist open neighbourhoods U_p around p and U_q around $q = F(p)$ such that $f = F|_{U_p} : U_p \rightarrow U_q$ is bijective and the inverse $f^{-1} : U_q \rightarrow U_p$ is a C^r -map. The differential $df^{-1}(q)$ of f^{-1} at q satisfies

$$df^{-1}(q) = (dF(p))^{-1}$$

i.e. it is the inverse of the differential $dF(p)$ of F at p .

Before stating the implicit function theorem we remind the reader of the following notions.

Definition 4.5. Let m, n be positive integers, U be an open subset of \mathbb{R}^n and $F : U \rightarrow \mathbb{R}^m$ be a C^1 -map. A point $p \in U$ is said to be **critical** for F if the differential

$$dF(p) : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

is not of full rank, and **regular** if it is not critical. A point $q \in F(U)$ is said to be a **regular value** of F if every point of the pre-image $F^{-1}(\{q\})$ of q is regular and a **critical value** otherwise.

Note that if $m \leq n$ then $p \in U$ is a regular point of

$$F = (F_1, \dots, F_m) : U \rightarrow \mathbb{R}^m$$

if and only if the gradients $\nabla F_1, \dots, \nabla F_m$ of the coordinate functions $F_1, \dots, F_m : U \rightarrow \mathbb{R}$ are linearly independent at p , or equivalently, the differential $dF(p)$ of F at p satisfies the following condition

$$\det(dF(p) \cdot dF(p)^t) \neq 0.$$

In differential geometry, the following important result is often called the **implicit function theorem**.

Theorem 4.6. Let $f : U \rightarrow \mathbb{R}$ be a C^1 -function defined on an open subset U of \mathbb{R}^3 and q be a regular value of f *i.e.*

$$(\nabla f)(p) \neq 0$$

for all p in $M = f^{-1}(\{q\})$. Then M is a regular surface in \mathbb{R}^3 .

PROOF. Let p be an arbitrary element of M . The gradient $\nabla f(p)$ at p is non-zero so we can, without loss of generality, assume that $f_z(p) \neq 0$. Then define the map $F : U \rightarrow \mathbb{R}^3$ by

$$F(x, y, z) \mapsto (x, y, f(x, y, z)).$$

Its differential $dF(p)$ at p satisfies

$$dF(p) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & f_z \end{bmatrix},$$

so the determinant $\det dF(p) = f_z$ is non-zero. Following the inverse mapping theorem there exist open neighbourhoods V around p and W around $F(p)$ such that the restriction $F|_V : V \rightarrow W$ of F to V is invertible. The inverse $(F|_V)^{-1} : W \rightarrow V$ is differentiable of the form

$$(u, v, w) \mapsto (u, v, g(u, v, w)),$$

where g is a real-valued function on W . It follows that the restriction

$$X = F^{-1}|_{\hat{W}} : \hat{W} \rightarrow \mathbb{R}^3$$

to the planar set

$$\hat{W} = \{(u, v, w) \in W \mid w = q\}$$

is differentiable. Since $X_u \times X_v \neq 0$ we see that $X : \hat{W} \rightarrow V \cap M$ is a local parametrisation of the open neighbourhood $V \cap M$ around p . Since p was chosen arbitrarily we have shown that M is a regular surface in \mathbb{R}^3 . \square

We shall now apply the implicit function theorem to construct examples of regular surfaces in \mathbb{R}^3 .

Example 4.7. Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ be the differentiable function given by

$$f(x, y, z) = x^2 + y^2 + z^2.$$

The gradient $\nabla f(p)$ of f at p satisfies $\nabla f(p) = 2p$, so each positive real number is a regular value for f . This means that the **sphere**

$$S_r^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\} = f^{-1}(\{r^2\})$$

of radius r is a regular surface in \mathbb{R}^3 .

Example 4.8. Let r, R be real numbers such that $0 < r < R$ and define the differentiable function

$$f : U = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \neq 0\} \rightarrow \mathbb{R}$$

by

$$f(x, y, z) = z^2 + (\sqrt{x^2 + y^2} - R)^2$$

and let T^2 be the pre-image

$$f^{-1}(\{r^2\}) = \{(x, y, z) \in U \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

The gradient ∇f of f at $p = (x, y, z)$ satisfies

$$\nabla f(p) = \frac{2}{\sqrt{x^2 + y^2}}(x(\sqrt{x^2 + y^2} - R), y(\sqrt{x^2 + y^2} - R), z\sqrt{x^2 + y^2}).$$

If $p \in T^2$ and $\nabla f(p) = 0$ then $z = 0$ and

$$\nabla f(p) = \frac{2r}{\sqrt{x^2 + y^2}}(x, y, 0) \neq 0.$$

This contradiction shows that r^2 is a regular value for f and that the **torus** T^2 is a regular surface in \mathbb{R}^3 .

We now introduce the useful notion of a regular parametrised surface in \mathbb{R}^3 . This is a map and should not be confused with a regular surface as a subset of \mathbb{R}^3 , introduced in Definition 4.1.

Definition 4.9. A differentiable map $X : U \rightarrow \mathbb{R}^3$ from an open subset U of \mathbb{R}^2 is said to be a **regular parametrised surface** in \mathbb{R}^3 if for each point $q \in U$

$$X_u(q) \times X_v(q) \neq 0.$$

Definition 4.10. Let M be a regular surface in \mathbb{R}^3 . A differentiable map $X : U \rightarrow M$ defined on an open subset of \mathbb{R}^2 is said to **parametrise** M if X is surjective and for each p in U there exists an open neighbourhood U_p of p such that $X|_{U_p} : U_p \rightarrow X(U_p)$ is a local parametrisation of M .

Example 4.11. We know that for $0 < r < R$ the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

is a regular surface in \mathbb{R}^3 . It is easily seen that T^2 is obtained by rotating the circle

$$\{(x, 0, z) \in \mathbb{R}^3 \mid z^2 + (x - R)^2 = r^2\}$$

in the (x, z) -plane around the z -axis. This rotation naturally induces the regular parametrised surface $X : \mathbb{R}^2 \rightarrow T^2$ with

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r \cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

This parametrises the regular surface T^2 as a subset of \mathbb{R}^3 .

Example 4.12. Let $\gamma = (r, 0, z) : I \rightarrow \mathbb{R}^3$ be differentiable curve in the (x, z) -plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. By rotating the curve around the z -axis we obtain a regular **surface of revolution** parametrised by $X : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix}.$$

The surface is regular because the vectors

$$X_u = \begin{bmatrix} \dot{r}(u) \cos v \\ \dot{r}(u) \sin v \\ \dot{z}(u) \end{bmatrix}, \quad X_v = \begin{bmatrix} -r(u) \sin v \\ r(u) \cos v \\ 0 \end{bmatrix}$$

are not only linearly independent but even orthogonal.

We now introduce the tangent space at a point of a regular surface and show that this is a two dimensional vector space.

Definition 4.13. Let M be a regular surface in \mathbb{R}^3 . A continuous map $\gamma : I \rightarrow M$, defined on an open interval I of the real line, is said to be a **differentiable curve** on M if it is differentiable as a map into \mathbb{R}^3 .

Definition 4.14. Let M be a regular surface in \mathbb{R}^3 and p be a point on M . Then the **tangent space** $T_p M$ of M at p is the set of all tangents $\dot{\gamma}(0)$ to differentiable curves $\gamma : I \rightarrow M$ such that $\gamma(0) = p$.

Proposition 4.15. Let M be a regular surface in \mathbb{R}^3 and p be a point on M . Then the tangent space $T_p M$ of M at p is a 2-dimensional real vector space.

PROOF. Let M be a regular surface in \mathbb{R}^3 , $p \in M$ and $X : U \rightarrow M$ be a local parametrisation of M such that $0 \in U$ and $X(0) = p$. Let $\alpha : I \rightarrow U$ be a differentiable curve in U such that $0 \in I$ and $\alpha(0) = 0 \in U$. Then the composition $\gamma = X \circ \alpha : I \rightarrow X(U)$ is a differentiable curve in $X(U)$ such that $\gamma(0) = p$. Since $X : U \rightarrow X(U)$ is a homeomorphism it is clear that any curve in $X(U)$ with $\gamma(0) = p$ can be obtained this way.

It follows from the chain rule that the tangent $\gamma'(0)$ of $\gamma : I \rightarrow M$ at p satisfies

$$\gamma'(0) = dX(0) \cdot \alpha'(0),$$

where $dX(0) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the differential of the local parametrisation $X : U \rightarrow M$. The differential is a linear map and the condition

$$X_u \times X_v \neq 0$$

implies that $dX(0)$ is of full rank i.e. the vectors

$$X_u(0) = dX(0) \cdot e_1 \quad \text{and} \quad X_v(0) = dX(0) \cdot e_2$$

are linearly independent. This shows that the image

$$\{dX(0) \cdot Z \mid Z \in \mathbb{R}^2\}$$

of $dX(0)$ is a two dimensional subspace of \mathbb{R}^3 . If $(a, b) \in \mathbb{R}^2$ then

$$\begin{aligned} dX(0) \cdot (a, b) &= dX(0) \cdot (ae_1 + be_2) \\ &= a dX(0) \cdot e_1 + b dX(0) \cdot e_2 \\ &= aX_u(0) + bX_v(0). \end{aligned}$$

It is clear that $T_p M$ is the space of all tangents $\gamma'(0)$ to differentiable curves $\gamma : I \rightarrow M$ in M such that $\gamma(0) = p$. \square

Example 4.16. Let $\gamma : I \rightarrow S^2$ be a curve into the unit sphere in \mathbb{R}^3 with $\gamma(0) = p$ and $\gamma'(0) = Z$. The curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \gamma'(t), \gamma(t) \rangle + \langle \gamma(t), \gamma'(t) \rangle = 0.$$

This means that $\langle Z, p \rangle = 0$ so every tangent vector $Z \in T_p S^m$ must be orthogonal to p . On the other hand if $Z \neq 0$ satisfies $\langle Z, p \rangle = 0$ then $\gamma : \mathbb{R} \rightarrow S^2$ with

$$\gamma : t \mapsto \cos(t|Z|) \cdot p + \sin(t|Z|) \cdot Z/|Z|$$

is a curve into S^2 with $\gamma(0) = p$ and $\gamma'(0) = Z$. This shows that the tangent plane $T_p S^2$ is given by

$$T_p S^2 = \{Z \in \mathbb{R}^3 \mid \langle p, Z \rangle = 0\}.$$

Example 4.17. For $0 < r < R$ let us parametrise the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

by $X : \mathbb{R}^2 \rightarrow T^2$ with

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r \cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

By differentiating we get a basis $\{X_v, X_u\}$ for the tangent plane $T_p T^2$ at $p = X(u, v)$ with

$$X_u = r \begin{bmatrix} -\sin u \cos v \\ -\sin u \sin v \\ \cos u \end{bmatrix}, \quad X_v = (R + r \cos u) \begin{bmatrix} -\sin v \\ \cos v \\ 0 \end{bmatrix}.$$

Definition 4.18. Let M be a regular surface in \mathbb{R}^3 . A real-valued function $f : M \rightarrow \mathbb{R}$ on M is said to be **differentiable** if for each local parametrisation $X : U \rightarrow M$ of M the composition $f \circ X : U \rightarrow \mathbb{R}$ is differentiable.

Example 4.19. For $0 < r < R$ let $f : T^2 \rightarrow \mathbb{R}$ be the real-valued function on the torus $T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$ given by

$$f : (x, y, z) \mapsto x.$$

For the natural parametrisation $X : \mathbb{R}^2 \rightarrow T^2$ with

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r \cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

of T^2 we see that $f \circ X : \mathbb{R}^2 \rightarrow \mathbb{R}$ is given by

$$f \circ X : (u, v) \mapsto (R + r \cos u) \cos v.$$

This is clearly differentiable.

Definition 4.20. A map $\phi : M_1 \rightarrow M_2$ between two regular surfaces in \mathbb{R}^3 is said to be **differentiable** if for all local parametrisations (U, X) of M_1 and (V, Y) of M_2 the map

$$Y^{-1} \circ \phi \circ X|_W : W \rightarrow \mathbb{R}^2,$$

defined on the open subset $W = X^{-1}(X(U) \cap \phi^{-1}(Y(V)))$ of \mathbb{R}^2 , is differentiable.

The next very useful proposition generalises a result from classical real analysis of several variables.

Proposition 4.21. *Let M_1 and M_2 be two regular surfaces in \mathbb{R}^3 . Let $\phi : U \rightarrow \mathbb{R}^3$ be a differentiable map defined on an open subset of \mathbb{R}^3 such that M_1 is contained in U and the image $\phi(M_1)$ is contained in M_2 . Then the restriction $\phi|_{M_1} : M_1 \rightarrow M_2$ is differentiable map from M_1 to M_2 .*

PROOF. See Exercise 4.2. □

Example 4.22. For $0 < r < R$ we have earlier parametrised the torus

$$T^2 = \{(x, y, z) \in \mathbb{R}^3 \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}$$

with the map $X : \mathbb{R}^2 \rightarrow T^2$ defined by

$$X : (u, v) \mapsto \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r \cos u \\ 0 \\ r \sin u \end{bmatrix}.$$

We can now map the torus T^2 into \mathbb{R}^3 with the following formula

$$N : \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} R + r \cos u \\ 0 \\ r \sin u \end{bmatrix} \mapsto \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}.$$

Then it is easy to see that this gives a well-defined map $N : T^2 \rightarrow S^2$ from the torus to the unit sphere

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}.$$

In the local coordinates (u, v) on the torus the map N is given by

$$N(u, v) = \begin{bmatrix} \cos u \cos v \\ \cos u \sin v \\ \sin u \end{bmatrix}.$$

It now follows from Proposition 4.21 that $N : T^2 \rightarrow S^2$ is differentiable.

Proposition 4.23. *Let $\phi_1 : M_1 \rightarrow M_2$ and $\phi_2 : M_2 \rightarrow M_3$ be differentiable maps between regular surfaces in \mathbb{R}^3 . Then the composition $\phi_2 \circ \phi_1 : M_1 \rightarrow M_3$ is differentiable.*

PROOF. See Exercise 4.4. \square

Definition 4.24. Two regular surfaces M_1 and M_2 in \mathbb{R}^3 are said to be **diffeomorphic** if there exists a bijective differentiable map $\phi : M_1 \rightarrow M_2$ such that the inverse $\phi^{-1} : M_2 \rightarrow M_1$ is differentiable. In that case the map ϕ is said to be a **diffeomorphism** between M_1 and M_2 .

Proposition 4.25. *Let M be a regular surface in \mathbb{R}^3 and $X : U \subset \mathbb{R}^2 \rightarrow M$ be a local parametrisation of M . Then the bijective continuous map $X : U \rightarrow X(U) \subset M$ is a diffeomorphism.*

PROOF. See Exercise 4.5. \square

Remember that if $F : U \rightarrow \mathbb{R}^m$ is a differentiable map defined on an open subset U of \mathbb{R}^n then its differential

$$dF(p) : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

at a point $p \in U$ is a linear map given by the $m \times n$ matrix

$$dF(p) = \begin{bmatrix} \partial F_1 / \partial x_1(p) & \dots & \partial F_1 / \partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m / \partial x_1(p) & \dots & \partial F_m / \partial x_n(p) \end{bmatrix}.$$

If $\gamma : \mathbb{R} \rightarrow U$ is a curve in U such that $\gamma(0) = p$ and $\gamma'(0) = Z$ then the composition $F \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}^m$ is a curve in \mathbb{R}^m and according to the chain rule we have

$$dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0},$$

which is the tangent vector of the curve $F \circ \gamma$ at the image point $F(p) \in \mathbb{R}^m$. This shows that the **differential** $dF(p)$ of F at p is the linear map given by the formula

$$dF(p) : \gamma'(0) = Z \mapsto dF(p) \cdot Z = \frac{d}{dt}(F \circ \gamma(t))|_{t=0}$$

mapping the tangent vectors at $p \in U$ to tangent vectors at the image point $F(p) \in \mathbb{R}^m$. This formula will now be generalised to the surface setting.

Proposition 4.26. *Let M_1 and M_2 be two regular surfaces in \mathbb{R}^3 , $p \in M_1$, $q \in M_2$ and $\phi : M_1 \rightarrow M_2$ be a differentiable map with $\phi(p) = q$. Then the formula*

$$d\phi_p : \gamma'(0) \mapsto \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0}$$

determines a well-defined linear map $d\phi_p : T_pM_1 \rightarrow T_qM_2$. Here $\gamma : I \rightarrow M_1$ is any differentiable curve satisfying $\gamma(0) = p$.

PROOF. Let $X : U \rightarrow M_1$ and $Y : V \rightarrow M_2$ be local parametrisations such that $X(0) = p$, $Y(0) = q$ and $\phi(X(U))$ contained in $Y(V)$. Then define

$$F = Y^{-1} \circ \phi \circ X : U \rightarrow \mathbb{R}^2$$

and let $\alpha : I \rightarrow U$ be a C^1 -curve with $\alpha(0) = 0$ and $\alpha'(0) = (a, b) \in \mathbb{R}^2$. If

$$\gamma = X \circ \alpha : I \rightarrow X(U)$$

then $\gamma(0) = p$ and

$$\gamma'(0) = dX(0) \cdot (a, b) = aX_u(0) + bX_v(0).$$

The image curve $\phi \circ \gamma : I \rightarrow Y(V)$ is given by $\phi \circ \gamma = Y \circ F \circ \alpha$ so the chain rule implies that

$$\begin{aligned} \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} &= dY(F(0)) \cdot \frac{d}{dt}(F \circ \alpha(t))|_{t=0} \\ &= dY(F(0)) \cdot dF(0) \cdot \alpha'(0). \end{aligned}$$

This means that $d\phi_p : T_pM_1 \rightarrow T_qM_2$ is given by

$$d\phi_p : (aX_u(0) + bX_v(0)) \mapsto dY(F(0)) \cdot dF(0) \cdot (a, b)$$

and hence clearly linear. \square

Definition 4.27. Let M_1 and M_2 be two regular surfaces in \mathbb{R}^3 , $p \in M_1$, $q \in M_2$ and $\phi : M_1 \rightarrow M_2$ be a differentiable map such that $\phi(p) = q$. The map $d\phi_p : T_pM_1 \rightarrow T_qM_2$ is called the **differential** or the **tangent map** of ϕ at p .

The classical **inverse mapping theorem** generalises as follows.

Theorem 4.28. *Let $\phi : M_1 \rightarrow M_2$ be a differentiable map between regular surfaces in \mathbb{R}^3 . If p is a point in M_1 , $q = \phi(p)$ and the differential*

$$d\phi_p : T_pM_1 \rightarrow T_qM_2$$

is bijective then there exist open neighborhoods U_p around p and U_q around q such that $\phi|_{U_p} : U_p \rightarrow U_q$ is bijective and the inverse $(\phi|_{U_p})^{-1} : U_q \rightarrow U_p$ is differentiable.

PROOF. See Exercise 4.8 \square

We now introduce the first fundamental form of a regular surface. This enables us to measure angles between tangent vectors, lengths of curves and even distances between points on the surface.

Definition 4.29. Let M be a regular surface in \mathbb{R}^3 and $p \in M$. Then we define the **first fundamental form** $I_p : T_p M \times T_p M \rightarrow \mathbb{R}$ of M at p by

$$I_p(Z, W) = \langle Z, W \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product in \mathbb{R}^3 restricted to the tangent plane $T_p M$ of M at p . Properties of the surface which only depend on its first fundamental form are called **inner properties**.

Definition 4.30. Let M be a regular surface in \mathbb{R}^3 and $\gamma : I \rightarrow M$ be a differentiable curve on M . Then the **length** $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{\langle \gamma'(t), \gamma'(t) \rangle} dt.$$

As we shall now see a regular surface in \mathbb{R}^3 has a natural distance function d . This gives (M, d) the structure of a **metric space**.

Proposition 4.31. Let M be a regular surface in \mathbb{R}^3 . For two points $p, q \in M$ let C_{pq} denote the set of differentiable curves $\gamma : [0, 1] \rightarrow M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the function $d : M \times M \rightarrow \mathbb{R}_0^+$ by

$$d(p, q) = \inf\{L(\gamma) \mid \gamma \in C_{pq}\}.$$

Then (M, d) is a metric space i.e. for all $p, q, r \in M$ we have

- (i) $d(p, q) \geq 0$,
- (ii) $d(p, q) = 0$ if and only if $p = q$,
- (iii) $d(p, q) = d(q, p)$,
- (iv) $d(p, q) \leq d(p, r) + d(r, q)$.

PROOF. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998). \square

Definition 4.32. A differentiable map $\phi : M_1 \rightarrow M_2$ between two regular surfaces in \mathbb{R}^3 is said to be **isometric** if for each $p \in M_1$ the differential $d\phi_p : T_p M_1 \rightarrow T_{\phi(p)} M_2$ preserves the first fundamental forms of the surfaces involved i.e.

$$\langle d\phi_p(Z), d\phi_p(W) \rangle = \langle Z, W \rangle$$

for all $Z, W \in T_p M_1$. An isometric diffeomorphism $\phi : M_1 \rightarrow M_2$ is called an **isometry**. Two regular surfaces M_1 and M_2 are said to be **isometric** if there exists an isometry $\phi : M_1 \rightarrow M_2$ between them.

Definition 4.33. A differentiable map $\phi : M_1 \rightarrow M_2$ between two regular surfaces in \mathbb{R}^3 is said to be **conformal** if there exists a differentiable function $\lambda : M_1 \rightarrow \mathbb{R}$ such that for each $p \in M$ the differential $d\phi_p : T_p M \rightarrow T_{\phi(p)} M$ satisfies

$$\langle d\phi_p(Z), d\phi_p(W) \rangle = e^{2\lambda} \langle Z, W \rangle$$

for all $Z, W \in T_p M$. Two regular surfaces M_1 and M_2 are said to be **conformally equivalent** if there exists a conformal diffeomorphism $\phi : M_1 \rightarrow M_2$ between them.

Let M be a regular surface in \mathbb{R}^3 and $X : U \rightarrow M$ be a local parametrisation of M . At each point $X(u, v)$ in $X(U)$ the tangent plane is generated by the vectors $X_u(u, v)$ and $X_v(u, v)$. For these we define the matrix-valued map $[DX] : U \rightarrow \mathbb{R}^{3 \times 2}$ by

$$[DX] = [X_u, X_v]$$

and the real-valued functions $E, F, G : U \rightarrow \mathbb{R}$ by the symmetric matrix

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX]$$

containing the scalar products

$$E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle = \langle X_v, X_u \rangle \quad \text{and} \quad G = \langle X_v, X_v \rangle.$$

This induces a so called **metric**

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2$$

in the parameter region U as follows: For each point $q \in U$ we have a scalar product $ds_q^2 : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$ds_q^2(z, w) = z \cdot \begin{bmatrix} E(q) & F(q) \\ F(q) & G(q) \end{bmatrix} \cdot w^t.$$

The following shows that the diffeomorphism X preserves the scalar products so it is actually an isometry.

Let $\alpha_1 = (u_1, v_1) : I \rightarrow U$ and $\alpha_2 = (u_2, v_2) : I \rightarrow U$ be two curves in U meeting at $\alpha_1(0) = q = \alpha_2(0)$. Further let $\gamma_1 = X \circ \alpha_1$ and $\gamma_2 = X \circ \alpha_2$ be the image curves in $X(U)$ meeting at $\gamma_1(0) = p = \gamma_2(0)$. Then the differential $dX(q)$ is given by

$$dX(q) : (a, b) = (a \cdot e_1 + b \cdot e_2) \mapsto aX_u(q) + bX_v(q)$$

so at q we have

$$ds_q^2(\alpha'_1, \alpha'_2) = \alpha'_1 \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot \alpha'_2{}^t$$

$$\begin{aligned}
&= \begin{bmatrix} u'_1 & v'_1 \end{bmatrix} \cdot \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot \begin{bmatrix} u'_2 \\ v'_2 \end{bmatrix} \\
&= Eu'_1u'_2 + F(u'_1v'_2 + u'_2v'_1) + Gv'_1v'_2 \\
&= \langle X_u, X_u \rangle u'_1u'_2 + \langle X_u, X_v \rangle (u'_1v'_2 + u'_2v'_1) + \langle X_v, X_v \rangle v'_1v'_2 \\
&= \langle u'_1X_u + v'_1X_v, u'_2X_u + v'_2X_v \rangle \\
&= \langle dX(\alpha'_1), dX(\alpha'_2) \rangle \\
&= \langle \gamma'_1, \gamma'_2 \rangle.
\end{aligned}$$

It now follows that the length of a curve $\alpha : I \rightarrow U$ in U is exactly the same as the length of the corresponding curve $\gamma = X \circ \alpha$ in $X(U)$. We have "pulled back" the first fundamental form on the surface $X(U)$ to a metric on the open subset U of \mathbb{R}^2 .

Deep Result 4.34. *Every regular surface M in \mathbb{R}^3 can locally be parametrised by **isothermal coordinates** i.e. for each point $p \in M$ there exists a local parametrisation $X : U \rightarrow M$ such that $p \in X(U)$*

$$E(u, v) = G(u, v) \quad \text{and} \quad F(u, v) = 0$$

for all $(u, v) \in U$.

PROOF. A complete twelve page proof can be found in the standard text: M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Publish or Perish (1979). \square

Definition 4.35. Let M be a regular surface in \mathbb{R}^3 and $X : U \rightarrow M$ be a local parametrisation of M where U is a measurable subset of the plane \mathbb{R}^2 . Then we define the **area** of $X(U)$ by

$$A(X(U)) = \int_U \sqrt{EG - F^2} dudv.$$

Exercises

Exercise 4.1. Determine whether the following subsets of \mathbb{R}^3 are regular surfaces or not.

$$\begin{aligned} M_1 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z\}, \\ M_2 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}, \\ M_3 &= \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}, \\ M_4 &= \{(x, y, z) \in \mathbb{R}^3 \mid x \sin z = y \cos z\}. \end{aligned}$$

Find a parametrisation for those which are regular surfaces in \mathbb{R}^3 .

Exercise 4.2. Prove Proposition 4.21.

Exercise 4.3. Prove that the map $\phi : T^2 \rightarrow S^2$ in Example 4.22 is differentiable.

Exercise 4.4. Prove Proposition 4.23.

Exercise 4.5. Prove Proposition 4.25.

Exercise 4.6. Construct a diffeomorphism $\phi : S^2 \rightarrow M$ between the unit sphere S^2 and the ellipsoid

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + 2y^2 + 3z^2 = 1\}.$$

Exercise 4.7. Let $U = \{(u, v) \in \mathbb{R}^2 \mid -\pi < u < \pi, 0 < v < 1\}$, define $X : U \rightarrow \mathbb{R}^3$ by $X(u, v) = (\sin u, \sin 2u, v)$ and set $M = X(U)$. Sketch M and show that X is differentiable, regular and injective but X^{-1} is not continuous. Is M a regular surface in \mathbb{R}^3 ?

Exercise 4.8. Find a proof for Theorem 4.28

Exercise 4.9. For $\alpha \in (0, \pi/2)$ define the parametrised surface M_α by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \rightarrow M$ by

$$X_\alpha(r, \theta) = \left(r \sin \alpha \cos\left(\frac{\theta}{\sin \alpha}\right), r \sin \alpha \sin\left(\frac{\theta}{\sin \alpha}\right), r \cos \alpha \right).$$

Calculate the first fundamental form of M_α and find an equation of the form $f_\alpha(x, y, z) = 0$ describing the surface.

Exercise 4.10. Find an isometric parametrisation $X : \mathbb{R}^2 \rightarrow M$ of the circular cylinder

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Exercise 4.11. Let M be the unit sphere S^2 with the two poles removed. Prove that Mercator's parametrization $X : \mathbb{R}^2 \rightarrow M$ of M with

$$X(u, v) = \left(\frac{\cos v}{\cosh u}, \frac{\sin v}{\cosh u}, \frac{\sinh u}{\cosh u} \right)$$

is conformal.

Exercise 4.12. Prove that the first fundamental form of a regular surface M in \mathbb{R}^3 is invariant under Euclidean motions.

Exercise 4.13. Let $X, Y : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the regular parametrised surfaces given by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$

$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

Calculate the first fundamental forms of X and Y . Find equations of the form $f(x, y, z) = 0$ describing the surfaces X and Y . Compare with Exercise 4.1.

Exercise 4.14. Calculate the area $A(T^2)$ of the torus

$$T^2 = \{(x, y, z) \in U \mid z^2 + (\sqrt{x^2 + y^2} - R)^2 = r^2\}.$$

Curvature

In this chapter we define the shape operator of an oriented surface and its second fundamental form. These measure the behaviour of the normal of the surface and lead us to the notions of normal curvature, Gaussian curvature and mean curvature.

From now on we assume, if not stated otherwise, that our curves and surfaces belong to the C^2 -category i.e. they can be parametrised locally by C^2 -maps.

Definition 5.1. Let M be a regular surface in \mathbb{R}^3 . A differentiable map $N : M \rightarrow S^2$ with values in the unit sphere is said to be a **Gauss map** for M if for each point $p \in M$ the image $N(p)$ is perpendicular to the tangent plane T_pM . The surface M is said to be **orientable** if such a Gauss map exists. A surface M equipped with a Gauss map is said to be **oriented**.

Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Let $p \in M$ and $\gamma : I \rightarrow M$ be a regular curve parametrised by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z \in T_pM$. Then the composition $N \circ \gamma : I \rightarrow S^2$ is a differentiable curve on the unit sphere and the linear differential $dN_p : T_pM \rightarrow T_{N(p)}S^2$ of N at p is given by the formula

$$dN_p : Z = \dot{\gamma}(0) \mapsto \frac{d}{ds}(N(\gamma(s)))|_{s=0} = dN_p(Z).$$

At the point p the second derivative $\ddot{\gamma}(0)$ has a natural decomposition

$$\ddot{\gamma}(0) = \ddot{\gamma}(0)^{\text{tan}} + \ddot{\gamma}(0)^{\text{norm}}$$

into its tangential part, contained in T_pM , and its normal part in the orthogonal complement $(T_pM)^\perp$ of T_pM . Along the curve $\gamma : I \rightarrow M$ the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot{\gamma}(s)$. This implies that

$$\begin{aligned} 0 &= \frac{d}{ds}(\langle \dot{\gamma}(s), N(\gamma(s)) \rangle) \\ &= \langle \ddot{\gamma}(s), N(\gamma(s)) \rangle + \langle \dot{\gamma}(s), dN_{\gamma(s)}(\dot{\gamma}(s)) \rangle. \end{aligned}$$

Hence the normal part of the second derivative $\ddot{\gamma}(0)$ satisfies

$$\begin{aligned}\ddot{\gamma}(0)^{\text{norm}} &= \langle \ddot{\gamma}(0), N(p) \rangle N(p) \\ &= -\langle \dot{\gamma}(0), dN_p(\dot{\gamma}(0)) \rangle N(p) \\ &= -\langle Z, dN_p(Z) \rangle N(p).\end{aligned}$$

This shows that the normal component of $\ddot{\gamma}(0)$ is completely determined by the value of $\dot{\gamma}(0)$ and the values of the Gauss map along any curve through p with tangent $\dot{\gamma}(0) = Z$ at p .

Since $N : M \rightarrow S^2$ is a Gauss map for the surface M and $p \in M$ we see that $N(p)$ is a unit normal to both the tangent planes T_pM and $T_{N(p)}S^2$ so we can make the identification $T_pM \cong T_{N(p)}S^2$.

Definition 5.2. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $p \in M$. Then the **shape operator**

$$S_p : T_pM \rightarrow T_pM$$

of M at p is the linear endomorphism given by

$$S_p(Z) = -dN_p(Z)$$

for all $Z \in T_pM$.

Proposition 5.3. Let M be an oriented regular surface with Gauss map $N : M \rightarrow S^2$ and $p \in M$. Then the shape operator $S_p : T_pM \rightarrow T_pM$ is symmetric i.e.

$$\langle S_p(Z), W \rangle = \langle Z, S_p(W) \rangle$$

for all $Z, W \in T_pM$.

PROOF. Let $X : U \rightarrow M$ be a local parametrisation of M such that $X(0) = p$ and let $N : X(U) \rightarrow S^2$ be the Gauss map on $X(U)$ given by

$$N(X(u, v)) = \pm \frac{X_u(u, v) \times X_v(u, v)}{|X_u(u, v) \times X_v(u, v)|}.$$

Then the normal vector $N(X(u, v))$ at the point $X(u, v)$ is orthogonal to the tangent plane $T_{X(u, v)}M$ so

$$0 = \frac{d}{dv} \langle N \circ X, X_u \rangle|_{(u, v)=0} = \langle dN_p(X_v), X_u \rangle + \langle N(p), X_{vu} \rangle$$

and

$$0 = \frac{d}{du} \langle N \circ X, X_v \rangle|_{(u, v)=0} = \langle dN_p(X_u), X_v \rangle + \langle N(p), X_{uv} \rangle.$$

By subtracting the second equation from the first one and employing the fact that $X_{uv} = X_{vu}$ we obtain

$$\langle dN_p(X_v), X_u \rangle = \langle X_v, dN_p(X_u) \rangle.$$

Since $\{X_u, X_v\}$ is a basis for the tangent plane T_pM , the symmetry of the linear endomorphism $dN_p : T_pM \rightarrow T_pM$ is a direct consequence of this last equation and the following obvious relations

$$\begin{aligned}\langle dN_p(X_u), X_u \rangle &= \langle X_u, dN_p(X_u) \rangle, \\ \langle dN_p(X_v), X_v \rangle &= \langle X_v, dN_p(X_v) \rangle.\end{aligned}$$

The statement now follows from the fact that $S_p = -dN_p$. \square

The symmetry of the shape operator has the following consequence.

Corollary 5.4. *Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $p \in M$. Then there exists an orthonormal basis $\{Z_1, Z_2\}$ for the tangent plane T_pM such that*

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2,$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

Definition 5.5. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $p \in M$. Then we define the **second fundamental form** $II_p : T_pM \times T_pM \rightarrow \mathbb{R}$ of M at p by

$$II_p(Z, W) = \langle S_p(Z), W \rangle.$$

Note that it is an immediate consequence of Proposition 5.3 that the second fundamental form is symmetric and bilinear.

Definition 5.6. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the **normal curvature** $\kappa_n(Z)$ of M at p in the direction of Z is defined by

$$\kappa_n(Z) = \langle \ddot{\gamma}(0), N(p) \rangle,$$

where $\gamma : I \rightarrow M$ is any curve parametrised by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$.

Proposition 5.7. *Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$, $p \in M$ and $Z \in T_pM$ with $|Z| = 1$. Then the normal curvature $\kappa_n(Z)$ of M at p in the direction of Z satisfies*

$$\kappa_n(Z) = \langle S_p(Z), Z \rangle = II_p(Z, Z).$$

PROOF. Let $\gamma : I \rightarrow M$ be a curve parametrised by arclength such that $\gamma(0) = p$ and $\dot{\gamma}(0) = Z$. Along the curve the normal $N(\gamma(s))$ is perpendicular to the tangent $\dot{\gamma}(s)$. This means that

$$\begin{aligned}0 &= \frac{d}{ds}(\langle \dot{\gamma}(s), N(\gamma(s)) \rangle) \\ &= \langle \ddot{\gamma}(s), N(\gamma(s)) \rangle + \langle \dot{\gamma}(s), dN_{\gamma(s)}(\dot{\gamma}(s)) \rangle.\end{aligned}$$

As a direct consequence we get

$$\begin{aligned}\kappa_n(Z) &= \langle \ddot{\gamma}(0), N(p) \rangle \\ &= -\langle Z, dN_p(Z) \rangle \\ &= \langle S_p(Z), Z \rangle \\ &= II_p(Z, Z).\end{aligned}$$

□

For an oriented regular surface M with Gauss map $N : M \rightarrow S^2$ and $p \in M$ let $T_p^1 M$ denote the unit circle in the tangent plane $T_p M$ i.e.

$$T_p^1 M = \{Z \in T_p M \mid |Z| = 1\}.$$

Then the real-valued function $\kappa_n : T_p^1 M \rightarrow \mathbb{R}$ is defined by

$$\kappa_n : Z \mapsto \kappa_n(Z).$$

The unit circle is compact and κ_n is continuous so there exist two directions $Z_1, Z_2 \in T_p^1 M$ such that

$$\kappa_1(p) = \kappa_n(Z_1) = \max_{Z \in T_p^1 M} \kappa_n(Z)$$

and

$$\kappa_2(p) = \kappa_n(Z_2) = \min_{Z \in T_p^1 M} \kappa_n(Z).$$

These are called **principal directions** at p and $\kappa_1(p), \kappa_2(p)$ the corresponding **principal curvatures**. A point $p \in M$ is said to be **umbilic** if $\kappa_1(p) = \kappa_2(p)$.

Theorem 5.8. *Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $p \in M$. Then $Z \in T_p^1 M$ is a principal direction at p if and only if it is an eigenvector for the shape operator $S_p : T_p M \rightarrow T_p M$.*

PROOF. Let $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent plane $T_p M$ of eigenvectors to S_p i.e.

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2$$

for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Then every unit vector $Z \in T_p^1 M$ can be written as

$$Z(\theta) = \cos \theta Z_1 + \sin \theta Z_2$$

and

$$\begin{aligned}\kappa_n(Z(\theta)) &= \langle S_p(\cos \theta Z_1 + \sin \theta Z_2), \cos \theta Z_1 + \sin \theta Z_2 \rangle \\ &= \cos^2 \theta \langle S_p(Z_1), Z_1 \rangle + \sin^2 \theta \langle S_p(Z_2), Z_2 \rangle \\ &\quad + \cos \theta \sin \theta (\langle S_p(Z_1), Z_2 \rangle + \langle S_p(Z_2), Z_1 \rangle)\end{aligned}$$

$$= \lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta.$$

If $\lambda_1 = \lambda_2$ then $\kappa_n(Z(\theta)) = \lambda_1$ for all $\theta \in \mathbb{R}$ so any direction is both principal and an eigenvector for the shape operator S_p .

If $\lambda_1 \neq \lambda_2$, then we can assume, without loss of generality, that $\lambda_1 > \lambda_2$. Then $Z(\theta)$ is a maximal principal direction if and only if $\cos^2 \theta = 1$ i.e. $Z = \pm Z_1$ and clearly a minimal principal direction if and only if $\sin^2 \theta = 1$ i.e. $Z = \pm Z_2$. \square

Definition 5.9. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Then we define the **Gaussian curvature** $K : M \rightarrow \mathbb{R}$ and the **mean curvature** $H : M \rightarrow \mathbb{R}$ by

$$K(p) = \det S_p \quad \text{and} \quad H(p) = \frac{1}{2} \text{trace } S_p,$$

respectively. The surface M is said to be **flat** if $K(p) = 0$ for all $p \in M$ and **minimal** if $H(p) = 0$ for all $p \in M$.

Let M be a regular surface in \mathbb{R}^3 , $p \in M$ and $\{Z_1, Z_2\}$ be an orthonormal basis for the tangent plane $T_p M$ at p such that

$$S_p(Z_1) = \lambda_1 Z_1 \quad \text{and} \quad S_p(Z_2) = \lambda_2 Z_2.$$

Further let $\alpha_1, \alpha_2 : I \rightarrow M$ be two curves, parametrised by arclength, meeting at p i.e. $\alpha_1(0) = p = \alpha_2(0)$, such that

$$\dot{\alpha}_1(0) = Z_1 \quad \text{and} \quad \dot{\alpha}_2(0) = Z_2.$$

Then the eigenvalues of the shape operator S_p satisfy

$$\lambda_1 = \langle S_p(Z_1), Z_1 \rangle = \langle \ddot{\alpha}_1(0), N(p) \rangle$$

and

$$\lambda_2 = \langle S_p(Z_2), Z_2 \rangle = \langle \ddot{\alpha}_2(0), N(p) \rangle.$$

If $K(p) = \lambda_1 \lambda_2 > 0$ then λ_1 and λ_2 have the same sign so the curves $\alpha_1, \alpha_2 : I \rightarrow M$ stay locally on the same side of the tangent plane. This means that the normal curvature $\kappa_n(Z)$ has the same sign independent of the direction $Z \in T_p M$ at p so any curve through the point p stays on the same side of the tangent plane.

If $K(p) = \lambda_1 \lambda_2 < 0$ then λ_1 and λ_2 have different signs so the curves $\alpha_1, \alpha_2 : I \rightarrow M$ stay locally on different sides of the tangent plane $T_p M$ at p .

Theorem 5.10. *Let M be a path-connected, oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Then the shape operator $S_p : T_p M \rightarrow T_p M$ vanishes for all $p \in M$ if and only if M is contained in a plane.*

PROOF. If M is contained in a plane, then the Gauss map is constant so $S_p = -dN_p = 0$ at any point $p \in M$.

Let us now assume that the shape operator vanishes identically i.e. $S_p = -dN_p = 0$ for all $p \in M$. Then fix $p \in M$, let q be an arbitrary point on M and $\gamma : I \rightarrow M$ be a curve such that $\gamma(0) = q$ and $\gamma(1) = p$. Then the real-valued function $f_q : I \rightarrow \mathbb{R}$ with

$$f_q(t) = \langle q - \gamma(t), N(\gamma(t)) \rangle$$

satisfies $f_q(0) = 0$ and

$$f'_q(t) = -\langle \gamma'(t), N(\gamma(t)) \rangle + \langle q - \gamma(t), dN_{\gamma(t)}(\gamma'(t)) \rangle = 0.$$

This implies that $f_q(t) = \langle q - \gamma(t), N(\gamma(t)) \rangle = 0$ for all $t \in I$, in particular,

$$f_q(1) = \langle q - p, N(p) \rangle = 0$$

for all $q \in M$. This shows that the surface is contained in the plane through p with normal $N(p)$. \square

We will now calculate the Gaussian curvature K and the mean curvature H of a surface in terms on a local parametrisation. Let M be an oriented surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Let $X : U \rightarrow M$ be a local parametrisation such that $X(0) = p \in M$. Then the tangent plane $T_p M$ is generated by X_u and X_v so there exists a 2×2 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

such that the shape operator $S_p : T_p M \rightarrow T_p M$ satisfies

$$\begin{aligned} S_p(aX_u + bX_v) &= aS_p(X_u) + bS_p(X_v) \\ &= a(a_{11}X_u + a_{21}X_v) + b(a_{12}X_u + a_{22}X_v) \\ &= (a_{11}a + a_{12}b)X_u + (a_{21}a + a_{22}b)X_v. \end{aligned}$$

This means that, with respect to the basis $\{X_u, X_v\}$, the shape operator S_p at p is given by

$$S_p : \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix}.$$

If we define the matrix-valued maps $[DX], [DN] : U \rightarrow \mathbb{R}^{3 \times 2}$ by

$$[DX] = [X_u, X_v] \quad \text{and} \quad [DN] = [N_u, N_v]$$

then it follows from the definition $S_p = -dN_p$ of the shape operator that

$$-[DN] = [DX] \cdot A.$$

Note that the 2×2 matrix

$$[DX]^t \cdot [DN]$$

is symmetric, since

$$\langle X_u, N_v \rangle = -\langle X_{uv}, N \rangle = -\langle X_{vu}, N \rangle = \langle X_v, N_u \rangle.$$

To the local parametrisation $X : U \rightarrow M$ we now associate the functions $e, f, g : U \rightarrow \mathbb{R}$ given by

$$\begin{aligned} \begin{bmatrix} e & f \\ f & g \end{bmatrix} &= -[DX]^t \cdot [DN] \\ &= [DX]^t \cdot [DX] \cdot A \\ &= \begin{bmatrix} E & F \\ F & G \end{bmatrix} \cdot A. \end{aligned}$$

This means that, with respect to the basis $\{X_u, X_v\}$, the matrix A corresponding to the shape operator $S_p : T_p M \rightarrow T_p M$ at $p \in M$ is given by

$$\begin{aligned} A &= \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix} \\ &= \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}. \end{aligned}$$

This implies that the Gaussian curvature K and the mean curvature H satisfy

$$K = \frac{eg - f^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

The principal curvatures λ_1 and λ_2 are the eigenvalues of the shape operator so they are the solutions to the polynomial characteristic equation

$$P(\lambda) = \det(A - \lambda \cdot I) = 0$$

or equivalently

$$\det\left(\begin{bmatrix} e & f \\ f & g \end{bmatrix} - \lambda \begin{bmatrix} E & F \\ F & G \end{bmatrix}\right) = 0.$$

Example 5.11. Let $\gamma = (r, 0, z) : I \rightarrow \mathbb{R}^3$ be a differentiable curve in the (x, z) -plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then $X : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix}$$

parametrises a regular surface of revolution M . The linearly independent and orthogonal tangent vectors

$$X_u = \begin{bmatrix} \dot{r}(u) \cos v \\ \dot{r}(u) \sin v \\ \dot{z}(u) \end{bmatrix}, \quad X_v = \begin{bmatrix} -r(u) \sin v \\ r(u) \cos v \\ 0 \end{bmatrix}$$

generate a Gauss map

$$N(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \dot{z}(u) \\ 0 \\ -\dot{r}(u) \end{bmatrix} = \begin{bmatrix} \dot{z}(u) \cos v \\ \dot{z}(u) \sin v \\ -\dot{r}(u) \end{bmatrix}.$$

Furthermore

$$[DX]^t = \begin{bmatrix} \dot{r}(u) \cos v & \dot{r}(u) \sin v & \dot{z}(u) \\ -r(u) \sin v & r(u) \cos v & 0 \end{bmatrix},$$

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} 1 & 0 \\ 0 & r(u)^2 \end{bmatrix}$$

and

$$\begin{bmatrix} e & f \\ f & g \end{bmatrix} = -[DN]^t \cdot [DX]$$

$$= \begin{bmatrix} -\ddot{z}(u) \cos v & -\ddot{z}(u) \sin v & \ddot{r}(u) \\ \dot{z}(u) \sin v & -\dot{z}(u) \cos v & 0 \end{bmatrix}$$

$$\cdot \begin{bmatrix} \dot{r}(u) \cos v & -r(u) \sin v \\ \dot{r}(u) \sin v & r(u) \cos v \\ \dot{z}(u) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)r(u) \end{bmatrix}.$$

This means that the 2×2 matrix A of the shape operator is given by

$$A = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \cdot \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

$$= \frac{1}{r(u)^2} \begin{bmatrix} r(u)^2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)r(u) \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{r}(u)\dot{z}(u) - \ddot{z}(u)\dot{r}(u) & 0 \\ 0 & -\dot{z}(u)/r(u) \end{bmatrix}.$$

Using the fact that the curve $(r, 0, z)$ is parametrised by arclength we get the following remarkably simple expression for the Gaussian curvature

$$K = \det A$$

$$\begin{aligned}
&= \frac{eg - f^2}{EG - F^2} \\
&= \frac{\dot{z}(u)r(u)(\ddot{z}(u)\dot{r}(u) - \ddot{r}(u)\dot{z}(u))}{r(u)^2} \\
&= \frac{\dot{r}(u)\dot{z}(u)\ddot{z}(u) - \ddot{r}(u)\dot{z}(u)^2}{r(u)} \\
&= \frac{\dot{r}(u)(-\dot{r}(u)\ddot{r}(u)) - \ddot{r}(u)(1 - \dot{r}(u)^2)}{r(u)} \\
&= -\frac{\ddot{r}(u)}{r(u)}.
\end{aligned}$$

This shows that the function $r : I \rightarrow \mathbb{R}$ satisfies the following second order linear ordinary differential equation

$$\ddot{r}(s) + K(s) \cdot r(s) = 0.$$

Theorem 5.12. *Let M be a path-connected oriented regular C^3 -surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. If every $p \in M$ is an umbilic point, then M is either contained in a plane or in a sphere.*

PROOF. Let $X : U \rightarrow M$ be a local parametrisation such that U is path-connected. Since each point in $X(U)$ is umbilic there exists a differentiable function $\kappa : U \rightarrow \mathbb{R}$ such that the shape operator is given by

$$S_p : (aX_u + bX_v) \mapsto \kappa(u, v)(aX_u + bX_v)$$

so in particular

$$(N \circ X)_u = -\kappa X_u \quad \text{and} \quad (N \circ X)_v = -\kappa X_v.$$

Furthermore

$$\begin{aligned}
0 &= (N \circ X)_{uv} - (N \circ X)_{vu} \\
&= (-\kappa X_u)_v - (-\kappa X_v)_u \\
&= -\kappa_v X_u - \kappa X_{uv} + \kappa_u X_v + \kappa X_{uv} \\
&= -\kappa_v X_u + \kappa_u X_v.
\end{aligned}$$

The vectors X_u and X_v are linearly independent so $\kappa_u = \kappa_v = 0$. The domain U is path-connected which means that κ is constant on U and hence on the whole of M since M is path-connected.

If $\kappa = 0$ then the shape operator vanishes and Theorem 5.10 tells us that the surface is contained in a plane.

If $\kappa \neq 0$ then we define the map $Y : U \rightarrow \mathbb{R}^3$ by

$$Y(u, v) = X(u, v) + \frac{1}{\kappa} N(u, v).$$

Then

$$dY = dX + \frac{1}{\kappa}dN = dX - \frac{1}{\kappa}\kappa dX = 0$$

so Y is constant and

$$|X - Y|^2 = \frac{1}{\kappa^2}.$$

This shows that $X(U)$ is contained in a sphere with centre Y and radius $1/\kappa$. Since M is path-connected the whole of M is contained in the same sphere. \square

Theorem 5.13. *Let M be a compact regular surface in \mathbb{R}^3 . Then there exists at least one point $p \in M$ such that the Gaussian curvature $K(p)$ is positive.*

PROOF. See Exercise 5.7. \square

Exercises

Exercise 5.1. Let U be an open subset of \mathbb{R}^3 and $q \in \mathbb{R}$ be a regular value of the differentiable function $f : U \rightarrow \mathbb{R}$. Prove that the regular surface $M = f^{-1}(\{q\})$ in \mathbb{R}^3 is orientable.

Exercise 5.2. Determine the Gaussian curvature and the mean curvature of the sphere $S_r^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = r^2\}$.

Exercise 5.3. Determine the Gaussian curvature and the mean curvature of the parametrised Enneper surface $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ given by

$$X(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right).$$

Exercise 5.4. Determine the Gaussian curvature and the mean curvature of the catenoid M parametrised by $X : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R}^3$ with

$$X(\theta, r) = \left(\frac{1+r^2}{2r} \cos \theta, \frac{1+r^2}{2r} \sin \theta, \log r\right).$$

Find an equation of the form $f(x, y, z) = 0$ describing the surface M . Compare with Exercise 4.13.

Exercise 5.5. Prove that the second fundamental form of an oriented regular surface M in \mathbb{R}^3 is invariant under rigid Euclidean motions.

Exercise 5.6. Let $X, Y : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the regular parametrised surfaces given by

$$X(u, v) = (\cosh u \cos v, \cosh u \sin v, u),$$

$$Y(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

Calculate the shape operators of X and Y and the corresponding principal curvatures κ_1, κ_2 . Compare with Exercise 4.13.

Exercise 5.7. Prove Theorem 5.13.

Exercise 5.8. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ be a regular curve, parametrised by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ , respectively. Let r be a positive real number and assume that the r -tube M around γ given by $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with

$$X(s, \theta) = \gamma(s) + r(\cos \theta \cdot n(s) + \sin \theta \cdot b(s))$$

is a regular surface in \mathbb{R}^3 . Determine the Gaussian curvature K of M in terms of $s, \theta, r, \kappa(s)$ and $\tau(s)$.

Exercise 5.9. Let M be a regular surface in \mathbb{R}^3 , $p \in M$ and $\{Z, W\}$ be an orthonormal basis for T_pM of eigenvectors for the shape operator $S_p : T_pM \rightarrow T_pM$. Let $\kappa_n(\theta)$ be the normal curvature of M at p in the direction of $\cos \theta Z + \sin \theta W$. Prove that the mean curvature H satisfies

$$H(p) = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta) d\theta.$$

Exercise 5.10. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Let $X : U \rightarrow M$ be a local parametrisation of M and $A(N \circ X(U))$ be the area of the image $N \circ X(U)$ on the unit sphere S^2 . Prove that

$$A(N \circ X(U)) = \int_{X(U)} |K| dA,$$

where K is the Gaussian curvature of M . Compare with Exercise 3.7.

Exercise 5.11. Let a be a positive real number and U be the open set

$$U = \{(x, y, z) \in \mathbb{R}^3 \mid 2a(x^2 + y^2) < z\}.$$

Prove that there does not exist a complete regular minimal surface M in \mathbb{R}^3 which is contained in U .

Theorema Egregium

In this chapter we prove the remarkable Theorema Egregium which tells us that the Gaussian curvature, of a regular surface, is actually completely determined by its first fundamental form.

Theorem 6.1. *Let M be a regular C^3 -surface in \mathbb{R}^3 . Then the Gaussian curvature K of M is determined by its first fundamental form.*

This result has a highly interesting consequence.

Corollary 6.2. *It is impossible to construct a distance preserving planar chart of the unit sphere S^2 .*

PROOF. If there existed a local parametrisation $X : U \rightarrow S^2$ of the unit sphere which was an isometry then the Gaussian curvature of the flat plane and the unit sphere would be the same. But we know that S^2 has constant curvature $K = 1$. \square

We shall now prove Theorem 6.1.

PROOF. Let M be a surface and $X : U \rightarrow M$ be a local parametrisation of M with first fundamental form determined by

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX].$$

The set $\{X_u, X_v\}$ is a basis for the tangent plane at each point $X(u, v)$ in $X(U)$. Applying the Gram-Schmidt process on this basis we get an orthonormal basis $\{Z, W\}$ for the tangent plane as follows:

$$\begin{aligned} Z &= \frac{X_u}{\sqrt{E}}, \\ \tilde{W} &= X_v - \langle X_v, Z \rangle Z \\ &= X_v - \frac{\langle X_v, X_u \rangle X_u}{\langle X_u, X_u \rangle} \\ &= X_v - \frac{F}{E} X_u \end{aligned}$$

and finally

$$W = \frac{\tilde{W}}{|\tilde{W}|} = \frac{\sqrt{E}}{\sqrt{EG - F^2}} \left(X_v - \frac{F}{E} X_u \right).$$

This means that there exist functions $a, b, c : U \rightarrow \mathbb{R}$ only depending on E, F, G such that

$$Z = aX_u \quad \text{and} \quad W = bX_u + cX_v.$$

If we define a local Gauss map $N : X(U) \rightarrow S^2$ by

$$N = \frac{X_u \times X_v}{|X_u \times X_v|} = Z \times W$$

then $\{Z, W, N\}$ is a positively oriented orthonormal basis for \mathbb{R}^3 along the open subset $X(U)$ of M . This means that the derivatives

$$Z_u, Z_v, W_u, W_v$$

satisfy the following system of equations

$$\begin{aligned} Z_u &= \langle Z_u, Z \rangle Z + \langle Z_u, W \rangle W + \langle Z_u, N \rangle N, \\ Z_v &= \langle Z_v, Z \rangle Z + \langle Z_v, W \rangle W + \langle Z_v, N \rangle N, \\ W_u &= \langle W_u, Z \rangle Z + \langle W_u, W \rangle W + \langle W_u, N \rangle N, \\ W_v &= \langle W_v, Z \rangle Z + \langle W_v, W \rangle W + \langle W_v, N \rangle N. \end{aligned}$$

Using the fact that $\{Z, W\}$ is orthonormal we can simplify to

$$\begin{aligned} Z_u &= \langle Z_u, W \rangle W + \langle Z_u, N \rangle N, \\ Z_v &= \langle Z_v, W \rangle W + \langle Z_v, N \rangle N, \\ W_u &= \langle W_u, Z \rangle Z + \langle W_u, N \rangle N, \\ W_v &= \langle W_v, Z \rangle Z + \langle W_v, N \rangle N. \end{aligned}$$

The following shows that $\langle Z_u, W \rangle$ is a function of $E, F, G : U \rightarrow \mathbb{R}$ and their first order derivatives.

$$\begin{aligned} \langle Z_u, W \rangle &= \langle (aX_u)_u, W \rangle \\ &= \langle a_u X_u + a X_{uu}, bX_u + cX_v \rangle \\ &= a_u b E + a_u c F + ab \langle X_{uu}, X_u \rangle + ac \langle X_{uu}, X_v \rangle \\ &= a_u b E + a_u c F + \frac{1}{2} ab E_u + ac \left(F_u - \frac{1}{2} E_v \right) \end{aligned}$$

It is easily seen that the same applies to $\langle Z_v, W \rangle$.

Employing Lemma 6.3 we now obtain

$$\begin{aligned} &\langle Z_u, W \rangle_v - \langle Z_v, W \rangle_u \\ &= \langle Z_{uv}, W \rangle + \langle Z_u, W_v \rangle - \langle Z_{vu}, W \rangle - \langle Z_v, W_u \rangle \\ &= \langle Z_u, W_v \rangle - \langle Z_v, W_u \rangle \end{aligned}$$

$$= K\sqrt{EG - F^2}.$$

Hence the Gaussian curvature K of M is given by

$$K = \frac{\langle Z_u, W \rangle_v - \langle Z_v, W \rangle_u}{\sqrt{EG - F^2}}$$

As an immediate consequence we see that K only depends on the functions E, F, G and their first and second order derivatives. Hence it is completely determined by the first fundamental form of M . \square

Lemma 6.3. *For the above situation we have*

$$\langle Z_u, W_v \rangle - \langle Z_v, W_u \rangle = K\sqrt{EG - F^2}.$$

PROOF. If A is the matrix for the shape operator in the basis $\{X_u, X_v\}$ then

$$-N_u = a_{11}X_u + a_{21}X_v \quad \text{and} \quad -N_v = a_{12}X_u + a_{22}X_v.$$

This means that

$$\begin{aligned} \langle N_u \times N_v, N \rangle &= \langle (a_{11}X_u + a_{21}X_v) \times (a_{12}X_u + a_{22}X_v), N \rangle \\ &= (a_{11}a_{22} - a_{12}a_{21})\langle X_u \times X_v, N \rangle \\ &= K\langle (\sqrt{EG - F^2})N, N \rangle \\ &= K\sqrt{EG - F^2}. \end{aligned}$$

We also have

$$\begin{aligned} \langle N_u \times N_v, N \rangle &= \langle N_u \times N_v, Z \times W \rangle \\ &= \langle N_u, Z \rangle \langle N_v, W \rangle - \langle N_u, W \rangle \langle N_v, Z \rangle \\ &= \langle Z_u, N \rangle \langle N, W_v \rangle - \langle W_u, N \rangle \langle N, Z_v \rangle \\ &= \langle Z_u, W_v \rangle - \langle Z_v, W_u \rangle. \end{aligned}$$

This proves the statement. \square

Deep Result 6.4. *Let M_1 and M_2 be two regular surfaces in \mathbb{R}^3 and $\phi : M_1 \rightarrow M_2$ be a diffeomorphism respecting their first and second fundamental forms, i.e.*

$$I_p(X, Y) = I_{\phi(p)}(d\phi(X), d\phi(Y))$$

and

$$II_p(X, Y) = II_{\phi(p)}(d\phi(X), d\phi(Y)),$$

for all $p \in M_1$ and $X, Y \in T_pM_1$. Then $\phi : M_1 \rightarrow M_2$ is the restriction $\phi = \Phi|_{M_1} : M_1 \rightarrow M_2$ of a Euclidean motion $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of \mathbb{R}^3 to the surface M_1 .

The proof of the last result is beyond the scope of these lecture notes. Here we need arguments from the theory of partial differential equations.

Exercises

Exercise 6.1. For $\alpha \in (0, \pi/2)$ define the parametrised surface M_α by $X_\alpha : \mathbb{R}^+ \times \mathbb{R} \rightarrow M$ by

$$X_\alpha(r, \theta) = \left(r \sin \alpha \cos\left(\frac{\theta}{\sin \alpha}\right), r \sin \alpha \sin\left(\frac{\theta}{\sin \alpha}\right), r \cos \alpha \right).$$

Calculate its Gaussian curvature K .

Exercise 6.2. Let M be a regular surface in \mathbb{R}^3 and $X : U \rightarrow M$ be an orthogonal parametrisation i.e. $F = 0$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right).$$

Exercise 6.3. Let M be a regular surface in \mathbb{R}^3 and $X : U \rightarrow M$ be an isothermal parametrisation i.e. $F = 0$ and $E = G$. Prove that the Gaussian curvature satisfies

$$K = -\frac{1}{2E} ((\log E)_{uu} + (\log E)_{vv}).$$

Determine the Gaussian curvature K in the cases when

$$E = \frac{4}{(1 + u^2 + v^2)^2}, \quad E = \frac{4}{(1 - u^2 - v^2)^2} \quad \text{or} \quad E = \frac{1}{u^2}.$$

Exercise 6.4. Equip \mathbb{R}^2 and \mathbb{R}^4 with their standard Euclidean scalar products. Prove that the parametrisation $X : \mathbb{R}^2 \rightarrow \mathbb{R}^4$

$$X(u, v) = (\cos u, \sin u, \cos v, \sin v)$$

of the compact torus M in \mathbb{R}^4 is isometric. What does this tell us about the Gaussian curvature of M ? Compare the result with Theorem 5.13.

Geodesics

In this chapter we introduce the notion of a geodesic on a regular surface in \mathbb{R}^3 . We show that locally they are the shortest paths between their endpoints. Geodesics generalise the straight lines in the Euclidean plane.

Let M be a regular surface in \mathbb{R}^3 and $\gamma : I \rightarrow M$ be a curve on M such that $\gamma(0) = p$. As we have seen earlier the second derivative $\gamma''(0)$ at p has a natural decomposition

$$\gamma''(0) = \gamma''(0)^{\text{tan}} + \gamma''(0)^{\text{norm}}$$

into its tangential part, contained in T_pM , and its normal part in the orthogonal complement T_pM^\perp .

Definition 7.1. Let M be a regular surface in \mathbb{R}^3 . A curve $\gamma : I \rightarrow M$ on M is said to be a **geodesic** if for all $t \in I$ the tangential part of the second derivative $\gamma''(t)$ vanishes i.e.

$$\gamma''(t)^{\text{tan}} = 0.$$

Example 7.2. Let $p \in S^2$ be a point on the unit sphere and $Z \in T_pS^2$ be a unit tangent vector. Then $\langle p, Z \rangle = 0$ so $\{p, Z\}$ is an orthonormal basis for a plane in \mathbb{R}^3 , through the origin, which intersects the sphere in a great circle. This circle is parametrised by the curve $\gamma : \mathbb{R} \rightarrow S^2$

$$\gamma(s) = \cos s \cdot p + \sin s \cdot Z.$$

Then for all $s \in I$ the second derivative $\ddot{\gamma}(s)$ satisfies

$$\ddot{\gamma}(s) = -\gamma(s) = -N(\gamma(s)),$$

where $N : S^2 \rightarrow S^2$ is the Gauss map pointing out of the unit sphere. This means that the tangential part $\ddot{\gamma}(s)^{\text{tan}}$ vanishes so the curve is a geodesic on S^2 .

Proposition 7.3. Let M be a regular surface in \mathbb{R}^3 and $\gamma : I \rightarrow M$ be a geodesic on M . Then the norm $|\gamma'| : I \rightarrow \mathbb{R}$ of the tangent γ' of γ is constant i.e. the curve is parametrised proportional to arclength.

PROOF. The statement is an immediate consequence of the following calculation

$$\begin{aligned}
\frac{d}{dt}|\gamma'(t)|^2 &= \frac{d}{dt}\langle\gamma'(t), \gamma'(t)\rangle \\
&= 2\langle\gamma''(t), \gamma'(t)\rangle \\
&= 2\langle\gamma''(t)^{\tan} + \gamma''(t)^{\text{norm}}, \gamma'(t)\rangle \\
&= 2\langle\gamma''(t)^{\tan}, \gamma'(t)\rangle \\
&= 0.
\end{aligned}$$

□

Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Along a curve $\gamma : I \rightarrow M$, parametrised by arclength, the two vectors $\dot{\gamma}$ and N are orthogonal and both of unit length, so the set

$$\{\dot{\gamma}(s), N(\gamma(s)), N(\gamma(s)) \times \dot{\gamma}(s)\}$$

is an orthonormal basis for \mathbb{R}^3 . This implies that the second derivative $\ddot{\gamma} : I \rightarrow \mathbb{R}^3$ has the orthogonal decomposition

$$\begin{aligned}
\ddot{\gamma} &= \langle\ddot{\gamma}, \dot{\gamma}\rangle\dot{\gamma} + \langle\ddot{\gamma}, N \times \dot{\gamma}\rangle(N \times \dot{\gamma}) + \langle\ddot{\gamma}, N\rangle N \\
&= \langle\ddot{\gamma}, N \times \dot{\gamma}\rangle(N \times \dot{\gamma}) + \langle\ddot{\gamma}, N\rangle N \\
&= \ddot{\gamma}^{\tan} + \ddot{\gamma}^{\text{norm}}.
\end{aligned}$$

Definition 7.4. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $\gamma : I \rightarrow M$ be a curve on M parametrised by arclength. Then we define the **geodesic curvature** $k_g : I \rightarrow \mathbb{R}$ of γ by

$$\kappa_g(s) = \langle\ddot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s)\rangle.$$

The set $\{\dot{\gamma}(s), N(\gamma(s)) \times \dot{\gamma}(s)\}$ is an orthonormal basis for the tangent plane $T_{\gamma(s)}M$ of M at $\gamma(s)$. The curve $\gamma : I \rightarrow M$ is parametrised by arclength so the second derivative is perpendicular to $\dot{\gamma}$. This means that

$$\kappa_g(s)^2 = |\ddot{\gamma}(s)^{\tan}|^2,$$

so the geodesic curvature is therefore a measure of how far the curve is from being a geodesic.

Corollary 7.5. Let M be an oriented regular surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$ and $\gamma : I \rightarrow M$ be a curve on M parametrised by arclength. Let $\kappa : I \rightarrow \mathbb{R}$ be the curvature of γ as a curve in \mathbb{R}^3 and $\kappa_n, \kappa_g : I \rightarrow \mathbb{R}$ be the normal and geodesic curvatures, respectively. Then

$$\kappa(s)^2 = \kappa_g(s)^2 + \kappa_n(s)^2.$$

PROOF. This is a direct consequence of the orthogonal decomposition

$$\ddot{\gamma}(s) = \ddot{\gamma}(s)^{\text{tan}} + \ddot{\gamma}(s)^{\text{norm}}.$$

□

Example 7.6. Let $\gamma = (r, 0, z) : I \rightarrow \mathbb{R}^3$ be a differentiable curve in the (x, z) -plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then $X : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix}$$

parametrises a surface of revolution M . The tangent plane is generated by the vectors

$$X_u = \begin{bmatrix} \dot{r}(u) \cos v \\ \dot{r}(u) \sin v \\ \dot{z}(u) \end{bmatrix}, \quad X_v = \begin{bmatrix} -r(u) \sin v \\ r(u) \cos v \\ 0 \end{bmatrix}.$$

For a fixed $v \in \mathbb{R}$ the curve $\gamma_1 : I \rightarrow M$ with

$$\gamma_1(u) = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix}$$

parametrises a **meridian** on M by arclength. It is easily seen that

$$\langle \ddot{\gamma}_1, X_u \rangle = \langle \ddot{\gamma}_1, X_v \rangle = 0.$$

This means that $\ddot{\gamma}_1^{\text{tan}} = 0$, so $\gamma_1 : I \rightarrow M$ is a geodesic.

For a fixed $u \in \mathbb{R}$ the curve $\gamma_2 : I \rightarrow M$ with

$$\gamma_2(v) = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix}$$

parametrises a **parallel** on M . A simple calculation yields

$$\langle \gamma_2'', X_u \rangle = -\dot{r}(u)r(u) \quad \text{and} \quad \langle \gamma_2'', X_v \rangle = 0.$$

This means that $\gamma_2 : I \rightarrow M$ is a geodesic if and only if $\dot{r}(u) = 0$.

The next result states the important **geodesic equations**.

Theorem 7.7. *Let M be a regular surface in \mathbb{R}^3 and $X : U \rightarrow M$ be a local parametrisation of M with*

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX].$$

If $(u, v) : I \rightarrow U$ is a C^2 -curve in U then the composition

$$\gamma = X \circ (u, v) : I \rightarrow X(U)$$

is a geodesic on M if and only if

$$\begin{aligned} \frac{d}{dt}(Eu' + Fv') &= \frac{1}{2}(E_u(u')^2 + 2F_u u'v' + G_u(v')^2), \\ \frac{d}{dt}(Fu' + Gv') &= \frac{1}{2}(E_v(u')^2 + 2F_v u'v' + G_v(v')^2). \end{aligned}$$

PROOF. The tangent vector of the curve $(u, v) : I \rightarrow U$ is given by $(u', v') = u'e_1 + v'e_2$ so for the tangent γ' of γ we have

$$\begin{aligned} \gamma' &= dX \cdot (u'e_1 + v'e_2) \\ &= u' dX \cdot e_1 + v' dX \cdot e_2 \\ &= u'X_u + v'X_v. \end{aligned}$$

Following the definition we see that $\gamma : I \rightarrow X(U)$ is a geodesic if and only if

$$\langle \gamma'', X_u \rangle = 0 \quad \text{and} \quad \langle \gamma'', X_v \rangle = 0.$$

The first equation gives

$$\begin{aligned} 0 &= \left\langle \frac{d}{dt}(u'X_u + v'X_v), X_u \right\rangle \\ &= \frac{d}{dt} \langle u'X_u + v'X_v, X_u \rangle - \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle \\ &= \frac{d}{dt}(Eu' + Fv') - \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle. \end{aligned}$$

This implies that

$$\begin{aligned} &\frac{d}{dt}(Eu' + Fv') \\ &= \frac{d}{dt} \langle u'X_u + v'X_v, X_u \rangle \\ &= \langle u'X_u + v'X_v, \frac{d}{dt}X_u \rangle \\ &= \langle u'X_u + v'X_v, u'X_{uu} + v'X_{uv} \rangle \\ &= (u')^2 \langle X_u, X_{uu} \rangle + u'v'(\langle X_u, X_{uv} \rangle + \langle X_v, X_{uu} \rangle) + (v')^2 \langle X_v, X_{uv} \rangle \\ &= \frac{1}{2}E_u(u')^2 + F_u u'v' + \frac{1}{2}G_u(v')^2. \end{aligned}$$

This gives us the first geodesic equation. The second one is obtained in exactly the same way. \square

Theorem 7.7 characterises geodesics as solutions to a second order non-linear system of ordinary differential equations. For this we have the following existence result.

Theorem 7.8. *Let M be a regular surface in \mathbb{R}^3 , $p \in M$ and $Z \in T_pM$. Then there exists a unique, locally defined, geodesic*

$$\gamma : (-\epsilon, \epsilon) \rightarrow M$$

satisfying the initial conditions $\gamma(0) = p$ and $\gamma'(0) = Z$.

PROOF. The proof is based on a second order consequence of the well-known theorem of Picard-Lindelöf formulated here as Fact 7.9. \square

Fact 7.9. *Let $f : U \rightarrow \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^{2n}$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2|$$

for all $(t, y_1), (t, y_2) \in U$. If $(t_0, (x_0, x_1)) \in U$ and $x_0, x_1 \in \mathbb{R}^n$ then there exists a unique local solution $x : I \rightarrow \mathbb{R}^n$ to the following initial value problem

$$x''(t) = f(t, x(t), x'(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

The following notion of completeness of a regular surface M in \mathbb{R}^3 is closely related to the existence part of Theorem 7.8.

Definition 7.10. A regular surface M in \mathbb{R}^3 is said to be **complete** if for each point $p \in M$ and each tangent vector $Z \in T_pM$ there exists a geodesic $\gamma : \mathbb{R} \rightarrow M$ defined on the whole real line such that $\gamma(0) = p$ and $\gamma'(0) = Z$.

Proposition 7.11. *Let M_1 and M_2 be two regular surfaces in \mathbb{R}^3 and $\phi : M_1 \rightarrow M_2$ be an isometric differentiable map. Then $\gamma_1 : I \rightarrow M_1$ is a geodesic on M_1 if and only if the composition $\gamma_2 = \phi \circ \gamma_1 : I \rightarrow M_2$ is a geodesic on M_2*

PROOF. See Exercise 7.7 \square

The next result is the famous Theorem of Clairaut.

Theorem 7.12. *Let M be a regular surface of revolution and $\gamma : I \rightarrow M$ be a geodesic on M parametrised by arclength. Let $r : I \rightarrow \mathbb{R}^+$ be the function describing the distance between a point $\gamma(s)$ and the axis of rotation and $\theta : I \rightarrow \mathbb{R}$ be the angle between $\dot{\gamma}(s)$ and the meridian through $\gamma(s)$. Then the product $r(s) \sin \theta(s)$ is constant along the geodesic.*

PROOF. Let the surface M be parametrised by $X : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$X(u, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(u) \\ 0 \\ z(u) \end{bmatrix} = \begin{bmatrix} r(u) \cos v \\ r(u) \sin v \\ z(u) \end{bmatrix},$$

where $(r, 0, z) : I \rightarrow \mathbb{R}^3$ is a differentiable curve in the (x, z) -plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$. Then

$$X_u = \begin{bmatrix} \dot{r}(u) \cos v \\ \dot{r}(u) \sin v \\ \dot{z}(u) \end{bmatrix}, \quad X_v = \begin{bmatrix} -r(u) \sin v \\ r(u) \cos v \\ 0 \end{bmatrix}$$

give

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} 1 & 0 \\ 0 & r(u)^2 \end{bmatrix}$$

so the set

$$\left\{ X_u, \frac{1}{r(u)} X_v \right\}$$

is an orthonormal basis for the tangent plane of M at $X(u, v)$. This means that the tangent $\dot{\gamma}(s)$ of the geodesic $\gamma : I \rightarrow M$ can be written as

$$\dot{\gamma}(s) = \cos \theta(s) \cdot X_u(u(s), v(s)) + \sin \theta(s) \cdot \frac{1}{r(u(s))} X_v(u(s), v(s)),$$

where $r(u(s))$ is the distance to the axes of revolution and $\theta(s)$ is the angle between $\dot{\gamma}(s)$ and the tangent $X_u(u(s), v(s))$ to the meridian. It follows that

$$\begin{aligned} X_u \times \dot{\gamma} &= X_u \times \left(\cos \theta \cdot X_u + \frac{\sin \theta}{r} \cdot X_v \right) \\ &= \frac{\sin \theta}{r} \cdot (X_u \times X_v) \end{aligned}$$

but also

$$\begin{aligned} X_u \times \dot{\gamma} &= X_u \times (\dot{u} \cdot X_u + \dot{v} \cdot X_v) \\ &= \dot{v} \cdot (X_u \times X_v). \end{aligned}$$

Hence

$$r(u(s))^2 \dot{v}(s) = r(u(s)) \sin \theta(s).$$

If we substitute the relations $E = 1$, $F = 0$, $G = r(u)^2$ and $u(s) = s$ into the second geodesic equation

$$\frac{d}{ds}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

we obtain

$$\frac{d}{ds}(r(s)^2\dot{v}(s)) = 0.$$

This tells us that the product $r(s) \sin \theta(s)$ is constant since

$$\frac{d}{ds}(r(s) \sin \theta(s)) = \frac{d}{ds}(r(s)^2\dot{v}(s)) = 0.$$

□

Example 7.13. Let M be a surface of revolution parametrised by $X : I \times \mathbb{R} \rightarrow M$ with

$$X(s, v) = \begin{bmatrix} \cos v & -\sin v & 0 \\ \sin v & \cos v & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} r(s) \\ 0 \\ z(s) \end{bmatrix} = \begin{bmatrix} r(s) \cos v \\ r(s) \sin v \\ z(s) \end{bmatrix}.$$

Here $(r, 0, z) : I \rightarrow \mathbb{R}^3$ is a differentiable curve in the (x, z) -plane such that $r(s) > 0$ and $\dot{r}(s)^2 + \dot{z}(s)^2 = 1$ for all $s \in I$.

In Example 5.11 we have proved that the Gaussian curvature K of M satisfies the equation

$$\ddot{r}(s) + K(s) \cdot r(s) = 0.$$

If we put $K \equiv -1$ and solve this linear ordinary differential equation we get the general solution $r(s) = ae^s + be^{-s}$. By the particular choice of $r, z : \mathbb{R}^+ \rightarrow \mathbb{R}$ satisfying

$$r(s) = e^{-s} \quad \text{and} \quad z(s) = \int_0^s \sqrt{1 - e^{-2t}} dt$$

we get a parametrisation $X : \mathbb{R}^+ \times \mathbb{R} \rightarrow M$ of the famous **pseudo-sphere**. The corresponding first fundamental form is

$$\begin{bmatrix} E_X & F_X \\ F_X & G_X \end{bmatrix} = [DX]^t \cdot [DX] = \begin{bmatrix} 1 & 0 \\ 0 & r(s)^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2s} \end{bmatrix}.$$

For convenience we introduce a new variable u satisfying

$$s(u) = \log u, \quad \text{or equivalently,} \quad u(s) = e^s.$$

This gives us a new parametrisation $Y : (1, \infty) \times \mathbb{R} \rightarrow M$ of the pseudo-sphere where $Y(u, v) = X(s(u), v)$. Then the chain rule gives

$$Y_u = s_u X_s = \frac{1}{u} X_s, \quad Y_v = X_v$$

and we get the following first fundamental form for Y

$$\begin{bmatrix} E_Y & F_Y \\ F_Y & G_Y \end{bmatrix} = [DY]^t \cdot [DY] = \frac{1}{u^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The corresponding metric satisfies

$$ds^2 = \frac{1}{u^2}(dv^2 + du^2).$$

It is clear that this extends to a metric defined in the upper half plane

$$H^2 = \{(v, u) \in \mathbb{R}^2 \mid u > 0\}.$$

This is called the **hyperbolic metric**. The hyperbolic plane (H^2, ds^2) is very interesting both for its rich geometry but also for its historic importance. It is a model for the famous **non-Euclidean geometry**.

We shall now determine the geodesics in the hyperbolic plane. Let

$$\gamma = (v, u) : I \rightarrow H^2$$

be a geodesic parametrised by arclength. Then $\dot{\gamma} = (\dot{v}, \dot{u})$ and

$$|\dot{\gamma}|_{H^2}^2 = ds^2(\dot{\gamma}, \dot{\gamma}) = \frac{1}{u^2}(\dot{v}^2 + \dot{u}^2) = 1,$$

or equivalently, $\dot{v}^2 + \dot{u}^2 = u^2$. If we now substitute the relations

$$E = \frac{1}{u^2}, \quad F = 0 \quad \text{and} \quad G = \frac{1}{u^2}$$

into the second geodesic equation

$$\frac{d}{ds}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

we obtain

$$\frac{d}{ds} \left(\frac{\dot{v}(s)}{u(s)^2} \right) = 0.$$

This implies that there exists a real constant R such that

$$\frac{dv}{ds} = \dot{v} = u^2 R.$$

If $R = 0$ we see that $\dot{v} = 0$ so the function v is constant. This means that the vertical lines $v = v_0$ in the upper half plane H^2 are geodesics. In this case, they are parametrised by $\gamma(s) = (e^s, v_0)$.

If $R \neq 0$ then we have

$$u^4 R^2 + \dot{u}^2 = u^2,$$

or equivalently,

$$\frac{du}{ds} = \dot{u} = \pm u \sqrt{1 - R^2 u^2}.$$

This means that

$$\frac{dv}{du} = \dot{v}/\dot{u} = \pm \frac{Ru}{\sqrt{1 - R^2 u^2}},$$

or equivalently,

$$dv = \pm \frac{Ru}{\sqrt{1 - R^2u^2}} du.$$

We integrate this to

$$R(v - v_0) = \pm \sqrt{1 - R^2u^2}$$

which implies

$$(v - v_0)^2 + u^2 = \frac{1}{R^2}.$$

This means that the geodesic is a half circle in H^2 with centre at $(v_0, 0)$ and radius $1/R$.

Deep Result 7.14 (David Hilbert 1901). *There does not exist an isometric embedding of the hyperbolic plane H^2 into the standard Euclidean \mathbb{R}^3 .*

We now characterise the geodesics as the critical points of the length functional. For this we need the following two definitions.

Definition 7.15. Let M be a regular surface in \mathbb{R}^3 and $\gamma : I \rightarrow M$ be a C^2 -curve on M . A **variation** of γ is a C^2 -map

$$\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$$

such that for each $t \in I$ we have $\Phi_0(t) = \Phi(0, t) = \gamma(t)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is said to be **proper** if for all $r \in (-\epsilon, \epsilon)$ we have $\Phi_r(a) = \gamma(a)$ and $\Phi_r(b) = \gamma(b)$.

Definition 7.16. Let M be a regular surface in \mathbb{R}^3 and $\gamma : I \rightarrow M$ be a C^2 -curve on M . For every compact subinterval $[a, b]$ of I we define the **length functional** $L_{[a,b]}$ by

$$L_{[a,b]}(\gamma) = \int_a^b |\gamma'(t)| dt.$$

A C^2 -curve $\gamma : I \rightarrow M$ is said to be a **critical point** for the length functional if every **proper variation** Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dr}(L_{[a,b]}(\Phi_r))|_{r=0} = 0.$$

Theorem 7.17. *Let $\gamma : I = [a, b] \rightarrow M$ be a C^2 -curve parametrised by arclength. Then γ is a critical point for the length functional if and only if it is a geodesic.*

PROOF. Let $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$ with $\Phi : (r, t) \mapsto \Phi(r, t)$ be a proper variation of $\gamma : I \rightarrow M$. Then, since $I = [a, b]$ is compact, we have

$$\begin{aligned}
& \frac{d}{dr} (L_{[a,b]}(\Phi_r))|_{r=0} \\
&= \frac{d}{dr} \left(\int_a^b |\dot{\gamma}_r(t)| dt \right) |_{r=0} \\
&= \int_a^b \frac{d}{dr} \left(\sqrt{\left\langle \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t} \right\rangle} \right) |_{r=0} dt \\
&= \int_a^b \left(\left\langle \frac{\partial^2 \Phi}{\partial r \partial t}, \frac{\partial \Phi}{\partial t} \right\rangle / \sqrt{\left\langle \frac{\partial \Phi}{\partial t}, \frac{\partial \Phi}{\partial t} \right\rangle} \right) |_{r=0} dt \\
&= \int_a^b \left\langle \frac{\partial^2 \Phi}{\partial t \partial r}, \frac{\partial \Phi}{\partial t} \right\rangle |_{r=0} dt \\
&= \int_a^b \left(\frac{d}{dt} \left(\left\langle \frac{\partial \Phi}{\partial r}, \frac{\partial \Phi}{\partial t} \right\rangle \right) - \left\langle \frac{\partial \Phi}{\partial r}, \frac{\partial^2 \Phi}{\partial t^2} \right\rangle \right) |_{r=0} dt \\
&= \left[\left\langle \frac{\partial \Phi}{\partial r}(0, t), \frac{\partial \Phi}{\partial t}(0, t) \right\rangle \right]_a^b - \int_a^b \left\langle \frac{\partial \Phi}{\partial r}(0, t), \frac{\partial^2 \Phi}{\partial t^2}(0, t) \right\rangle dt.
\end{aligned}$$

The variation is proper, so

$$\frac{\partial \Phi}{\partial r}(0, a) = \frac{\partial \Phi}{\partial r}(0, b) = 0.$$

Furthermore

$$\frac{\partial^2 \Phi}{\partial t^2}(0, t) = \ddot{\gamma}(t),$$

so

$$\frac{d}{dr} (L_{[a,b]}(\Phi_r))|_{r=0} = - \int_a^b \left\langle \frac{\partial \Phi}{\partial r}(0, t), \ddot{\gamma}(t)^{\text{tan}} \right\rangle dt.$$

The last integral vanishes for every proper variation Φ of γ if and only if $\ddot{\gamma}(t)^{\text{tan}} = 0$ for all $t \in I$ i.e. γ is a geodesic. \square

Let M be a regular surface in \mathbb{R}^3 , $p \in M$ and

$$T_p^1 M = \{e \in T_p M \mid |e| = 1\}$$

be the unit circle in the tangent plane $T_p M$. Then every non-zero tangent vector $Z \in T_p M$ can be written as

$$Z = r_Z \cdot e_Z,$$

where $r_Z = |Z|$ and $e_Z = Z/|Z| \in T_p^1 M$. For a unit tangent vector $e \in T_p^1 M$ let

$$\gamma_e : (-a_e, b_e) \rightarrow M$$

be the maximal geodesic such that $a_e, b_e \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_e(0) = p$ and $\dot{\gamma}_e(0) = e$. It can be shown that the real number

$$\epsilon_p = \inf\{a_e, b_e \mid e \in T_p^1 M\}.$$

is positive so the open ball

$$B_{\epsilon_p}^2(0) = \{Z \in T_p M \mid |Z| < \epsilon_p\}$$

is non-empty. The **exponential map** $\exp_p : B_{\epsilon_p}^2(0) \rightarrow M$ at p is defined by

$$\exp_p : Z \mapsto \begin{cases} p & \text{if } Z = 0 \\ \gamma_{e_Z}(r_Z) & \text{if } Z \neq 0. \end{cases}$$

Note that for a unit tangent $e \in T_p^1 M$ the line segment

$$\lambda_e : (-\epsilon_p, \epsilon_p) \rightarrow T_p M$$

with $\lambda_e : t \mapsto t \cdot e$ is mapped onto the geodesic γ_e i.e. locally we have $\gamma_e(t) = \exp_p \circ \lambda_e(t)$.

One can prove that the map \exp_p is differentiable and it follows from its definition that the differential

$$d(\exp_p)_0 : T_p M \rightarrow T_p M$$

is the identity map for the tangent plane $T_p M$. Then Theorem 4.28 tells us that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B_{r_p}^2(0)$ and $V_p = \exp_p(U_p)$ then the restriction

$$\exp_p|_{U_p} : U_p \rightarrow V_p,$$

of the exponential map \exp_p at p to U_p , is a diffeomorphism parametrising the open subset V_p of M .

Example 7.18. Let S^2 be the unit sphere in \mathbb{R}^3 and $p = (1, 0, 0)$ be the north pole. Then the unit circle in the tangent plane $T_p S^2$ is given by

$$T_p^1 S^2 = \{(0, \cos \theta, \sin \theta) \mid \theta \in \mathbb{R}\}.$$

The exponential map $\exp_p : T_p S^2 \rightarrow S^2$ of S^2 at p is defined by

$$\exp_p : s(0, \cos \theta, \sin \theta) \mapsto \cos s \cdot (1, 0, 0) + \sin s \cdot (0, \cos \theta, \sin \theta).$$

This is clearly injective on the open ball

$$B_{\pi}^2(0) = \{Z \in T_p S^2 \mid |Z| < \pi\}$$

and the geodesic

$$\gamma : s \mapsto \exp_p(s(0, \cos \theta, \sin \theta))$$

is the shortest path between p and $\gamma(r)$ as long as $r < \pi$.

Theorem 7.19. *Let M be a regular surface in \mathbb{R}^3 . Then the geodesics are locally the shortest paths between their endpoints.*

PROOF. For $p \in M$, choose $r > 0$ such that the restriction

$$\phi = \exp_p|_U : U \rightarrow \exp_p(U),$$

of the exponential map at p to the open disc $U = B_r^2(0)$ in the tangent plane T_pM , is a diffeomorphism onto the image $\exp_p(U)$. Then define the metric ds^2 on U such that ϕ is an isometry i.e. for any two vector fields X, Y on U we have

$$ds^2(X, Y) = \langle d\phi(X), d\phi(Y) \rangle.$$

It then follows from the construction of the exponential map, that the geodesics in U through the origin $0 = \phi^{-1}(p)$ are exactly the lines

$$\lambda_Z : t \mapsto t \cdot Z$$

where $Z \in T_pM$.

Now let $q \in U \setminus \{0\}$ and $\lambda_q : [0, 1] \rightarrow U$ be the curve $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \rightarrow U$ be an arbitrary curve in U such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along the curve σ we define two vector fields $\hat{\sigma}$ and σ'_{rad} by

$$\hat{\sigma} : t \mapsto \sigma(t) \quad \text{and} \quad \sigma'_{\text{rad}} : t \mapsto \frac{ds^2(\sigma'(t), \sigma(t))}{ds^2(\sigma(t), \sigma(t))} \cdot \sigma(t).$$

Note that $\sigma'_{\text{rad}}(t)$ is the radial projection of the tangent $\sigma'(t)$ of the curve $\sigma(t)$ onto the line generated by the vector $\sigma(t)$. This means that

$$\begin{aligned} |\sigma'_{\text{rad}}(t)|^2 &= \frac{ds^2(\sigma'(t), \sigma(t))^2 \cdot ds^2(\sigma(t), \sigma(t))}{ds^2(\sigma(t), \sigma(t))^2} \\ &= \frac{ds^2(\sigma'(t), \sigma(t))^2}{ds^2(\sigma(t), \sigma(t))}, \end{aligned}$$

so

$$|\sigma'_{\text{rad}}(t)| = \frac{|ds^2(\sigma'(t), \sigma(t))|}{|\sigma(t)|}.$$

Further

$$\begin{aligned} \frac{d}{dt}|\sigma(t)| &= \frac{d}{dt}\sqrt{ds^2(\sigma(t), \sigma(t))} \\ &= \frac{2 \cdot ds^2(\sigma'(t), \sigma(t))}{2 \cdot \sqrt{ds^2(\sigma(t), \sigma(t))}} \\ &= \frac{ds^2(\sigma'(t), \sigma(t))}{|\sigma(t)|}. \end{aligned}$$

Combining these two relations we obtain

$$|\sigma'_{\text{rad}}(t)| \geq \frac{d}{dt}|\sigma(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\sigma'(t)| dt \\ &\geq \int_0^1 |\sigma'_{\text{rad}}(t)| dt \\ &\geq \int_0^1 \frac{d}{dt}|\sigma(t)| dt \\ &= |\sigma(1)| - |\sigma(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact λ_q is the shortest path connecting p and q . \square

Exercises

Exercise 7.1. Describe the geodesics on the circular cylinder

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}.$$

Exercise 7.2. Find four different geodesics, as geometric curves, passing through the point $p = (1, 0, 0)$ on the one-sheeted hyperboloid

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}.$$

Exercise 7.3. Find four different geodesics passing through the point $p = (0, 0, 0)$ on the surface

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid xy(x^2 - y^2) = z\}.$$

Exercise 7.4. Let $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the parametrised surface in \mathbb{R}^3 given by

$$X(u, v) = (u \cos v, u \sin v, v).$$

Determine for which values of $\alpha \in \mathbb{R}$ the curve $\gamma_\alpha : \mathbb{R} \rightarrow M$ with

$$\gamma_\alpha(t) = X(t, \alpha t) = (t \cos(\alpha t), t \sin(\alpha t), \alpha t)$$

is a geodesic on M .

Exercise 7.5. Let $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be the parametrised surface in \mathbb{R}^3 given by

$$X(u, v) = (u, v, \sin u \cdot \sin v).$$

Determine for which values of $\theta \in \mathbb{R}$ the curve $\gamma_\theta : \mathbb{R} \rightarrow M$ with

$$\gamma_\theta(t) = X(t \cdot \cos \theta, t \cdot \sin \theta)$$

is a geodesic on M .

Exercise 7.6. Let $\gamma : I \rightarrow \mathbb{R}^3$ be a regular curve, parametrised by arclength, with non-vanishing curvature and n, b denote the principal normal and the binormal of γ , respectively. Let $r \in \mathbb{R}^+$ such that the r -tube M around γ given by $X : I \times \mathbb{R} \rightarrow \mathbb{R}^3$ with

$$X(s, \theta) \mapsto \gamma(s) + r(\cos \theta \cdot n(s) + \sin \theta \cdot b(s))$$

is a regular surface. Show that for each $s \in I$ the circle $\gamma_s : \mathbb{R} \rightarrow \mathbb{R}^3$, with $\gamma_s(\theta) = X(s, \theta)$, is a geodesic on the surface.

Exercise 7.7. Find a proof of Proposition 7.11.

Exercise 7.8. Let M be the regular surface in \mathbb{R}^3 parametrised by $X : \mathbb{R} \times (-1, 1) \rightarrow \mathbb{R}^3$ with

$$X(u, v) = 2(\cos u, \sin u, 0) + v \sin(u/2)(0, 0, 1) \\ + v \cos(u/2)(\cos u, \sin u, 0).$$

Determine whether the curve $\gamma : \mathbb{R} \rightarrow M$ defined by

$$\gamma : t \mapsto X(t, 0)$$

is a geodesic or not. Is the surface M orientable ?

Exercise 7.9. Let M be the regular surface in \mathbb{R}^3 given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1\}.$$

Show that $v = (-1, 3, -\sqrt{2})$ is a tangent vector to M at $p = (\sqrt{2}, 0, 1)$. Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{R} \rightarrow M$ be the geodesic which is uniquely determined by $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Determine the value

$$\inf_{s \in \mathbb{R}} \gamma_3(s).$$

Exercise 7.10. Let M be a regular surface in \mathbb{R}^3 such that every geodesic $\gamma : I \rightarrow M$ is contained in a plane. Show that M is either contained in a plane or in a sphere.

Exercise 7.11. The regular surface M in \mathbb{R}^3 is parametrised by $X : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ with

$$X : (u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u).$$

Let $\gamma = (\gamma_1, \gamma_2, \gamma_3) : \mathbb{R} \rightarrow M$ be the geodesic on M satisfying

$$\gamma(0) = (3, 0, 0) \quad \text{and} \quad \gamma'(0) = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Determine the value

$$\inf_{s \in \mathbb{R}} (\gamma_1^2(s) + \gamma_2^2(s)).$$

The Gauss-Bonnet Theorems

In this chapter we prove three different versions of the famous Gauss-Bonnet theorem.

Theorem 8.1. *Let M be an oriented regular C^3 -surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Let $X : U \rightarrow M$ be a local parametrisation of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \rightarrow X(U)$ parametrise a regular, simple, closed and positively oriented C^2 -curve on $X(U)$ by arclength. Let $\text{Int}(\gamma)$ be the interior of γ and $\kappa_g : \mathbb{R} \rightarrow \mathbb{R}$ be its geodesic curvature. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s) ds = 2\pi - \int_{\text{Int}(\gamma)} K dA,$$

where K is the Gaussian curvature of M .

PROOF. Let $\{Z, W\}$ be the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$. Along the curve $\gamma : \mathbb{R} \rightarrow X(U)$ we define the angle $\theta : \mathbb{R} \rightarrow \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies

$$\dot{\gamma}(s) = \cos \theta(s)Z(s) + \sin \theta(s)W(s).$$

Then

$$\begin{aligned} N \times \dot{\gamma} &= N \times (\cos \theta Z + \sin \theta W) \\ &= \cos \theta (N \times Z) + \sin \theta (N \times W) \\ &= \cos \theta W - \sin \theta Z. \end{aligned}$$

and for the second derivative $\ddot{\gamma}$ we have

$$\ddot{\gamma} = \dot{\theta}(-\sin \theta Z + \cos \theta W) + \cos \theta \dot{Z} + \sin \theta \dot{W}$$

so the geodesic curvature satisfies

$$\begin{aligned} \kappa_g &= \langle N \times \dot{\gamma}, \ddot{\gamma} \rangle \\ &= \dot{\theta} \langle -\sin \theta Z + \cos \theta W, -\sin \theta Z + \cos \theta W \rangle \\ &\quad + \langle -\sin \theta Z + \cos \theta W, \cos \theta \dot{Z} + \sin \theta \dot{W} \rangle \\ &= \dot{\theta} - \langle Z, \dot{W} \rangle. \end{aligned}$$

If we integrate the geodesic curvature $\kappa_g : \mathbb{R} \rightarrow \mathbb{R}$ over one period we get

$$\begin{aligned} \int_0^L \kappa_g(s) ds &= \int_0^L \dot{\theta}(s) ds - \int_0^L \langle Z(s), \dot{W}(s) \rangle ds \\ &= \theta(L) - \theta(0) - \int_0^L \langle Z(s), \dot{W}(s) \rangle ds \\ &= 2\pi - \int_0^L \langle Z(s), \dot{W}(s) \rangle ds. \end{aligned}$$

Let $\alpha = X^{-1} \circ \gamma : \mathbb{R} \rightarrow U$ be the inverse image of the curve γ in the simply connected parameter region U . The curve α is simple, closed and positively oriented. Utilising Lemma 6.3 and Green's theorem we now get

$$\begin{aligned} \int_0^L \langle Z(s), \dot{W}(s) \rangle ds &= \int_{\gamma} \langle Z, \dot{u}W_u + \dot{v}W_v \rangle ds \\ &= \int_{\alpha} \langle Z, W_u \rangle du + \langle Z, W_v \rangle dv \\ &= \int_{\text{Int}(\alpha)} (\langle Z, W_v \rangle_u - \langle Z, W_u \rangle_v) dudv \\ &= \int_{\text{Int}(\alpha)} (\langle Z_u, W_v \rangle + \langle Z, W_{uv} \rangle \\ &\quad - \langle Z_v, W_u \rangle - \langle Z, W_{vu} \rangle) dudv \\ &= \int_{\text{Int}(\alpha)} (\langle Z_u, W_v \rangle - \langle Z_v, W_u \rangle) dudv \\ &= \int_{\text{Int}(\alpha)} K \sqrt{EG - F^2} dudv \\ &= \int_{\text{Int}(\gamma)} K dA. \end{aligned}$$

This proves the statement. \square

Corollary 8.2. *Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ parametrise a regular, simple, closed and positively oriented C^2 -curve by arclength. If $L \in \mathbb{R}^+$ is the period of γ then*

$$\int_0^L \kappa_g(s) ds = 2\pi,$$

where $\kappa_g : \mathbb{R} \rightarrow \mathbb{R}$ is the geodesic curvature of γ .

The reader should compare the result of Corollary 8.2 with Exercise 2.7 and Exercise 3.7.

Definition 8.3. Let M be a regular surface in \mathbb{R}^3 . A periodic continuous curve $\gamma : \mathbb{R} \rightarrow M$ of period $L \in \mathbb{R}^+$ is said to parametrise a **simple piecewise regular polygon** on M if

- (1) $\gamma(t) = \gamma(t^*)$ if and only if $(t - t^*) \in L \cdot \mathbb{Z}$,
- (2) there exists a subdivision

$$0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = L$$

of the interval $[0, L]$ such that $\gamma|_{(t_i, t_{i+1})} : (t_i, t_{i+1}) \rightarrow M$ is smooth for $i = 0, \dots, n-1$, and

- (3) the one-sided derivatives

$$\dot{\gamma}^-(t_i) = \lim_{t \rightarrow t_i^-} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}, \quad \dot{\gamma}^+(t_i) = \lim_{t \rightarrow t_i^+} \frac{\gamma(t_i) - \gamma(t)}{t_i - t}$$

exist, are non-zero and not parallel.

The next result generalises Theorem 8.1

Theorem 8.4. Let M be an oriented regular C^3 -surface in \mathbb{R}^3 with Gauss map $N : M \rightarrow S^2$. Let $X : U \rightarrow M$ be a local parametrisation of M such that $X(U)$ is connected and simply connected. Let $\gamma : \mathbb{R} \rightarrow X(U)$ parametrise a positively oriented, simple and piecewise regular C^2 -polygon on M by arclength. Let $\text{Int}(\gamma)$ be the interior of γ and $\kappa_g : \mathbb{R} \rightarrow \mathbb{R}$ be its geodesic curvature on each regular piece. If $L \in \mathbb{R}^+$ is the period of γ then

$$\int_0^L \kappa_g(s) ds = \sum_{i=1}^n \alpha_i - (n-2)\pi - \int_{\text{Int}(\gamma)} K dA$$

where K is the Gaussian curvature of M and $\alpha_1, \dots, \alpha_n$ are the inner angles at the n corner points.

PROOF. Let $\{Z, W\}$ the orthonormal basis which we obtain by applying the Gram-Schmidt process on the basis $\{X_u, X_v\}$. Let \mathcal{D} be the discrete subset of \mathbb{R} corresponding to the corner points of $\gamma(\mathbb{R})$. Along the regular arcs of $\gamma : \mathbb{R} \rightarrow X(U)$ we define an angle $\theta : \mathbb{R} \setminus \mathcal{D} \rightarrow \mathbb{R}$ such that the unit tangent vector $\dot{\gamma}$ satisfies

$$\dot{\gamma}(s) = \cos \theta(s)Z(s) + \sin \theta(s)W(s).$$

We have seen earlier that in this case the geodesic curvature is given by $\kappa_g = \dot{\theta} - \langle Z, \dot{W} \rangle$ and integration over one period gives

$$\int_0^L \kappa_g(s) ds = \int_0^L \dot{\theta}(s) ds - \int_0^L \langle Z(s), \dot{W}(s) \rangle ds.$$

As a consequence of Green's theorem we have

$$\int_0^L \langle Z(s), \dot{W}(s) \rangle ds = \int_{\text{Int}(\gamma)} K dA.$$

The integral over the derivative $\dot{\theta}$ splits up into integrals over each regular arc

$$\int_0^L \dot{\theta}(s) ds = \sum_{i=1}^n \int_{s_{i-1}}^{s_i} \dot{\theta}(s) ds$$

which measures the change of angle with respect to the orthonormal basis $\{Z, W\}$ along each arc. At each corner point the tangent jumps by the angle $(\pi - \alpha_i)$ where α_i is the corresponding inner angle. When moving around the curve once the changes along the arcs and the jumps at the corner points add up to 2π . Hence

$$2\pi = \int_0^L \dot{\theta}(s) ds + \sum_{i=1}^n (\pi - \alpha_i).$$

This proves the statement \square

Definition 8.5. A C^2 -polygon on a regular surface M in \mathbb{R}^3 is said to be **geodesic** if its edges are geodesics.

It should be noted that if the piecewise regular polygon in Theorem 8.4 is geodesic then the formula simplifies to

$$(1) \quad \sum_{i=1}^n \alpha_i = (n-2)\pi + \int_{\text{Int}(\gamma)} K dA.$$

This has some interesting consequences.

Example 8.6. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle in the flat Euclidean plane. As a direct consequence of equation (1) we have the following classical result

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi.$$

Example 8.7. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle Δ on the unit sphere S^2 with constant curvature $K \equiv 1$. Then equation (1) gives

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi + A(\Delta) > \pi.$$

Here $A(\Delta)$ is the area of the triangle.

Example 8.8. Let $\alpha_1, \alpha_2, \alpha_3$ be the angles of a geodesic triangle Δ in the hyperbolic plane H^2 with constant curvature $K \equiv -1$. Then equation (1) gives

$$\alpha_1 + \alpha_2 + \alpha_3 = \pi - A(\Delta) < \pi.$$

It follows from Theorem 7.8 that the angles are positive. This implies that the area must satisfy the inequality $A(\Delta) < \pi$.

We complete our journey with the global Gauss-Bonnet theorem.

Theorem 8.9. *Let M be an orientable and compact regular C^3 -surface in \mathbb{R}^3 . If K is the Gaussian curvature of M then*

$$\int_M K dA = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of the surface.

PROOF. Let $\mathcal{T} = \{T_1, \dots, T_F\}$ be a triangulation of the surface M such that each T_k is a geodesic triangle contained in the image $X_k(U_k)$ of a local parametrisation $X_k : U_k \rightarrow M$. Then the integral of the Gaussian curvature K over M splits

$$\int_M K dA = \sum_{k=1}^F \int_{T_k} K dA$$

into the sum of integrals over each triangle $T_k \in \mathcal{T}$.

According to Theorem 8.4 we now have

$$\int_{T_k} K dA = \sum_{i=1}^{n_k} \alpha_{ki} + (2 - n_k)\pi$$

for each triangle T_k . By adding these relations we obtain

$$\begin{aligned} \int_M K dA &= \sum_{k=1}^F \left((2 - n_k)\pi + \sum_{i=1}^{n_k} \alpha_{ki} \right) \\ &= 2\pi F - 2\pi E + \sum_{k=1}^F \sum_{i=1}^{n_k} \alpha_{ki} \\ &= 2\pi(F - E + V). \end{aligned}$$

This proves the statement. □

Exercises

Exercise 8.1. Let M be a regular surfaces in \mathbb{R}^3 diffeomorphic to the torus. Show that there exists a point $p \in M$ where the Gaussian curvature $K(p)$ is negative.

Exercise 8.2. The regular surface M in \mathbb{R}^3 is given by

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 - z^2 = 1 \text{ and } -1 < z < 1\}.$$

Determine the value of the integral

$$\int_M K dA,$$

where K is the Gaussian curvature of M .

Exercise 8.3. For $r \in \mathbb{R}^+$ let the surface Σ_r be given by

$$\Sigma_r = \{(x, y, z) \in \mathbb{R}^3 \mid z = \cos \sqrt{x^2 + y^2}, x^2 + y^2 < r^2, x, y > 0\}.$$

Determine the value of the integral

$$\int_{\Sigma_r} K dA,$$

where K is the Gaussian curvature of Σ_r .

Exercise 8.4. For $n \geq 1$ let M_n be the regular surface in \mathbb{R}^3 given by

$$M_n = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = (1 + z^{2n})^2, 0 < z < 1\}.$$

Determine the value of the integral

$$\int_{M_n} K dA,$$

where K is the Gaussian curvature of M_n .