LINEAR DIFFERENCE EQUATIONS

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(First Draft, November 1994)

In these notes we shall by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural, real and complex numbers, respectively. All the definitions and most of the results mentioned below can be formulated both for the real and the complex numbers. Here we have chosen to present the complex versions, but if the same also applies to the real case we have added the symbol $[\mathbb{R}]$ in the corresponding header.

Definition 0.1. $[\mathbb{R}]$ A **complex valued sequence** is a function $f : \mathbb{N} \to \mathbb{C}$. We denote by $\mathbb{C}^\infty$ the set of all complex valued sequences i.e.

$$\mathbb{C}^\infty = \{f|\ f : \mathbb{N} \to \mathbb{C}\}.$$

For two elements $f, g \in \mathbb{C}^\infty$ and $\alpha \in \mathbb{C}$ we define the addition $+$ and the product $\cdot$ by

i. $(f + g)(n) = f(n) + g(n)$ for all $n \in \mathbb{N}$,
ii. $(\alpha \cdot f)(n) = \alpha \cdot f(n)$ for all $n \in \mathbb{N}$.

Exercise 0.2. $[\mathbb{R}]$ Prove that the triple $(\mathbb{C}^\infty, +, \cdot)$ is a complex linear space.

Remark 0.3. $[\mathbb{R}]$ If $n \in \mathbb{N}$ and $f \in \mathbb{C}^\infty$ we often write f_n instead of $f(n)$.

Definition 0.4. $[\mathbb{R}]$ A **linear difference equation of degree k with constant complex coefficients** is an equation of the form

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = g_n, \tag{1}$$

where $c_0, \dots, c_k$ are complex numbers such that $c_0 \neq 0 \neq c_k$, $g \in \mathbb{C}^\infty$ is a sequence and $a_n, \dots, a_{n-k}$ unknown complex numbers. If $g \equiv 0$ then the equation is said to be **homogeneous**, but **non-homogeneous** otherwise.

Definition 0.5. [ℝ] A **solution** to equation (1) is a sequence $f \in \mathbb{C}^\infty$ such that

$$c_k f_n + c_{k-1} f_{n-1} + \cdots + c_0 f_{n-k} = g_n,$$

for all natural numbers $n \geq k$

1. The Homogeneous Case

Proposition 1.1. [ℝ] Let $c_0, \ldots, c_k$ be complex numbers, such that $c_0 \neq 0 \neq c_k$ and V be the set of all solutions to the homogeneous linear difference equation $c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0$ i.e.

$$V = \{f \in \mathbb{C}^\infty |\ c_k f_n + c_{k-1} f_{n-1} + \cdots + c_0 f_{n-k} = 0 \text{ for } n \geq k\}.$$

Then V is a linear subspace of $\mathbb{C}^\infty$.

Proof: [ℝ] Let $f, g \in \mathbb{C}^\infty$ be solutions to the equation and $\alpha, \beta \in \mathbb{C}$, then

$$\begin{aligned} & c_k(\alpha f_n + \beta g_n) + c_{k-1}(\alpha f_{n-1} + \beta g_{n-1}) + \cdots + c_0(\alpha f_{n-k} + \beta g_{n-k}) \\ = \ & \alpha(c_k f_n + c_{k-1} f_{n-1} + \cdots + c_0 f_{n-k}) + \beta(c_k g_n + c_{k-1} g_{n-1} + \cdots + c_0 g_{n-k}) \\ = \ & 0. \end{aligned}$$

This proves that for every $\alpha, \beta \in \mathbb{C}$ the sequence $(\alpha f + \beta g)$ is an element of V, which implies that V is a linear subspace of $\mathbb{C}^\infty$.

□

Proposition 1.2. [ℝ] Let $c_0, \ldots, c_k, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ be complex numbers, such that $c_0 \neq 0 \neq c_k$. Then the linear homogeneous difference equation

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0, \tag{2}$$

has a unique solution $f \in \mathbb{C}^\infty$ which satisfies the following initial conditions:

$$f_0 = \alpha_0, f_1 = \alpha_1, \ldots, f_{k-1} = \alpha_{k-1}.$$

In particular the linear space V of solutions to the homogeneous equation (2) can be described by k parameters, so $\dim(V) \leq k$.

Proof: [ℝ] We have assumed that $c_k \neq 0$, so for each $n \geq k$ we have

$$f_n = -(c_{k-1} f_{n-1} + \cdots + c_0 f_{n-k})/c_k.$$

The function values $f_0, f_1, \ldots, f_{k-1}$ are given by the initial conditions, so it follows from the induction principle that the sequence $f \in \mathbb{C}^\infty$ is uniquely determined. $\square$

Definition 1.3. [ℝ] For complex numbers $c_0, \ldots, c_k$, such that $c_0 \neq 0 \neq c_k$ we call

$$p(\lambda) = c_k\lambda^k + c_{k-1}\lambda^{k-1} + \cdots + c_1\lambda + c_0$$

the **characteristic polynomial** of the homogeneous linear difference equation

$$c_k a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = 0.$$

Proposition 1.4. [ℝ] Let λ_0 be a non-zero complex number. Then the sequence $f : n \mapsto \lambda_0^n$ is a solution to the homogeneous linear difference equation

$$c_k a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = 0.$$

if and only if λ_0 is a root of the corresponding characteristic polynomial.

Proof: The result follows directly from

$$\begin{aligned} & c_k f_n + c_{k-1} f_{n-1} + \cdots + c_1 f_{n-k+1} + c_0 f_{n-k} \\ = \; & c_k\lambda^n + c_{k-1}\lambda^{n-1} + \cdots + c_1\lambda^{n-k+1} + c_0\lambda^{n-k} \\ = \; & \lambda^{n-k}(c_k\lambda^k + \cdots + c_1\lambda + c_0). \end{aligned}$$

$\square$

The following result is well-known but rather messy to prove in its full generality. The reader should note that we have left out the symbol [ℝ].

Proposition 1.5. Let $c_0, \ldots, c_k, \alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ be complex numbers, such that $c_0 \neq 0 \neq c_k$ and $\lambda_1, \ldots, \lambda_t$ be the roots of the characteristic polynomial

$$p(\lambda) = (c_k\lambda^k + \cdots + c_1\lambda + c_0),$$

with multiplicities $m_1, \ldots, m_t$. Then the $k = m_1 + m_2 + \cdots + m_t$ sequences

$$\begin{array}{llll} f_{1,1} : n \mapsto \lambda_1^n & f_{1,2} : n \mapsto n\lambda_1^n & \ldots & f_{1,m_1} : n \mapsto n^{m_1-1}\lambda_1^n \\ f_{2,1} : n \mapsto \lambda_2^n & f_{2,2} : n \mapsto n\lambda_2^n & \ldots & f_{2,m_2} : n \mapsto n^{m_2-1}\lambda_2^n \\ & \vdots & & \\ f_{t,1} : n \mapsto \lambda_t^n & f_{t,2} : n \mapsto n\lambda_t^n & \ldots & f_{t,m_t} : n \mapsto n^{m_t-1}\lambda_t^n \end{array}$$

are linearly independent solutions to the homogeneous linear difference equation

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0.$$

Exercise 1.6. Prove Proposition (1.5) for the special cases when $k = 1, 2, 3$.

The next Theorem gives a nice description of the solution space in the complex case. For the real case see Theorem (1.10).

Theorem 1.7. *Let $c_0, \ldots, c_k$ be complex numbers, such that $c_0 \neq 0 \neq c_k$. Then the set V of all solutions to the homogeneous linear difference equation*

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0$$

is a k-dimensional complex linear subspace of $\mathbb{C}^\infty$.

Proof: This follows directly from Propositions (1.1), (1.2) and (1.5). □

Lemma 1.8. *Let the real numbers $c_0, c_1, \ldots, c_k$ satisfying $c_0 \neq 0 \neq c_k$ be the coefficients for the homogeneous linear difference equations*

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0. \tag{3}$$

Then the following conditions are equivalent

i. $f \in \mathbb{C}^\infty$ *is a solution to* (3),
ii. $\bar{f} \in \mathbb{C}^\infty$ *is a solution to* (3),
iii. $\mathrm{Re}\, f \in \mathbb{R}^\infty$ *and* $\mathrm{Im}\, f \in \mathbb{R}^\infty$ *are solutions to* (3).

Proof: The sequence $f \in \mathbb{C}^\infty$ is a solution to equation (3) if and only if $c_k f_n + c_{k-1} f_{n-1} + \cdots + c_0 f_{n-k} = 0$ for all $n \geq k$. By conjugating this equation and using the fact that the coefficients are all real we obtain the equivalent equation $c_k \bar{f}_n + c_{k-1} \bar{f}_{n-1} + \cdots + c_0 \bar{f}_{n-k} = 0$ which holds for all $n \geq k$. But this means that the sequence $\bar{f} \in \mathbb{C}^\infty$ is a solution to equation (3). From this we conclude that i. and ii. are equivalent.

As a direct consequence of the definitions of the real and imaginary parts of $f \in \mathbb{C}^\infty$ we have

$$\mathrm{Re}\, f = (f + \bar{f})/2 \qquad \mathrm{Im}\, f = (f - \bar{f})/2i \tag{4}$$

and

$$f = \mathrm{Re}\ f + i\ \mathrm{Im}\ f \qquad\qquad \bar{f} = \mathrm{Re}\ - i\ \mathrm{Im}\ f \tag{5}$$

It follows from the fact that the solution space V is complex linear and equations (4) that i. and ii. imply iii. Similarily we can conclude from equations (5) that iii. implies i. and ii.

□

Example 1.9. Let α and β be real numbers. Then the characteristic polynomial of the linear difference equation

$$a_n - 2\alpha a_{n-1} + (\alpha^2 + \beta^2) a_{n-2} = 0 \tag{6}$$

is given by

$$p(\lambda) = \lambda^2 - 2\alpha\lambda + (\alpha^2 + \beta^2).$$

This polynomial has the complex roots $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$. According to Proposition (1.5) the sequences $f : n \mapsto (\alpha + i\beta)^n$ and $\bar{f} : n \mapsto (\alpha - i\beta)^n$ form a basis for the two dimensional solution space V. Writing $\alpha + i\beta$ in polar form $re^{i\theta}$ we see that the two sequences are given by

$$f : n \mapsto r^n e^{in\theta} \qquad \text{and} \qquad \bar{f} : n \mapsto r^n e^{-in\theta}.$$

Following Lemma (1.8) we see that the real valued sequences

$$\mathrm{Re}\ f : n \mapsto r^n \cos n\theta \qquad \text{and} \qquad \mathrm{Im}\ f : n \mapsto r^n \sin n\theta$$

are linearily independent solutions to the homogeneous linear difference equation (6) with real coefficients.

The next result corresponds to Theorem (1.7) and gives a description for the solution space in the real case.

Theorem 1.10. Let $c_0, \dots, c_k$ be real numbers, such that $c_0 \neq 0 \neq c_k$. Then the set V of all solutions to the homogeneous linear difference equation

$$c_k a_n + c_{k-1} a_{n-1} + \cdots + c_0 a_{n-k} = 0$$

is a k-dimensional real linear subspace of $\mathbb{R}^\infty$.

Proof: The coefficient of the characteristic polynomial are all real, which means that its roots are either real or come in pairs $\{(\alpha+i\beta),(\alpha-i\beta)\}$. Then we can use the method of Example (1.9) on the independent solution of Proposition (1.5) to obtain k independent solutions in $\mathbb{R}^\infty$. It then follows from the real version of Propositions (1.1) and (1.2) that the solution space is a k-dimensional real subspace of $\mathbb{R}^\infty$. □

2. THE NON-HOMOGENEOUS CASE

Lemma 2.1. [$\mathbb{R}$] Let $c_0,\dots,c_k$ be complex numbers, such that $c_0\neq 0\neq c_k$ and $g\in\mathbb{C}^\infty$. If $h\in\mathbb{C}^\infty$ is a solution to the non-homogeneous equation

$$c_k a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = g_n,$$

then the following conditions are equivalent:

i. $f\in\mathbb{C}^\infty$ is a solution to the corresponding homogeneous equation,
ii. $(f+h)\in\mathbb{C}^\infty$ is a solution to the non-homogeneous equation.

Proof: This result follows directly from

$$\begin{aligned} & c_k(f+h)_n + c_{k-1}(f+h)_{n-1} + \cdots + c_0(f+h)_{n-k} - g_n \\ = \; & (c_kf_n + c_{k-1}f_{n-1} + \cdots + c_0f_{n-k}) + (c_kh_n + c_{k-1}h_{n-1} + \cdots + c_0h_{n-k}) - g_n \\ = \; & c_kf_n + c_{k-1}f_{n-1} + \cdots + c_0f_{n-k}. \end{aligned}$$

□

Theorem 2.2. [$\mathbb{R}$] Let $c_0,\dots,c_k$ be complex numbers, such that $c_0\neq 0\neq c_k$, $g\in\mathbb{C}^\infty$ be a sequence and $h\in\mathbb{C}^\infty$ be a solution to the non-homogeneous linear difference equation

$$c_k a_n + c_{k-1}a_{n-1} + \cdots + c_0 a_{n-k} = g_n.$$

Further let V be the solution space to the corresponding homogeneous equation. Then the set S_g of all solutions to the non-homogeneous equation is given by

$$S_g = h + V = \{h+f\in\mathbb{C}^\infty \mid f\in V\}.$$

Proof: This result is a direct consequence of Lemma (2.1). □

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