LUNDS UNIVERSITET

Matematik NF

Tentamensskrivning
Topologi
Fredagen den 21 mars 2003
Skrivtid: 14.00–19.00

Inga hjälpmedel. Använd institutionens papper, skriv bara på den ena sidan och högst en uppgift på varje papper. Fyll i omslaget fullständigt och skriv initialer på varje ark. Skriv tydligt. Ge klara och kortfattade motiveringar, rita gärna figur i förekommande fall.

1. Let (X, d) be a metric space and define $\tilde{d} : X \times X \to \mathbb{R}_0^+$ by $\tilde{d}(x, y) = d^{1/3}(x, y)$. Prove that $(X, \tilde{d})$ is a metric space homeomorphic to (X, d).

2. Consider on the set $\mathbb{N}$ of natural numbers the family τ_0 of sets $A \subset \mathbb{N}$ with the property that the complement of A, $\mathbb{N} \setminus A$, contains finitely many elements. Prove that τ_0 is the basis of a topological space $(\mathbb{N}, \tau)$. Is the resulting topology Hausdorff? Is it induced by a metric on $\mathbb{N}$?

3. Let (X, τ) be a topological space. Prove that if A, $B \subset X$ are connected sets with nonempty intersection, then their union $A \cup B$ is also connected. Show by means of a counterexample that this statement is not necessarily true if the sets are disjoint (that is, with no common elements). Is it always true that if A, $B \subset X$ are disjoint connected sets, their union is not connected?

4. Consider the metric space $C[0, 1]$ of all continuous functions $f : [0, 1] \to \mathbb{R}$, endowed with the metric

$$d(f, g) = \sup_{x \in [0,1]} |f(x) - g(x)|, \quad f, g \in C[0, 1].$$

Prove that the set of all polynomials with real coefficients in $[0, 1]$ and of degree less than 2003 is compact in $C[0, 1]$. Is the set of all polynomials with real coefficients belonging to the interval $[0, 1]$ a compact set in $C[0, 1]$?

5. Let (X, d) be a metric space.
(i) Prove that a pathwise connected set $A \subset X$ is connected.
(ii) In the case of the real line $\mathbb{R}$ equipped with its standard metric, show that pathwise connectedness is equivalent to connectedness.
(iii) In the case of the plane $\mathbb{R}^2$ equipped with its standard metric, prove that for open sets pathwise connectedness is equivalent to connectedness. Is this statement true for compact sets?