Lecture Notes in Mathematics

An Introduction to Riemannian Geometry

Sigmundur Gudmundsson

(Lund University)

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Preface

These lecture notes grew out of an M.Sc. course on differential geometry which I gave at the University of Leeds 1992. Their main purpose is to introduce the beautiful theory of Riemannian Geometry a still very active area of mathematical research. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work. Of special interest are the classical Lie groups allowing concrete calculations of many of the abstract notions on the menu.

The study of Riemannian geometry is rather meaningless without some basic knowledge on Gaussian geometry that i.e. the geometry of curves and surfaces in 3-dimensional space. For this I recommend the excellent textbook: M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall (1976).

These lecture notes are written for students with a good understanding of linear algebra, real analysis of several variables, the classical theory of ordinary differential equations and some topology. The most important results stated in the text are also proved there. Other are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course.

For further reading I recommend the very interesting textbook: M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).

I am very grateful to my many enthusiastic students who throughout the years have contributed to the text by finding numerous typing errors and giving many useful comments on the presentation.

Norra Nöbbelöv, 17 February 2012

Sigmundur Gudmundsson

Contents

CHAPTER 1

Introduction

On the 10th of June 1854 Georg Friedrich Bernhard Riemann (1826-1866) gave his famous "Habilitationsvortrag" in the Colloquium of the Philosophical Faculty at Göttingen. His talk "Über die Hypothesen, welche der Geometrie zu Grunde liegen" is often said to be the most important in the history of differential geometry. Johann Carl Friedrich Gauss (1777-1855), at the age of 76, was in the audience and is said to have been very impressed by his former student.

Riemann's revolutionary ideas generalized the geometry of surfaces which had been studied earlier by Gauss, Bolyai and Lobachevsky. Later this lead to an exact definition of the modern concept of an abstract Riemannian manifold.

The development of the 20th century has turned Riemannian geometry into one of the most important parts of modern mathematics. For an excellent survey of this vast field we recommend the following work written by one of the superstars in the field: Marcel Berger, *A Panoramic View of Riemannian Geometry*, Springer (2003).

CHAPTER 2

Differentiable Manifolds

In this chapter we introduce the important notion of a differentiable manifold. This generalizes curves and surfaces in $\mathbb{R}^3$ studied in classical differential geometry. Our manifolds are modelled on the standard differentiable structure on the vector spaces $\mathbb{R}^m$ via compatible local charts. We give many examples, study their submanifolds and differentiable maps between manifolds.

For a natural number m let $\mathbb{R}^m$ be the m-dimensional real vector space equipped with the topology induced by the standard Euclidean metric d on $\mathbb{R}^m$ given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \ldots + (x_m - y_m)^2}.$$

For positive natural numbers n, r and an open subset U of $\mathbb{R}^m$ we shall by $C^r(U, \mathbb{R}^n)$ denote the r-times continuously **differentiable** maps from U to $\mathbb{R}^n$. By **smooth** maps $U \to \mathbb{R}^n$ we mean the elements of

$$C^\infty(U, \mathbb{R}^n) = \bigcap_{r=1}^{\infty} C^r(U, \mathbb{R}^n).$$

The set of **real analytic** maps from U to $\mathbb{R}^n$ will be denoted by $C^\omega(U, \mathbb{R}^n)$. For the theory of real analytic maps we recommend the book: S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser (1992).

Definition 2.1. Let $(M, \mathcal{T})$ be a topological Hausdorff space with a countable basis. Then M is said to be a **topological manifold** if there exists a natural number m and for each point $p \in M$ an open neighbourhood U of p and a continuous map $x : U \to \mathbb{R}^m$ which is a homeomorphism onto its image $x(U)$ which is an open subset of $\mathbb{R}^m$. The pair (U, x) is called a (local) **chart** (or local **coordinates**) on M. The natural number m is called the **dimension** of M. To denote that the dimension of M is m we write M^m.

According to Definition 2.1 a topological manifold M is locally homeomorphic to the standard $\mathbb{R}^m$ for some natural number m. We shall now define a differentiable structure on M via local charts and turn M into a differentiable manifold.

Definition 2.2. Let M be a topological manifold. Then a C^r**-atlas** on M is a collection

$$\mathcal{A} = \{(U_\alpha, x_\alpha)|\ \alpha \in I\}$$

of local charts on M such that $\mathcal{A}$ covers the whole of M i.e.

$$M = \bigcup_\alpha U_\alpha$$

and for all $\alpha, \beta \in I$ the corresponding **transition maps**

$$x_\beta \circ x_\alpha^{-1}|_{x_\alpha(U_\alpha \cap U_\beta)} : x_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^m$$

are r-times continuously differentiable.

A chart (U, x) on M is said to be **compatible** with a C^r-atlas $\mathcal{A}$ on M if the union $\mathcal{A} \cup \{(U, x)\}$ is a C^r-atlas on M. A C^r-atlas $\hat{\mathcal{A}}$ is said to be **maximal** if it contains all the charts that are compatible with it. A maximal atlas $\hat{\mathcal{A}}$ on M is also called a C^r-**structure** on M. The pair $(M, \hat{\mathcal{A}})$ is said to be a C^r-**manifold**, or a **differentiable manifold** of class C^r, if M is a topological manifold and $\hat{\mathcal{A}}$ is a C^r-structure on M. A differentiable manifold is said to be **smooth** if its transition maps are C^∞ and **real analytic** if they are C^ω.

It should be noted that a given C^r-atlas $\mathcal{A}$ on a topological manifold M determines a unique C^r-structure $\hat{\mathcal{A}}$ on M containing $\mathcal{A}$. It simply consists of all charts compatible with $\mathcal{A}$.

Example 2.3. For the standard topological space $(\mathbb{R}^m, \mathcal{T})$ we have the trivial C^ω-atlas

$$\mathcal{A} = \{(\mathbb{R}^m, x)|\ x : p \mapsto p\}$$

inducing the standard C^ω-structure $\hat{\mathcal{A}}$ on $\mathbb{R}^m$.

Example 2.4. Let S^m denote the unit sphere in $\mathbb{R}^{m+1}$ i.e.

$$S^m = \{p \in \mathbb{R}^{m+1}|\ p_1^2 + \cdots + p_{m+1}^2 = 1\}$$

equipped with the subset topology induced by the standard $\mathcal{T}$ on $\mathbb{R}^{m+1}$. Let N be the north pole $N = (1, 0) \in \mathbb{R} \times \mathbb{R}^m$ and S be the south pole $S = (-1, 0)$ on S^m, respectively. Put $U_N = S^m \setminus \{N\}$, $U_S = S^m \setminus \{S\}$ and define $x_N : U_N \to \mathbb{R}^m$, $x_S : U_S \to \mathbb{R}^m$ by

$$x_N : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 - p_1}(p_2, \dots, p_{m+1}),$$

$$x_S : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 + p_1}(p_2, \dots, p_{m+1}).$$

Then the transition maps

$$x_S \circ x_N^{-1}, x_N \circ x_S^{-1} : \mathbb{R}^m \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$$

are given by

$$x \mapsto \frac{x}{|x|^2}$$

so $\mathcal{A} = \{(U_N, x_N), (U_S, x_S)\}$ is a C^ω-atlas on S^m. The C^ω-manifold $(S^m, \hat{\mathcal{A}})$ is called the standard m-dimensional **sphere**.

Another interesting example of a differentiable manifold is the m-dimensional real projective space $\mathbb{R}P^m$.

Example 2.5. On the set $\mathbb{R}^{m+1} \setminus \{0\}$ we define the equivalence relation $\equiv$ by:

$$p \equiv q \text{ if and only if there exists a } \lambda \in \mathbb{R}^* \text{ such that } p = \lambda q.$$

Let $\mathbb{R}P^m$ be the quotient space $(\mathbb{R}^{m+1} \setminus \{0\})/\equiv$ and

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$$

be the natural projection mapping a point $p \in \mathbb{R}^{m+1} \setminus \{0\}$ to the equivalence class $[p] \in \mathbb{R}P^m$ i.e. the line

$$[p] = \{\lambda p \in \mathbb{R}^{m+1} | \ \lambda \in \mathbb{R}^*\}$$

through the origin generated by p. Equip $\mathbb{R}P^m$ with the quotient topology induced by π and $\mathcal{T}$ on $\mathbb{R}^{m+1}$. For $k \in \{1, \dots, m+1\}$ define the open subset

$$U_k = \{[p] \in \mathbb{R}P^m | \ p_k \neq 0\}$$

of $\mathbb{R}P^m$ and the charts $x_k : U_k \to \mathbb{R}^m$ by

$$x_k : [p] \mapsto (\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k}).$$

If $[p] \equiv [q]$ then $p = \lambda q$ for some $\lambda \in \mathbb{R}^*$ so $p_l/p_k = q_l/q_k$ for all l. This means that the map x_k is well defined for all k. The corresponding transition maps

$$x_k \circ x_l^{-1}|_{x_l(U_l \cap U_k)} : x_l(U_l \cap U_k) \to \mathbb{R}^m$$

are given by

$$(\frac{p_1}{p_l}, \dots, \frac{p_{l-1}}{p_l}, 1, \frac{p_{l+1}}{p_l}, \dots, \frac{p_{m+1}}{p_l}) \mapsto (\frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k})$$

so the collection

$$\mathcal{A} = \{(U_k, x_k) | \ k = 1, \dots, m+1\}$$

is a C^ω-atlas on $\mathbb{R}P^m$. The differentiable manifold $(\mathbb{R}P^m, \hat{\mathcal{A}})$ is called the m-dimensional **real projective space**.

Example 2.6. Let $\hat{\mathbb{C}}$ be the extended complex plane given by

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and put $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $U_0 = \mathbb{C}$ and $U_\infty = \hat{\mathbb{C}} \setminus \{0\}$. Then define the local coordinates $x_0 : U_0 \to \mathbb{C}$ and $x_\infty : U_\infty \to \mathbb{C}$ on $\hat{\mathbb{C}}$ by $x_0 : z \mapsto z$ and $x_\infty : w \mapsto 1/w$, respectively. The corresponding transition maps

$$x_\infty \circ x_0^{-1}, x_0 \circ x_\infty^{-1} : \mathbb{C}^* \to \mathbb{C}^*$$

are both given by $z \mapsto 1/z$ so $\mathcal{A} = \{(U_0, x_0), (U_\infty, x_\infty)\}$ is a C^ω-atlas on $\hat{\mathbb{C}}$. The real analytic manifold $(\hat{\mathbb{C}}, \hat{\mathcal{A}})$ is called the **Riemann sphere**.

For the product of two differentiable manifolds we have the following interesting result.

Proposition 2.7. *Let $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r. Let $M = M_1 \times M_2$ be the product space with the product topology. Then there exists an atlas $\mathcal{A}$ on M making $(M, \hat{\mathcal{A}})$ into a differentiable manifold of class C^r and the dimension of M satisfies*

$$\dim M = \dim M_1 + \dim M_2.$$

Proof. See Exercise 2.1. □

The concept of a submanifold of a given differentiable manifold will play an important role as we go along and we shall be especially interested in the connection between the geometry of a submanifold and that of its ambient space.

Definition 2.8. Let $m, n \in \mathbb{N}$ be natural numbers such that $1 \leq m \leq n$ and $(N^n, \hat{\mathcal{B}})$ be a C^r-manifold. A subset M of N is said to be a **submanifold** of N if for each point $p \in M$ there exists a chart $(U_p, x_p) \in \hat{\mathcal{B}}$ such that $p \in U_p$ and $x_p : U_p \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ satisfies

$$x_p(U_p \cap M) = x_p(U_p) \cap (\mathbb{R}^m \times \{0\}).$$

The natural number $(n - m)$ is called the **codimension** of M in N.

Proposition 2.9. *Let $m, n \in \mathbb{N}$ be natural numbers such that $1 \leq m \leq n$ and $(N^n, \hat{\mathcal{B}})$ be a C^r-manifold. Let M be a submanifold of N equipped with the subset topology and $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^m$ be the natural projection onto the first factor. Then*

$$\mathcal{A} = \{(U_p \cap M, (\pi \circ x_p)|_{U_p \cap M}) |\ p \in M\}$$

is a C^r-atlas for M. In particular, the pair $(M, \hat{\mathcal{A}})$ is an m-dimensional C^r-manifold. The differentiable structure $\hat{\mathcal{A}}$ on the submanifold M of N is called the **induced structure** *of $\hat{\mathcal{B}}$.*

Proof. See Exercise 2.2. □

Remark 2.10. Our next aim is to prove the implicit function Theorem 2.14 which is a useful tool for the construction of submanifolds of $\mathbb{R}^n$. For this we use the classical inverse function theorem stated below. Note that if

$$F : U \subset \mathbb{R}^n \to \mathbb{R}^m$$

is a differentiable map defined on an open subset U of $\mathbb{R}^n$ then its differential $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ at the point $p \in U$ is a linear map given by the $m \times n$ matrix

$$dF_p = \begin{pmatrix} \partial F_1/\partial x_1(p) & \dots & \partial F_1/\partial x_n(p) \\ \vdots & & \vdots \\ \partial F_m/\partial x_1(p) & \dots & \partial F_m/\partial x_n(p) \end{pmatrix}.$$

If $\gamma : \mathbb{R} \to U$ is a curve in U such that $\gamma(0) = p$ and $\dot\gamma(0) = v \in \mathbb{R}^n$ then the composition $F \circ \gamma : \mathbb{R} \to \mathbb{R}^m$ is a curve in $\mathbb{R}^m$ and according to the chain rule we have

$$dF_p \cdot v = \frac{d}{ds}(F \circ \gamma(s))|_{s=0},$$

which is the tangent vector of the curve $F \circ \gamma$ at $F(p) \in \mathbb{R}^m$.

Hence the differential dF_p can be seen as a linear map that maps tangent vectors at $p \in U$ to tangent vectors at the image $F(p) \in \mathbb{R}^m$. This will later be generalized to the manifold setting.

Fact 2.11 (The Inverse Function Theorem)**.** *Let U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^n$ be a C^r-map. If $p \in U$ and the differential*

$$dF_p : \mathbb{R}^n \to \mathbb{R}^n$$

of F at p is invertible then there exist open neighbourhoods U_p around p and U_q around $q = F(p)$ such that $\hat F = F|_{U_p} : U_p \to U_q$ is bijective and the inverse $(\hat F)^{-1} : U_q \to U_p$ is a C^r-map. The differential $(d\hat F^{-1})_q$ of $\hat F^{-1}$ at q satisfies

$$(d\hat F^{-1})_q = (dF_p)^{-1}$$

i.e. it is the inverse of the differential dF_p of F at p.

Before stating the implicit function theorem we remind the reader of the definition of the following notions.

Definition 2.12. Let m, n be positive natural numbers, U be an open subset of $\mathbb{R}^n$ and $F : U \to \mathbb{R}^m$ be a C^r-map. A point $p \in U$ is said to be **critical** for F if the differential

$$dF_p : \mathbb{R}^n \to \mathbb{R}^m$$

is not of full rank, and **regular** if it is not critical. A point $q \in F(U)$ is said to be a **regular value** of F if every point of the pre-image $F^{-1}(\{q\})$ of q is regular and a **critical value** otherwise.

Remark 2.13. Note that if $m, n \in \mathbb{Z}^+$ such that $m \leq n$ then $p \in U$ is a regular point of

$$F = (F_1, \dots, F_m) : U \to \mathbb{R}^m$$

if and only if the gradients $\text{grad}F_1, \dots, \text{grad}F_m$ of the coordinate functions $F_1, \dots, F_m : U \to \mathbb{R}$ are linearly independent at p, or equivalently, the differential dF_p of F at p satisfies the following condition

$$\det(dF_p \cdot (dF_p)^t) \neq 0.$$

Theorem 2.14 (The Implicit Function Theorem)**.** *Let m, n be natural numbers such that $m < n$ and $F : U \to \mathbb{R}^m$ be a C^r-map from an open subset U of $\mathbb{R}^n$. If $q \in F(U)$ is a regular value of F then the pre-image $F^{-1}(\{q\})$ of q is an $(n-m)$-dimensional submanifold of $\mathbb{R}^n$ of class C^r.*

Proof. Let p be an element of $F^{-1}(\{q\})$ and K_p be the kernel of the differential dF_p i.e. the $(n-m)$-dimensional subspace of $\mathbb{R}^n$ given by $K_p = \{v \in \mathbb{R}^n | \ dF_p \cdot v = 0\}$. Let $\pi_p : \mathbb{R}^n \to \mathbb{R}^{n-m}$ be a linear map such that $\pi_p|_{K_p} : K_p \to \mathbb{R}^{n-m}$ is bijective, $\pi_p|_{K_p^\perp} = 0$ and define the map $G_p : U \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ by

$$G_p : x \mapsto (F(x), \pi_p(x)).$$

Then the differential $(dG_p)_p : \mathbb{R}^n \to \mathbb{R}^n$ of G_p, with respect to the decompositions $\mathbb{R}^n = K_p^\perp \oplus K_p$ and $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$, is given by

$$(dG_p)_p = \begin{pmatrix} dF_p|_{K_p^\perp} & 0 \\ 0 & \pi_p \end{pmatrix},$$

hence bijective. It now follows from the inverse function theorem that there exist open neighbourhoods V_p around p and W_p around $G_p(p)$ such that $\hat{G}_p = G_p|_{V_p} : V_p \to W_p$ is bijective, the inverse $\hat{G}_p^{-1} : W_p \to V_p$ is C^r, $d(\hat{G}_p^{-1})_{G_p(p)} = (dG_p)_p^{-1}$ and $d(\hat{G}_p^{-1})_y$ is bijective for all $y \in W_p$. Now put $\tilde{U}_p = F^{-1}(\{q\}) \cap V_p$ then

$$\tilde{U}_p = \hat{G}_p^{-1}((\{q\} \times \mathbb{R}^{n-m}) \cap W_p)$$

so if $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ is the natural projection onto the second factor, then the map

$$\tilde{x}_p = \pi \circ G_p|_{\tilde{U}_p} : \tilde{U}_p \to (\{q\} \times \mathbb{R}^{n-m}) \cap W_p \to \mathbb{R}^{n-m}$$

is a chart on the open neighbourhood $\tilde{U}_p$ of p. The point $q \in F(U)$ is a regular value so the set

$$\mathcal{B} = \{(\tilde{U}_p, \tilde{x}_p) |\ p \in F^{-1}(\{q\})\}$$

is a C^r-atlas for $F^{-1}(\{q\})$. □

Employing the implicit function theorem we yield the following interesting examples of the m-dimensional sphere S^m and its tangent bundle TS^m as differentiable submanifolds of $\mathbb{R}^{m+1}$ and $\mathbb{R}^{2m+2}$, respectively.

Example 2.15. Let $F : \mathbb{R}^{m+1} \to \mathbb{R}$ be the C^ω-map given by

$$F : (p_1, \dots, p_{m+1}) \mapsto \sum_{i=1}^{m+1} p_i^2.$$

The differential dF_p of F at p is given by $dF_p = 2p$, so

$$dF_p \cdot (dF_p)^t = 4|p|^2 \in \mathbb{R}.$$

This means that $1 \in \mathbb{R}$ is a regular value of F so the fibre

$$S^m = \{p \in \mathbb{R}^{m+1} |\ |p|^2 = 1\} = F^{-1}(\{1\})$$

of F is an m-dimensional submanifold of $\mathbb{R}^{m+1}$. It is the standard m-dimensional sphere introduced in Example 2.4.

Example 2.16. Let $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \to \mathbb{R}^2$ be the C^ω-map defined by $F : (p, v) \mapsto ((|p|^2 - 1)/2, \langle p, v \rangle)$. The differential $dF_{(p,v)}$ of F at (p, v) satisfies

$$dF_{(p,v)} = \begin{pmatrix} p & 0 \\ v & p \end{pmatrix}.$$

Hence

$$\det(dF \cdot (dF)^t) = |p|^2(|p|^2 + |v|^2) = 1 + |v|^2 > 0$$

on $F^{-1}(\{0\})$. This means that

$$F^{-1}(\{0\}) = \{(p, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} |\ \ |p|^2 = 1 \ \text{ and } \ \langle p, v \rangle = 0\}$$

which we denote by TS^m is a $2m$-dimensional submanifold of $\mathbb{R}^{2m+2}$. We shall later see that TS^m is what is called the tangent bundle of the m-dimensional sphere.

We shall now apply the implicit function theorem to construct the important orthogonal group $\mathbf{O}(m)$ as a submanifold of the set of the real vector space of $m \times m$ matrices $\mathbb{R}^{m\times m}$.

Example 2.17. Let $\mathrm{Sym}(\mathbb{R}^m)$ be the $m(m+1)/2$ dimensional linear subspace of $\mathbb{R}^{m\times m}$ consisting of all symmetric $m \times m$ matrices

$$\mathrm{Sym}(\mathbb{R}^m) = \{y \in \mathbb{R}^{m\times m} |\ y^t = y\}.$$

Let $F : \mathbb{R}^{m\times m} \to \mathrm{Sym}(\mathbb{R}^m)$ be the map defined by

$$F : x \mapsto x^t x.$$

If $\gamma : I \to \mathbb{R}^{m\times m}$ is a curve in $\mathbb{R}^{m\times m}$ then

$$\frac{d}{ds}(F \circ \gamma(s)) = \dot\gamma(s)^t \gamma(s) + \gamma(s)^t \dot\gamma(s),$$

so the differential dF_x of F at $x \in \mathbb{R}^{m\times m}$ satisfies

$$dF_x : X \mapsto X^t x + x^t X.$$

This means that for an arbitrary element p in

$$\mathbf{O}(m) = F^{-1}(\{e\}) = \{p \in \mathbb{R}^{m\times m} |\ p^t p = e\}$$

and $Y \in \mathrm{Sym}(\mathbb{R}^m)$ we have $dF_p(pY/2) = Y$. Hence the differential dF_p is surjective, so the identity matrix $e \in \mathrm{Sym}(\mathbb{R}^m)$ is a regular value for F. Following the implicit function theorem $\mathbf{O}(m)$ is a submanifold of $\mathbb{R}^{m\times m}$ of dimension $m(m-1)/2$. The set $\mathbf{O}(m)$ is the well known **orthogonal group**.

The concept of a differentiable map $U \to \mathbb{R}^n$, defined on an open subset of $\mathbb{R}^m$, can be generalized to mappings between manifolds. We shall see that the most important properties of these objects in the classical case are also valid in the manifold setting.

Definition 2.18. Let $(M^m, \hat{\mathcal{A}}_1)$ and $(N^n, \hat{\mathcal{A}}_2)$ be two C^r-manifolds. A map $\phi : M \to N$ is said to be **differentiable** of class C^r if for all charts $(U, x) \in \hat{\mathcal{A}}_1$ and $(V, y) \in \hat{\mathcal{A}}_2$ the map

$$y \circ \phi \circ x^{-1}|_{x(U\cap\phi^{-1}(V))} : x(U \cap \phi^{-1}(V)) \subset \mathbb{R}^m \to \mathbb{R}^n$$

is of class C^r. A differentiable map $\gamma : I \to M$ defined on an open interval of $\mathbb{R}$ is called a differentiable **curve** in M. A differentiable map $f : M \to \mathbb{R}$ with values in $\mathbb{R}$ is called a differentiable **function** on M. The set of smooth functions defined on M is denoted by $C^\infty(M)$.

It is an easy exercise, using Definition 2.18, to prove the following result concerning the composition of differentiable maps between manifolds.

Proposition 2.19. *Let $(M_1, \hat{\mathcal{A}}_1), (M_2, \hat{\mathcal{A}}_2), (M_3, \hat{\mathcal{A}}_3)$ be C^r-manifolds and $\phi : (M_1, \hat{\mathcal{A}}_1) \to (M_2, \hat{\mathcal{A}}_2)$, $\psi : (M_2, \hat{\mathcal{A}}_2) \to (M_3, \hat{\mathcal{A}}_3)$ be differentiable maps of class C^r. Then the composition $\psi \circ \phi : (M_1, \hat{\mathcal{A}}_1) \to (M_3, \hat{\mathcal{A}}_3)$ is a differentiable map of class C^r.*

PROOF. See Exercise 2.5. □

Definition 2.20. Two manifolds $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ of class C^r are said to be **diffeomorphic** if there exists a bijective C^r-map $\phi : M_1 \to M_2$, such that the inverse $\phi^{-1} : M_2 \to M_1$ is of class C^r. In that case the map ϕ is said to be a **diffeomorphism** between $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$.

It can be shown that the 2-dimensional sphere S^2 in $\mathbb{R}^3$ and the Riemann sphere, introduced earlier, are diffeomorphic, see Exercise 2.7.

Definition 2.21. Two C^r-structures $\hat{\mathcal{A}}_1$ and $\hat{\mathcal{A}}_2$ on the same topological manifold M are said to be **different** if the identity map $\text{id}_M : (M, \hat{\mathcal{A}}_1) \to (M, \hat{\mathcal{A}}_2)$ is not a diffeomorphism.

It can be seen that even the real line $\mathbb{R}$ carries different differentiable structures, see Exercise 2.6.

Deep Result 2.22. *Let $(M_1^m, \hat{\mathcal{A}}_1)$, $(M_2^m, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r and of equal dimensions. If M_1 and M_2 are homeomorphic as topological spaces and $m \leq 3$ then $(M_1, \hat{\mathcal{A}}_1)$ and $(M_2, \hat{\mathcal{A}}_2)$ are diffeomorphic.*

The following remarkable result was proved by John Milnor in his famous paper: *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962-972.

Deep Result 2.23. *The 7-dimensional sphere S^7 has exactly 28 different differentiable structures.*

The next very useful proposition generalizes a classical result from the real analysis of several variables.

Proposition 2.24. *Let $(N_1, \hat{\mathcal{A}}_1)$ and $(N_2, \hat{\mathcal{A}}_2)$ be two differentiable manifolds of class C^r and M_1, M_2 be submanifolds of N_1 and N_2, respectively. If $\phi : N_1 \to N_2$ is a differentiable map of class C^r such that $\phi(M_1)$ is contained in M_2, then the restriction $\phi|_{M_1} : M_1 \to M_2$ is differentiable of class C^r.*

PROOF. See Exercise 2.8. □

Example 2.25. The result of Proposition 2.24 can be used to show that the following maps are all smooth.

(i) $\phi_1 : S^2 \subset \mathbb{R}^3 \to S^3 \subset \mathbb{R}^4$, $\phi_1 : (x, y, z) \mapsto (x, y, z, 0)$,
(ii) $\phi_2 : S^3 \subset \mathbb{C}^2 \to S^2 \subset \mathbb{C}\times\mathbb{R}$, $\phi_2 : (z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$,
(iii) $\phi_3 : \mathbb{R}^1 \to S^1 \subset \mathbb{C}$, $\phi_3 : t \mapsto e^{it}$,
(iv) $\phi_4 : \mathbb{R}^{m+1} \setminus \{0\} \to S^m$, $\phi_4 : x \mapsto x/|x|$,
(v) $\phi_5 : \mathbb{R}^{m+1} \setminus \{0\} \to \mathbb{R}P^m$, $\phi_5 : x \mapsto [x]$,
(vi) $\phi_6 : S^m \to \mathbb{R}P^m$, $\phi_6 : x \mapsto [x]$.

In differential geometry we are especially interested in differentiable manifolds carrying a group structure compatible with their differentiable structure. Such manifolds are named after the famous mathematician Sophus Lie (1842-1899) and will play an important role throughout this work.

Definition 2.26. A **Lie group** is a smooth manifold G with a group structure $\cdot$ such that the map $\rho : G \times G \to G$ with

$$\rho : (p, q) \mapsto p \cdot q^{-1}$$

is smooth. For an element p in G the **left translation** by p is the map $L_p : G \to G$ defined by $L_p : q \mapsto p \cdot q$.

Example 2.27. Let $(\mathbb{R}^m, +, \cdot)$ be the m-dimensional vector space equipped with its standard differential structure. Then $(\mathbb{R}^m, +)$ with $\rho : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ given by

$$\rho : (p, q) \mapsto p - q$$

is a Lie group.

Corollary 2.28. *Let G be a Lie group and p be an element of G. Then the left translation $L_p : G \to G$ is a smooth diffeomorphism.*

Proof. See Exercise 2.10 □

Proposition 2.29. *Let $(G, \cdot)$ be a Lie group and K be a submanifold of G which is a subgroup. Then $(K, \cdot)$ is a Lie group.*

Proof. The statement is a direct consequence of Definition 2.26 and Proposition 2.24. □

The set of non-zero complex numbers $\mathbb{C}^*$ together with the standard multiplication $\cdot$ forms a Lie group $(\mathbb{C}^*, \cdot)$. The unit circle $(S^1, \cdot)$ is an interesting compact Lie subgroup of $(\mathbb{C}^*, \cdot)$. Another subgroup is the set of the non-zero real numbers $(\mathbb{R}^*, \cdot)$ containing the positive real numbers $(\mathbb{R}^+, \cdot)$ and the 0-dimensional sphere $(S^0, \cdot)$ as subgroups.

Example 2.30. Let $\mathbb{H}$ be the set of **quaternions** defined by

$$\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\}$$

equipped with the addition $+$, multiplication $\cdot$ and conjugation $\bar{\ }$

(i) $\overline{(z + wj)} = \bar{z} - wj$,
(ii) $(z_1 + w_1 j) + (z_2 + w_2 j) = (z_1 + z_2) + (w_1 + w_2)j$,
(iii) $(z_1 + w_1 j) \cdot (z_2 + w_2 j) = (z_1 z_2 - w_1 \bar{w}_2) + (z_1 w_2 + w_1 \bar{z}_2)j$

extending the standard operations on $\mathbb{R}$ and $\mathbb{C}$ as subsets of $\mathbb{H}$. Then it is easily seen that the non-zero quaternions $(\mathbb{H}^*, \cdot)$ form a Lie group. On $\mathbb{H}$ we define a scalar product

$$\mathbb{H} \times \mathbb{H} \to \mathbb{H}, \qquad (p, q) \mapsto p \cdot \bar{q}$$

and a real valued norm given by $|p|^2 = p \cdot \bar{p}$. Then the 3-dimensional unit sphere S^3 in $\mathbb{H} \cong \mathbb{R}^4$ with the restricted multiplication forms a compact Lie subgroup $(S^3, \cdot)$ of $(\mathbb{H}^*, \cdot)$. They are both non-abelian.

We shall now introduce some of the classical real and complex matrix Lie groups. As a reference on this topic we recommend the wonderful book: A. W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser (2002).

Example 2.31. Let Nil^3 be the subset of $\mathbb{R}^{3\times 3}$ given by

$$Nil^3 = \{\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \mid x, y, z \in \mathbb{R}\}.$$

Then Nil^3 has a natural differentiable structure determined by the global coordinates $\phi : Nil^3 \to \mathbb{R}^3$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if $*$ is the standard matrix multiplication, then $(Nil^3, *)$ is a Lie group.

Example 2.32. Let Sol^3 be the subset of $\mathbb{R}^{3\times 3}$ given by

$$Sol^3 = \{\begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \mid x, y, z \in \mathbb{R}\}.$$

Then Sol^3 has a natural differentiable structure determined by the global coordinates $\phi : Sol^3 \to \mathbb{R}^3$ with

$$\phi : \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if $*$ is the standard matrix multiplication, then $(Sol^3, *)$ is a Lie group.

Example 2.33. The set of invertible real $m \times m$ matrices

$$\mathbf{GL}_m(\mathbb{R}) = \{A \in \mathbb{R}^{m\times m} |\ \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the **real general linear group** and its neutral element e is the identity matrix. The subset $\mathbf{GL}_m(\mathbb{R})$ of $\mathbb{R}^{m\times m}$ is open so $\dim \mathbf{GL}_m(\mathbb{R}) = m^2$.

As a subgroup of $\mathbf{GL}_m(\mathbb{R})$ we have the **real special linear group** $\mathbf{SL}_m(\mathbb{R})$ given by

$$\mathbf{SL}_m(\mathbb{R}) = \{A \in \mathbb{R}^{m\times m} |\ \det A = 1\}.$$

We will show in Example 3.10 that the dimension of the submanifold $\mathbf{SL}_m(\mathbb{R})$ of $\mathbb{R}^{m\times m}$ is $m^2 - 1$.

Another subgroup of $\mathbf{GL}_m(\mathbb{R})$ is the **orthogonal group**

$$\mathbf{O}(m) = \{A \in \mathbb{R}^{m\times m} |\ A^t A = e\}.$$

As we have already seen in Example 2.17 the dimension of $\mathbf{O}(m)$ is $m(m-1)/2$.

As a subgroup of $\mathbf{O}(m)$ and $\mathbf{SL}_m(\mathbb{R})$ we have the **special orthogonal group $\mathbf{SO}(m)$** which is defined as

$$\mathbf{SO}(m) = \mathbf{O}(m) \cap \mathbf{SL}_m(\mathbb{R}).$$

It can be shown that $\mathbf{O}(m)$ is diffeomorphic to $\mathbf{SO}(m) \times \mathbf{O}(1)$, see Exercise 2.9. Note that $\mathbf{O}(1) = \{\pm 1\}$ so $\mathbf{O}(m)$ can be seen as two copies of $\mathbf{SO}(m)$. This means that

$$\dim \mathbf{SO}(m) = \dim \mathbf{O}(m) = m(m-1)/2.$$

Example 2.34. The set of invertible complex $m \times m$ matrices

$$\mathbf{GL}_m(\mathbb{C}) = \{A \in \mathbb{C}^{m\times m} |\ \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the **complex general linear group** and its neutral element e is the identity matrix. The subset $\mathbf{GL}_m(\mathbb{C})$ of $\mathbb{C}^{m\times m}$ is open so $\dim(\mathbf{GL}_m(\mathbb{C})) = 2m^2$.

As a subgroup of $\mathbf{GL}_m(\mathbb{C})$ we have the **complex special linear group** $\mathbf{SL}_m(\mathbb{C})$ given by

$$\mathbf{SL}_m(\mathbb{C}) = \{A \in \mathbb{C}^{m\times m} |\ \det A = 1\}.$$

The dimension of the submanifold $\mathbf{SL}_m(\mathbb{C})$ of $\mathbb{C}^{m\times m}$ is $2(m^2-1)$.

Another subgroup of $\mathbf{GL}_m(\mathbb{C})$ is the **unitary group** $\mathbf{U}(m)$ given by

$$\mathbf{U}(m) = \{A \in \mathbb{C}^{m\times m} |\ \bar{A}^t A = e\}.$$

Calculations similar to those for the orthogonal group show that the dimension of $\mathbf{U}(m)$ is m^2.

As a subgroup of $\mathbf{U}(m)$ and $\mathbf{SL}_m(\mathbb{C})$ we have the **special unitary group** $\mathbf{SU}(m)$ which is defined as

$$\mathbf{SU}(m) = \mathbf{U}(m) \cap \mathbf{SL}_m(\mathbb{C}).$$

It can be shown that $\mathbf{U}(1)$ is diffeomorphic to the circle S^1 and that $\mathbf{U}(m)$ is diffeomorphic to $\mathbf{SU}(m) \times \mathbf{U}(1)$, see Exercise 2.9. This means that $\dim \mathbf{SU}(m) = m^2 - 1$.

For the rest of this manuscript we shall assume, when not stating otherwise, that our manifolds and maps are smooth i.e. in the C^∞-category.

Exercises

Exercise 2.1. Find a proof for Proposition 2.7.

Exercise 2.2. Find a proof for Proposition 2.9.

Exercise 2.3. Let S^1 be the unit circle in the complex plane $\mathbb{C}$ given by $S^1 = \{z \in \mathbb{C} |\ |z|^2 = 1\}$. Use the maps $x : \mathbb{C} \setminus \{i\} \to \mathbb{C}$ and $y : \mathbb{C} \setminus \{-i\} \to \mathbb{C}$ with

$$x : z \mapsto \frac{i+z}{1+iz}, \quad y : z \mapsto \frac{1+iz}{i+z}$$

to show that S^1 is a 1-dimensional submanifold of $\mathbb{C} \cong \mathbb{R}^2$.

Exercise 2.4. Use the implicit function theorem to show that the m-dimensional **torus**

$$T^m = \{z \in \mathbb{C}^m |\ |z_1| = \dots = |z_m| = 1\}$$

is a differentiable submanifold of $\mathbb{C}^m \cong \mathbb{R}^{2m}$.

Exercise 2.5. Find a proof of Proposition 2.19.

Exercise 2.6. Equip the real line $\mathbb{R}$ with the standard topology and for each odd integer $k \in \mathbb{Z}^+$ let $\hat{\mathcal{A}}_k$ be the C^ω-structure defined on $\mathbb{R}$ by the atlas

$$\mathcal{A}_k = \{(\mathbb{R}, x_k) |\ x_k : p \mapsto p^k\}.$$

Prove that the differentiable structures $\hat{\mathcal{A}}_k$ are all different but that the differentiable manifolds $(\mathbb{R}, \hat{\mathcal{A}}_k)$ are all diffeomorphic.

Exercise 2.7. Prove that the 2-dimensional sphere S^2 as a differentiable submanifold of the standard $\mathbb{R}^3$ and the Riemann sphere $\hat{\mathbb{C}}$ are diffeomorphic.

Exercise 2.8. Find a proof of Proposition 2.24.

Exercise 2.9. Let the spheres S^1, S^3 and the Lie groups $\mathbf{SO}(n)$, $\mathbf{O}(n)$, $\mathbf{SU}(n)$, $\mathbf{U}(n)$ be equipped with their standard differentiable structures introduced above. Use Proposition 2.24 to prove the following diffeomorphisms

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$
$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

Exercise 2.10. Find a proof of Corollary 2.28.

Exercise 2.11. Let $(G, *)$ and $(H, \cdot)$ be two Lie groups. Prove that the product manifold $G \times H$ has the structure of a Lie group.

CHAPTER 3

The Tangent Space

In this chapter we introduce the notion of the tangent space T_pM of a differentiable manifold M at a point $p \in M$. This is a vector space of the same dimension as M. We start by studying the standard $\mathbb{R}^m$ and show how a tangent vector v at a point $p \in \mathbb{R}^m$ can be interpreted as a first order linear differential operator, annihilating constants, when acting on real valued functions locally defined around p.

Let $\mathbb{R}^m$ be the m-dimensional real vector space with the standard differentiable structure. If p is a point in $\mathbb{R}^m$ and $\gamma : I \to \mathbb{R}^m$ is a C^1-curve such that $\gamma(0) = p$ then the **tangent vector**

$$\dot{\gamma}(0) = \lim_{t \to 0} \frac{\gamma(t) - \gamma(0)}{t}$$

of γ at p is an element of $\mathbb{R}^m$. Conversely, for an arbitrary element v of $\mathbb{R}^m$ we can easily find a curve $\gamma : I \to \mathbb{R}^m$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. One example is given by

$$\gamma : t \mapsto p + t \cdot v.$$

This shows that the **tangent space**, i.e. the space of tangent vectors, at the point $p \in \mathbb{R}^m$ can be identified with $\mathbb{R}^m$.

We shall now describe how first order differential operators annihilating constants can be interpreted as tangent vectors. For a point p in $\mathbb{R}^m$ we denote by $\varepsilon(p)$ the set of differentiable real-valued functions defined locally around p. Then it is well known from multi-variable analysis that if $v \in \mathbb{R}^m$ and $f \in \varepsilon(p)$ then the **directional derivative** $\partial_v f$ of f at p in the direction of v is given by

$$\partial_v f = \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t}.$$

Furthermore the operator ∂ has the following properties:

$$\begin{aligned} \partial_v(\lambda \cdot f + \mu \cdot g) &= \lambda \cdot \partial_v f + \mu \cdot \partial_v g, \\ \partial_v(f \cdot g) &= \partial_v f \cdot g(p) + f(p) \cdot \partial_v g, \\ \partial_{(\lambda \cdot v + \mu \cdot w)} f &= \lambda \cdot \partial_v f + \mu \cdot \partial_w f \end{aligned}$$

for all $\lambda, \mu \in \mathbb{R}$, $v, w \in \mathbb{R}^m$ and $f, g \in \varepsilon(p)$.

Definition 3.1. For a point p in $\mathbb{R}^m$ let $T_p\mathbb{R}^m$ be the set of first order linear differential operators at p annihilating constants i.e. the set of mappings $\alpha : \varepsilon(p) \to \mathbb{R}$ such that

(i) $\alpha(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \alpha(f) + \mu \cdot \alpha(g)$,
(ii) $\alpha(f \cdot g) = \alpha(f) \cdot g(p) + f(p) \cdot \alpha(g)$

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$.

The set of diffential operators $T_p\mathbb{R}^m$ carries the structure of a real vector space. This is given by the addition $+$ and the multiplication $\cdot$ by real numbers satisfying

$$\begin{aligned}(\alpha + \beta)(f) &= \alpha(f) + \beta(f),\\ (\lambda \cdot \alpha)(f) &= \lambda \cdot \alpha(f)\end{aligned}$$

for all $\alpha, \beta \in T_p\mathbb{R}^m$, $f \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

The above mentioned properties of the operator ∂ show that we have a well defined linear map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ given by

$$\Phi : v \mapsto \partial_v.$$

Theorem 3.2. *For a point p in $\mathbb{R}^m$ the linear map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ defined by $\Phi : v \mapsto \partial_v$ is a vector space isomorphism.*

Proof. Let $v, w \in \mathbb{R}^m$ such that $v \neq w$. Choose an element $u \in \mathbb{R}^m$ such that $\langle u, v\rangle \neq \langle u, w\rangle$ and define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(x) = \langle u, x\rangle$. Then $\partial_v f = \langle u, v\rangle \neq \langle u, w\rangle = \partial_w f$ so $\partial_v \neq \partial_w$. This proves that the map Φ is injective.

Let α be an arbitrary element of $T_p\mathbb{R}^m$. For $k = 1, \dots, m$ let $\hat{x}_k : \mathbb{R}^m \to \mathbb{R}$ be the map given by

$$\hat{x}_k : (x_1, \dots, x_m) \mapsto x_k$$

and put $v_k = \alpha(\hat{x}_k)$. For the constant function $1 : (x_1, \dots, x_m) \mapsto 1$ we have

$$\alpha(1) = \alpha(1 \cdot 1) = 1 \cdot \alpha(1) + 1 \cdot \alpha(1) = 2 \cdot \alpha(1),$$

so $\alpha(1) = 0$. By the linearity of α it follows that $\alpha(c) = 0$ for any constant $c \in \mathbb{R}$. Let $f \in \varepsilon(p)$ and following Lemma 3.3 locally write

$$f(x) = f(p) + \sum_{k=1}^{m} (\hat{x}_k(x) - p_k) \cdot \psi_k(x),$$

where $\psi_k \in \varepsilon(p)$ with

$$\psi_k(p) = \frac{\partial f}{\partial x_k}(p).$$

We can now apply the differential operator $\alpha \in T_p\mathbb{R}^m$ and yield

$$\begin{aligned}
\alpha(f) &= \alpha(f(p) + \sum_{k=1}^m (\hat{x}_k - p_k) \cdot \psi_k) \\
&= \alpha(f(p)) + \sum_{k=1}^m \alpha(\hat{x}_k - p_k) \cdot \psi_k(p) + \sum_{k=1}^m (\hat{x}_k(p) - p_k) \cdot \alpha(\psi_k) \\
&= \sum_{k=1}^m v_k \frac{\partial f}{\partial x_k}(p) \\
&= \langle v, \mathrm{grad} f_p \rangle \\
&= \partial_v f,
\end{aligned}$$

where $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. This means that $\Phi(v) = \partial_v = \alpha$ so the map $\Phi : \mathbb{R}^m \to T_p\mathbb{R}^m$ is surjective and hence a vector space isomorphism. □

Lemma 3.3. *Let p be a point in $\mathbb{R}^m$ and $f : U \to \mathbb{R}$ be a function defined on an open ball around p. Then for each $k = 1, 2, \dots, m$ there exist functions $\psi_k : U \to \mathbb{R}$ such that*

$$f(x) = f(p) + \sum_{k=1}^m (x_k - p_k) \cdot \psi_k(x) \quad \text{and} \quad \psi_k(p) = \frac{\partial f}{\partial x_k}(p)$$

for all $x \in U$.

Proof. It follows from the fundamental theorem of calculus that

$$\begin{aligned}
f(x) - f(p) &= \int_0^1 \frac{\partial}{\partial t}(f(p + t(x-p)))dt \\
&= \sum_{k=1}^m (x_k - p_k) \cdot \int_0^1 \frac{\partial f}{\partial x_k}(p + t(x-p))dt.
\end{aligned}$$

The statement then immediately follows by setting

$$\psi_k(x) = \int_0^1 \frac{\partial f}{\partial x_k}(p + t(x-p))dt.$$

□

Remark 3.4. Let p be a point in $\mathbb{R}^m$, $v \in T_p\mathbb{R}^m$ be a tangent vector at p and $f : U \to \mathbb{R}$ be a C^1-function defined on an open subset U of $\mathbb{R}^m$ containing p. Let $\gamma : I \to U$ be a curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The identification given by Theorem 3.2 tells us that v acts on f by

$$v(f) = \partial_v(f) = \langle v, \mathrm{grad} f_p \rangle = df_p(\dot{\gamma}(0)) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}.$$

This implies that the real number $v(f)$ is independent of the choice of the curve γ as long as $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

As a direct consequence of Theorem 3.2 we have the following useful result.

Corollary 3.5. *Let p be a point in $\mathbb{R}^m$ and $\{e_k|\ k = 1, \dots, m\}$ be a basis for $\mathbb{R}^m$. Then the set $\{\partial_{e_k}|\ k = 1, \dots, m\}$ is a basis for the tangent space $T_p\mathbb{R}^m$ at p.*

We shall now use the ideas presented above to generalize to the manifold setting. Let M be a differentiable manifold and for a point $p \in M$ let $\varepsilon(p)$ denote the set of differentiable functions defined on an open neighborhood of p.

Definition 3.6. Let M be a differentiable manifold and p be a point on M. A **tangent vector** X_p at p is a map $X_p : \varepsilon(p) \to \mathbb{R}$ such that

(i) $X_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot X_p(f) + \mu \cdot X_p(g)$,
(ii) $X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$

for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$. The set of tangent vectors at p is called the **tangent space** at p and denoted by T_pM.

The tangent space T_pM of M at p has the structure of a real vector space. The addition $+$ and the multiplication $\cdot$ by real numbers are simply given by

$$\begin{aligned} (X_p + Y_p)(f) &= X_p(f) + Y_p(f), \\ (\lambda \cdot X_p)(f) &= \lambda \cdot X_p(f) \end{aligned}$$

for all $X_p, Y_p \in T_pM,\ f \in \varepsilon(p)$ and $\lambda \in \mathbb{R}$.

Remark 3.7. Let M be an m-dimensional manifold and (U, x) be a local chart around $p \in M$. Then the differential $dx_p : T_pM \to T_{x(p)}\mathbb{R}^m$ is a bijective linear map so for a given element $X_p \in T_pM$ there exists a tangent vector v in $\mathbb{R}^m$ such that $dx_p(X_p) = v$. The image $x(U)$ is an open subset of $\mathbb{R}^m$ containing $x(p)$ so we can find a curve $c : I \to x(U)$ with $c(0) = x(p)$ and $\dot{c}(0) = v$. Then the composition $\gamma = x^{-1} \circ c : I \to U$ is a curve in M through p since $\gamma(0) = p$. The element $d(x^{-1})_{x(p)}(v)$ of the tangent space T_pM denoted by $\dot{\gamma}(0)$ is called the **tangent** to the curve γ at p. It follows from the relation

$$\dot{\gamma}(0) = d(x^{-1})_{x(p)}(v) = X_p$$

that the tangent space T_pM can be thought of as the set of all tangents to curves through the point p.

If $f : U \to \mathbb{R}$ is a C^1-function defined locally on U then it follows from Definition 3.14 that

$$\begin{aligned} X_p(f) &= (dx_p(X_p))(f \circ x^{-1}) \\ &= \frac{d}{dt}(f \circ x^{-1} \circ c(t))|_{t=0} \\ &= \frac{d}{dt}(f \circ \gamma(t))|_{t=0} \end{aligned}$$

It should be noted that the real number $X_p(f)$ is independent of the choice of the chart (U, x) around p and the curve $c : I \to x(U)$ as long as $\gamma(0) = p$ and $\dot\gamma(0) = X_p$.

We shall now determine the tangent spaces of some of the explicit differentiable manifolds introduced in Chapter 2.

Example 3.8. Let $\gamma : I \to S^m$ be a curve into the m-dimensional unit sphere in $\mathbb{R}^{m+1}$ with $\gamma(0) = p$ and $\dot\gamma(0) = X$. The curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \dot\gamma(t), \gamma(t) \rangle + \langle \gamma(t), \dot\gamma(t) \rangle = 0.$$

This means that $\langle p, X \rangle = 0$ so every tangent vector $X \in T_pS^m$ must be orthogonal to p. On the other hand if $X \neq 0$ satisfies $\langle p, X \rangle = 0$ then $\gamma : \mathbb{R} \to S^m$ with

$$\gamma : t \mapsto \cos(t|X|) \cdot p + \sin(t|X|) \cdot X/|X|$$

is a curve into S^m with $\gamma(0) = p$ and $\dot\gamma(0) = X$. This shows that the tangent space T_pS^m is actually given by

$$T_pS^m = \{X \in \mathbb{R}^{m+1} |\ \langle p, X \rangle = 0\}.$$

Proposition 3.9. *Let* $\text{Exp} : \mathbb{C}^{m\times m} \to \mathbb{C}^{m\times m}$ *be the well-known exponential map for complex matrices given by the converging power series*

$$\text{Exp} : Z \mapsto \sum_{k=0}^{\infty} \frac{Z^k}{k!}.$$

If $Z, W \in \mathbb{C}^{m\times m}$ *then*

(i) $\det(\text{Exp}(Z)) = \exp(\text{trace} Z)$,
(ii) $\text{Exp}(\bar Z^t) = \overline{\text{Exp}(Z)}^t$, *and*
(iii) $\text{Exp}(Z + W) = \text{Exp}(Z)\text{Exp}(W)$ *when ever* $ZW = WZ$.

Proof. See Exercise 3.2 □

The real general linear group $\mathbf{GL}_m(\mathbb{R})$ is an open subset of $\mathbb{R}^{m\times m}$ so its tangent space $T_p\mathbf{GL}_m(\mathbb{R})$ at any point p is simply $\mathbb{R}^{m\times m}$. The tangent space $T_e\mathbf{SL}_m(\mathbb{R})$ of the special linear group $\mathbf{SL}_m(\mathbb{R})$ at the neutral element e can be determined as follows.

Example 3.10. If X is a matrix in $\mathbb{R}^{m\times m}$ with $\text{trace}X = 0$ then define a curve $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ by

$$A : s \mapsto \text{Exp}(sX).$$

Then $A(0) = e$, $\dot{A}(0) = X$ and

$$\det(A(s)) = \det(\text{Exp}(sX)) = \exp(\text{trace}(sX)) = \exp(0) = 1.$$

This shows that A is a curve into the special linear group $\mathbf{SL}_m(\mathbb{R})$ and that X is an element of the tangent space $T_e\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ at the neutral element e. Hence the linear space

$$\{X \in \mathbb{R}^{m\times m} |\ \text{trace}X = 0\}$$

of dimension $m^2 - 1$ is contained in the tangent space $T_e\mathbf{SL}_m(\mathbb{R})$.

The curve given by $s \mapsto \text{Exp}(se) = \exp(s)e$ is not contained in $\mathbf{SL}_m(\mathbb{R})$ so the dimension of $T_e\mathbf{SL}_m(\mathbb{R})$ is at most $m^2 - 1$. This shows that

$$T_e\mathbf{SL}_m(\mathbb{R}) = \{X \in \mathbb{R}^{m\times m} |\ \text{trace}X = 0\}.$$

Example 3.11. Let $\gamma : I \to \mathbf{O}(m)$ be a curve into the orthogonal group $\mathbf{O}(m)$ such that $\gamma(0) = e$. Then $\gamma(s)^t\gamma(s) = e$ for all $s \in I$ and differentiation gives

$$\{\dot{\gamma}(s)^t\gamma(s) + \gamma(s)^t\dot{\gamma}(s)\}|_{s=0} = 0$$

or equivalently $\dot{\gamma}(0)^t + \dot{\gamma}(0) = 0$. This means that each tangent vector of $\mathbf{O}(m)$ at e is a skew-symmetric matrix.

On the other hand, for an arbitrary real skew-symmetric matrix X define the curve $A : \mathbb{R} \to \mathbb{R}^{m\times m}$ by $A : s \mapsto \text{Exp}(sX)$. Then

$$\begin{aligned} A(s)^tA(s) &= \text{Exp}(sX)^t\text{Exp}(sX) \\ &= \text{Exp}(sX^t)\text{Exp}(sX) \\ &= \text{Exp}(s(X^t + X)) \\ &= \text{Exp}(0) \\ &= e. \end{aligned}$$

This shows that A is a curve on the orthogonal group, $A(0) = e$ and $\dot{A}(0) = X$ so X is an element of $T_e\mathbf{O}(m)$. Hence

$$T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} |\ X^t + X = 0\}.$$

The dimension of $T_e\mathbf{O}(m)$ is therefore $m(m-1)/2$. The orthogonal group $\mathbf{O}(m)$ is diffeomorphic to $\mathbf{SO}(m) \times \{\pm 1\}$ so $\dim(\mathbf{SO}(m)) = \dim(\mathbf{O}(m))$ hence

$$T_e\mathbf{SO}(m) = T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} |\ X^t + X = 0\}.$$

We have proved the following result.

Proposition 3.12. *Let e be the neutral element of the classical real Lie groups $\mathbf{GL}_m(\mathbb{R})$, $\mathbf{SL}_m(\mathbb{R})$, $\mathbf{O}(m)$, $\mathbf{SO}(m)$. Then their tangent spaces at e are given by*

$$\begin{aligned} T_e\mathbf{GL}_m(\mathbb{R}) &= \mathbb{R}^{m\times m} \\ T_e\mathbf{SL}_m(\mathbb{R}) &= \{X \in \mathbb{R}^{m\times m} |\ \mathrm{trace} X = 0\} \\ T_e\mathbf{O}(m) &= \{X \in \mathbb{R}^{m\times m} |\ X^t + X = 0\} \\ T_e\mathbf{SO}(m) &= T_e\mathbf{O}(m) \cap T_e\mathbf{SL}_m(\mathbb{R}) = T_e\mathbf{O}(m) \end{aligned}$$

For the classical complex Lie groups similar methods can be used to prove the following.

Proposition 3.13. *Let e be the neutral element of the classical complex Lie groups $\mathbf{GL}_m(\mathbb{C})$, $\mathbf{SL}_m(\mathbb{C})$, $\mathbf{U}(m)$, and $\mathbf{SU}(m)$. Then their tangent spaces at e are given by*

$$\begin{aligned} T_e\mathbf{GL}_m(\mathbb{C}) &= \mathbb{C}^{m\times m} \\ T_e\mathbf{SL}_m(\mathbb{C}) &= \{Z \in \mathbb{C}^{m\times m} |\ \mathrm{trace} Z = 0\} \\ T_e\mathbf{U}(m) &= \{Z \in \mathbb{C}^{m\times m} |\ \bar{Z}^t + Z = 0\} \\ T_e\mathbf{SU}(m) &= T_e\mathbf{U}(m) \cap T_e\mathbf{SL}_m(\mathbb{C}). \end{aligned}$$

Proof. See Exercise 3.4 □

Definition 3.14. Let $\phi : M \to N$ be a differentiable map between manifolds. Then the **differential** $d\phi_p$ of ϕ at a point p in M is the map $d\phi_p : T_pM \to T_{\phi(p)}N$ such that for all $X_p \in T_pM$ and $f \in \varepsilon(\phi(p))$

$$(d\phi_p(X_p))(f) = X_p(f \circ \phi).$$

Remark 3.15. Let M and N be differentiable manifolds, $p \in M$ and $\phi : M \to N$ be a smooth map. Further let $\gamma : I \to M$ be a curve on M such that $\gamma(0) = p$ and $\dot\gamma(0) = X_p$. Let $c : I \to N$ be the curve $c = \phi \circ \gamma$ in N with $c(0) = \phi(p)$ and put $Y_{\phi(p)} = \dot c(0)$. Then it is an immediate consequence of Definition 3.14 that for each function $f \in \varepsilon(q)$ defined locally around q

$$(d\phi_p(X_p))(f) \;=\; X_p(f \circ \phi)$$

$$\begin{aligned} &= \frac{d}{dt}(f \circ \phi \circ \gamma(t))|_{t=0} \\ &= \frac{d}{dt}(f \circ c(t))|_{t=0} \\ &= Y_{\phi(p)}(f). \end{aligned}$$

Hence $d\phi_p(X_p) = Y_{\phi(p)}$ or equivalently $d\phi_p(\dot\gamma(0)) = \dot c(0)$. This result should be compared with Remark 2.10.

Proposition 3.16. *Let $\phi : M_1 \to M_2$ and $\psi : M_2 \to M_3$ be differentiable maps between manifolds, then for each $p \in M_1$ we have*

(i) *the map $d\phi_p : T_pM_1 \to T_{\phi(p)}M_2$ is linear,*
(ii) *if $id_{M_1} : M_1 \to M_1$ is the identity map, then $d(id_{M_1})_p = id_{T_pM_1}$,*
(iii) *$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$.*

Proof. The only non-trivial statement is the relation (iii) which is called the **chain rule**. If $X_p \in T_pM_1$ and $f \in \varepsilon(\psi \circ \phi(p))$, then

$$\begin{aligned} (d\psi_{\phi(p)} \circ d\phi_p)(X_p)(f) &= (d\psi_{\phi(p)}(d\phi_p(X_p)))(f) \\ &= (d\phi_p(X_p))(f \circ \psi) \\ &= X_p(f \circ \psi \circ \phi) \\ &= (d(\psi \circ \phi)_p(X_p))(f). \end{aligned}$$

□

Corollary 3.17. *Let $\phi : M \to N$ be a diffeomorphism with inverse $\psi = \phi^{-1} : N \to M$. Then the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ at p is bijective and $(d\phi_p)^{-1} = d\psi_{\phi(p)}$.*

Proof. The statement is a direct consequence of the following relations

$$d\psi_{\phi(p)} \circ d\phi_p = d(\psi \circ \phi)_p = d(\mathrm{id}_M)_p = \mathrm{id}_{T_pM},$$
$$d\phi_p \circ d\psi_{\phi(p)} = d(\phi \circ \psi)_{\phi(p)} = d(\mathrm{id}_N)_{\phi(p)} = \mathrm{id}_{T_{\phi(p)}N}.$$

□

We are now ready to prove the following interesting result. This is of course a direct generalization of the corresponding result in the classical theory for surfaces in $\mathbb{R}^3$.

Theorem 3.18. *Let M^m be an m-dimensional differentable manifold and p be a point in M. Then the tangent space T_pM at p is an m-dimensional real vector space.*

Proof. Let (U, x) be a local chart on M. Then the linear map $dx_p : T_pM \to T_{x(p)}\mathbb{R}^m$ is a vector space isomorphism. The statement now follows from Theorem 3.2 and Corollary 3.17. □

Proposition 3.19. *Let M^m be a differentiable manifold, (U, x) be a local chart on M and $\{e_k | \ k = 1, \dots, m\}$ be the canonical basis for $\mathbb{R}^m$. For an arbitrary point p in U we define $(\frac{\partial}{\partial x_k})_p$ in T_pM by*

$$\big(\frac{\partial}{\partial x_k}\big)_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \partial_{e_k}(f \circ x^{-1})(x(p)).$$

Then the set

$$\{(\frac{\partial}{\partial x_k})_p \mid k = 1, 2, \dots, m\}$$

is a basis for the tangent space T_pM of M at p.

Proof. The local chart $x : U \to x(U)$ is a diffeomorphism and the differential $(dx^{-1})_{x(p)} : T_{x(p)}\mathbb{R}^m \to T_pM$ of the inverse $x^{-1} : x(U) \to U$ satisfies

$$\begin{aligned} (dx^{-1})_{x(p)}(\partial_{e_k})(f) &= \partial_{e_k}(f \circ x^{-1})(x(p)) \\ &= \big(\frac{\partial}{\partial x_k}\big)_p(f) \end{aligned}$$

for all $f \in \varepsilon(p)$. The statement is then a direct consequence of Corollary 3.5. □

The rest of this chapter is devoted to the introduction of special types of differentiable maps, the so-called immersions, submersions and embeddings.

Definition 3.20. A differentiable map $\phi : M \to N$ between manifolds is said to be an **immersion** if for each $p \in M$ the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ is injective. An **embedding** is an immersion $\phi : M \to N$ which is a homeomorphism onto its image $\phi(M)$.

For positive integers m, n with $m < n$ we have the inclusion map $\phi : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ given by

$$\phi : (x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_{m+1}, 0, \dots, 0).$$

The differential $d\phi_x$ at x is injective since $d\phi_x(v) = (v, 0)$. The map ϕ is obviously a homeomorphism onto its image $\phi(\mathbb{R}^{m+1})$ hence an embedding. It is easily seen that even the restriction $\phi|_{S^m} : S^m \to S^n$ of ϕ to the m-dimensional unit sphere S^m in $\mathbb{R}^{m+1}$ is an embedding.

Definition 3.21. Let M be an m-dimensional differentiable manifold and U be an open subset of $\mathbb{R}^m$. An immersion $\phi : U \to M$ is called a local **parametrization** of M.

If M is a differentiable manifold and (U, x) a chart on M then the inverse $x^{-1} : x(U) \to U$ of x is a parametrization of the subset U of M.

Example 3.22. Let S^1 be the unit circle in the complex plane $\mathbb{C}$. For a non-zero integer $k \in \mathbb{Z}$ define $\phi_k : S^1 \to \mathbb{C}$ by $\phi_k : z \mapsto z^k$. For a point $w \in S^1$ let $\gamma_w : \mathbb{R} \to S^1$ be the curve with $\gamma_w : t \mapsto we^{it}$. Then $\gamma_w(0) = w$ and $\dot{\gamma}_w(0) = iw$. For the differential of ϕ_k we have

$$(d\phi_k)_w(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_k \circ \gamma_w(t))|_{t=0} = \frac{d}{dt}(w^k e^{ikt})|_{t=0} = kiw^k \neq 0.$$

This shows that the differential $(d\phi_k)_w : T_w S^1 \cong \mathbb{R} \to T_{w^k}\mathbb{C} \cong \mathbb{R}^2$ is injective, so the map ϕ_k is an immersion. It is easily seen that ϕ_k is an embedding if and only if $k = \pm 1$.

Example 3.23. Let $q \in S^3$ be a quaternion of unit length and $\phi_q : S^1 \to S^3$ be the map defined by $\phi_q : z \mapsto qz$. For $w \in S^1$ let $\gamma_w : \mathbb{R} \to S^1$ be the curve given by $\gamma_w(t) = we^{it}$. Then $\gamma_w(0) = w$, $\dot{\gamma}_w(0) = iw$ and $\phi_q(\gamma_w(t)) = qwe^{it}$. By differentiating we yield

$$d\phi_q(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_q(\gamma_w(t)))|_{t=0} = \frac{d}{dt}(qwe^{it})|_{t=0} = qiw.$$

Then $|d\phi_q(\dot{\gamma}_w(0))| = |qwi| = |q||w| = 1 \neq 0$ implies that the differential $d\phi_q$ is injective. It is easily checked that the immersion ϕ_q is an embedding.

In the next example we construct an interesting embedding of the real projective space $\mathbb{R}P^m$ into the vector space $\mathrm{Sym}(\mathbb{R}^{m+1})$ of the real symmetric $(m+1) \times (m+1)$ matrices.

Example 3.24. Let m be a positive integer and S^m be the m-dimensional unit sphere in $\mathbb{R}^{m+1}$. For a point $p \in S^m$ let

$$L_p = \{\lambda p \in \mathbb{R}^{m+1} | \ \lambda \in \mathbb{R}\}$$

be the line through the origin generated by p and $\rho_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ be the reflection about the line L_p. Then ρ_p is an element of $\mathrm{End}(\mathbb{R}^{m+1})$ i.e. the set of linear endomorphisms of $\mathbb{R}^{m+1}$ which can be identified with $\mathbb{R}^{(m+1)\times(m+1)}$. It is easily checked that the reflection about the line L_p is given by

$$\rho_p : q \mapsto 2\langle q, p\rangle p - q.$$

It then follows from the equations

$$\rho_p(q) = 2\langle q, p\rangle p - q = 2p\langle p, q\rangle - q = (2pp^t - e)q$$

that the matrix in $\mathbb{R}^{(m+1)\times(m+1)}$ corresponding to ρ_p is just

$$(2pp^t - e).$$

We shall now show that the map $\phi : S^m \to \mathbb{R}^{(m+1)\times(m+1)}$ given by

$$\phi : p \mapsto \rho_p$$

is an immersion. Let p be an arbitrary point on S^m and $\alpha, \beta : I \to S^m$ be two curves meeting at p, that is $\alpha(0) = p = \beta(0)$, with $X = \dot\alpha(0)$ and $Y = \dot\beta(0)$. For $\gamma \in \{\alpha, \beta\}$ we have

$$\phi \circ \gamma : t \mapsto (q \mapsto 2\langle q, \gamma(t)\rangle\gamma(t) - q)$$

so

$$\begin{aligned} (d\phi)_p(\dot\gamma(0)) &= \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} \\ &= (q \mapsto 2\langle q, \dot\gamma(0)\rangle\gamma(0) + 2\langle q, \gamma(0)\rangle\dot\gamma(0)). \end{aligned}$$

This means that

$$d\phi_p(X) = (q \mapsto 2\langle q, X\rangle p + 2\langle q, p\rangle X)$$

and

$$d\phi_p(Y) = (q \mapsto 2\langle q, Y\rangle p + 2\langle q, p\rangle Y).$$

If $X \neq Y$ then $d\phi_p(X) \neq d\phi_p(Y)$ so the differential $d\phi_p$ is injective, hence the map $\phi : S^m \to \mathbb{R}^{(m+1)\times(m+1)}$ is an immersion.

If the points $p, q \in S^m$ are linearly independent, then the lines L_p and L_q are different. But these are just the eigenspaces of ρ_p and ρ_q with the eigenvalue $+1$, respectively. This shows that the linear endomorphisms ρ_p, ρ_q of $\mathbb{R}^{m+1}$ are different in this case.

On the other hand, if p and q are parallel then $p = \pm q$ hence $\rho_p = \rho_q$. This means that the image $\phi(S^m)$ can be identified with the quotient space $S^m/\equiv$ where $\equiv$ is the equivalence relation defined by

$$x \equiv y \text{ if and only if } x = \pm y.$$

This of course is the real projective space $\mathbb{R}P^m$ so the map ϕ induces an embedding $\psi : \mathbb{R}P^m \to \text{Sym}(\mathbb{R}^{m+1})$ with

$$\psi : [p] \to \rho_p.$$

For each $p \in S^m$ the reflection $\rho_p : \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$ about the line L_p satisfies

$$\rho_p^t \cdot \rho_p = e.$$

This shows that the image $\psi(\mathbb{R}P^m) = \phi(S^m)$ is not only contained in the linear space $\text{Sym}(\mathbb{R}^{m+1})$ but also in the orthogonal group $\mathbf{O}(m+1)$ which we know is a submanifold of $\mathbb{R}^{(m+1)\times(m+1)}$

The following result was proved by Hassler Whitney in his famous paper, *Differentiable Manifolds*, Ann. of Math. **37** (1936), 645-680.

Deep Result 3.25. *For $1 \leq r \leq \infty$ let M be an m-dimensional C^r-manifold. Then there exists a C^r-embedding $\phi : M \to \mathbb{R}^{2m+1}$ into the $(2m+1)$-dimensional real vector space $\mathbb{R}^{2m+1}$.*

The classical inverse function theorem generalizes to the manifold setting as follows.

Theorem 3.26 (The Inverse Function Theorem)**.** *Let $\phi : M \to N$ be a differentiable map between manifolds with* $\dim M = \dim N$*. If p is a point in M such that the differential $d\phi_p : T_pM \to T_{\phi(p)}N$ at p is bijective then there exist open neighborhoods U_p around p and U_q around $q = \phi(p)$ such that $\psi = \phi|_{U_p} : U_p \to U_q$ is bijective and the inverse $\psi^{-1} : U_q \to U_p$ is differentiable.*

Proof. See Exercise 3.8 □

We shall now generalize the classical implicit function theorem to manifolds. For this we need the following definition.

Definition 3.27. Let m, n be positive natural numbers and $\phi : N^n \to M^m$ be a differentiable map between manifolds. A point $p \in N$ is said to be **critical** for ϕ if the differential

$$d\phi_p : T_pN \to T_{\phi(p)}M$$

is not of full rank, and **regular** if it is not critical. A point $q \in \phi(N)$ is said to be a **regular value** of ϕ if every point of the pre-image $\phi^{-1}(\{q\})$ of $\{q\}$ is regular and a **critical value** otherwise.

Theorem 3.28 (The Implicit Function Theorem)**.** *Let $\phi : N^n \to M^m$ be a differentiable map between manifolds such that $n > m$. If $q \in \phi(N)$ is a regular value, then the pre-image $\phi^{-1}(\{q\})$ of q is an $(n-m)$-dimensional submanifold of N^n. The tangent space $T_p\phi^{-1}(\{q\})$ of $\phi^{-1}(\{q\})$ at p is the kernel of the differential $d\phi_p$ i.e. $T_p\phi^{-1}(\{q\}) =$* Ker $d\phi_p$.

Proof. Let (V_q, ψ_q) be a chart on M with $q \in V_q$ and $\psi_q(q) = 0$. For a point $p \in \phi^{-1}(\{q\})$ we choose a chart (U_p, ψ_p) on N such that $p \in U_p$, $\psi_p(p) = 0$ and $\phi(U_p) \subset V_q$. The differential of the map

$$\hat{\phi} = \psi_q \circ \phi \circ \psi_p^{-1}|_{\psi_p(U_p)} : \psi_p(U_p) \to \mathbb{R}^m$$

at the point 0 is given by

$$d\hat{\phi}_0 = (d\psi_q)_q \circ d\phi_p \circ (d\psi_p^{-1})_0 : T_0\mathbb{R}^n \to T_0\mathbb{R}^m.$$

The pairs (U_p, ψ_p) and (V_q, ψ_q) are charts so the differentials $(d\psi_q)_q$ and $(d\psi_p^{-1})_0$ are bijective. This means that the differential $d\hat{\phi}_0$ is surjective since $d\phi_p$ is. It then follows from the implicit function theorem 2.14 that $\psi_p(\phi^{-1}(\{q\}) \cap U_p)$ is an $(n-m)$-dimensional submanifold of $\psi_p(U_p)$. Hence $\phi^{-1}(\{q\}) \cap U_p$ is an $(n-m)$-dimensional submanifold of U_p. This is true for each point $p \in \phi^{-1}(\{q\})$ so we have proven that $\phi^{-1}(\{q\})$ is an $(n-m)$-dimensional submanifold of N^n.

Let $\gamma : I \to \phi^{-1}(\{q\})$ be a curve, such that $\gamma(0) = p$. Then

$$(d\phi)_p(\dot\gamma(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = \frac{dq}{dt}|_{t=0} = 0.$$

This implies that $T_p\phi^{-1}(\{q\})$ is contained in and has the same dimension as the kernel of $d\phi_p$, so $T_p\phi^{-1}(\{q\}) = \text{Ker } d\phi_p$. □

Definition 3.29. For positive integers m, n with $m \leq n$ a map $\phi : N^n \to M^m$ between two manifolds is said to be a **submersion** if for each $p \in N$ the differential $d\phi_p : T_pN \to T_{\phi(p)}M$ is surjective.

If $m, n \in \mathbb{N}$ such that $m \leq n$ then we have the projection map $\pi : \mathbb{R}^n \to \mathbb{R}^m$ given by $\pi : (x_1, \dots, x_n) \mapsto (x_1, \dots, x_m)$. Its differential $d\pi_x$ at a point x is surjective since

$$d\pi_x(v_1, \dots, v_n) = (v_1, \dots, v_m).$$

This means that the projection is a submersion. An important submersion between spheres is given by the following.

Example 3.30. Let S^3 and S^2 be the unit spheres in $\mathbb{C}^2$ and $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$, respectively. The **Hopf map** $\phi : S^3 \to S^2$ is given by

$$\phi : (x, y) \mapsto (2x\bar y, |x|^2 - |y|^2).$$

For $p \in S^3$ the **Hopf circle** C_p through p is given by

$$C_p = \{e^{i\theta}(x, y) |\ \theta \in \mathbb{R}\}.$$

The following shows that the Hopf map is constant along each Hopf circle

$$\begin{aligned}
\phi(e^{i\theta}(x, y)) &= (2e^{i\theta}xe^{-i\theta}\bar y, |e^{i\theta}x|^2 - |e^{i\theta}y|^2) \\
&= (2x\bar y, |x|^2 - |y|^2) \\
&= \phi((x, y)).
\end{aligned}$$

The map ϕ and its differential $d\phi_p : T_pS^3 \to T_{\phi(p)}S^2$ are surjective for each $p \in S^3$. This implies that each point $q \in S^2$ is a regular value and the fibres of ϕ are 1-dimensional submanifolds of S^3. They are actually the great circles given by

$$\phi^{-1}(\{(2x\bar y, |x|^2 - |y|^2)\}) = \{e^{i\theta}(x, y) |\ \theta \in \mathbb{R}\}.$$

This means that the 3-dimensional sphere S^3 is a disjoint union of great circles

$$S^3 = \bigcup_{q \in S^2} \phi^{-1}(\{q\}).$$

Exercises

Exercise 3.1. Let p be an arbitrary point on the unit sphere S^{2n+1} of $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$. Determine the tangent space T_pS^{2n+1} and show that it contains an n-dimensional complex subspace of $\mathbb{C}^{n+1}$.

Exercise 3.2. Find a proof for Proposition 3.9 in your local library.

Exercise 3.3. Prove that the matrices

$$X_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis for the tangent space $T_e\mathbf{SL}_2(\mathbb{R})$ of the real special linear group $\mathbf{SL}_2(\mathbb{R})$ at the neutral element e. For each $k = 1, 2, 3$ find an explicit formula for the curve $\gamma_k : \mathbb{R} \to \mathbf{SL}_2(\mathbb{R})$ given by

$$\gamma_k : s \mapsto \text{Exp}(sX_k).$$

Exercise 3.4. Find a proof for Proposition 3.13.

Exercise 3.5. Prove that the matrices

$$Z_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad Z_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

form a basis for the tangent space $T_e\mathbf{SU}(2)$ of the special unitary group $\mathbf{SU}(2)$ at the neutral element e. For each $k = 1, 2, 3$ find an explicit formula for the curve $\gamma_k : \mathbb{R} \to \mathbf{SU}(2)$ given by

$$\gamma_k : s \mapsto \text{Exp}(sZ_k).$$

Exercise 3.6. For each $k \in \mathbb{N}_0$ define $\phi_k : \mathbb{C} \to \mathbb{C}$ and $\psi_k : \mathbb{C}^* \to \mathbb{C}$ by $\phi_k, \psi_k : z \mapsto z^k$. For which $k \in \mathbb{N}_0$ are ϕ_k, ψ_k immersions, submersions or embeddings.

Exercise 3.7. Prove that the map $\phi : \mathbb{R}^m \to \mathbb{C}^m$ given by

$$\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$$

is a parametrization of the m-dimensional torus T^m in $\mathbb{C}^m$.

Exercise 3.8. Find a proof for Theorem 3.26.

Exercise 3.9. Prove that the Hopf-map $\phi : S^3 \to S^2$ with $\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2)$ is a submersion.

CHAPTER 4

The Tangent Bundle

In this chapter we introduce the tangent bundle TM of a differentiable manifold M. Intuitively, this is the object that we get by glueing at each point p of M the corresponding tangent space T_pM. The differentiable structure on M induces a natural differentiable structure on the tangent bundle TM turning it into a differentiable manifold.

We have already seen that for a point $p \in \mathbb{R}^m$ the **tangent space** $T_p\mathbb{R}^m$ can be identified with the m-dimensional vector space $\mathbb{R}^m$. This means that if we at each point $p \in \mathbb{R}^m$ glue the tangent space $T_p\mathbb{R}^m$ to $\mathbb{R}^m$ we obtain the so called **tangent bundle** of $\mathbb{R}^m$

$$T\mathbb{R}^m = \{(p, v)|\ p \in \mathbb{R}^m \ \text{ and } \ v \in T_p\mathbb{R}^m\}.$$

For this we have the **natural projection** $\pi : T\mathbb{R}^m \to \mathbb{R}^m$ defined by

$$\pi : (p, v) \mapsto p$$

and for each point $p \in M$ the fibre $\pi^{-1}(\{p\})$ over p is precisely the tangent space $T_p\mathbb{R}^m$ at p.

Classically, a **vector field** X on $\mathbb{R}^m$ is a smooth map $X : \mathbb{R}^m \to \mathbb{R}^m$ but we would like to view it as a map $X : \mathbb{R}^m \to T\mathbb{R}^m$ into the tangent bundle and with abuse of notation write

$$X : p \mapsto (p, X(p)).$$

Following Proposition 3.19 two vector fields $X, Y : \mathbb{R}^m \to T\mathbb{R}^m$ can be written as

$$X = \sum_{k=1}^{m} a_k \frac{\partial}{\partial x_k} \ \text{ and } \ Y = \sum_{k=1}^{m} b_k \frac{\partial}{\partial x_k},$$

where $a_k, b_k : \mathbb{R}^m \to \mathbb{R}$ are smooth functions defined on $\mathbb{R}^m$. If $f : \mathbb{R}^m \to \mathbb{R}$ is another such function the **commutator** $[X, Y]$ acts on f as follows:

$$\begin{aligned} [X, Y](f) &= X(Y(f)) - Y(X(f)) \\ &= \sum_{k,l=1}^{m} \big(a_k \frac{\partial}{\partial x_k}(b_l \frac{\partial}{\partial x_l}) - b_k \frac{\partial}{\partial x_k}(a_l \frac{\partial}{\partial x_l})\big)(f) \end{aligned}$$

$$= \sum_{k,l=1}^{m} \big(a_k \frac{\partial b_l}{\partial x_k}\frac{\partial}{\partial x_l} + a_k b_l \frac{\partial^2}{\partial x_k \partial x_l}$$
$$-b_k \frac{\partial a_l}{\partial x_k}\frac{\partial}{\partial x_l} - b_k a_l \frac{\partial^2}{\partial x_k \partial x_l}\big)(f)$$
$$= \sum_{k,l=1}^{m} \big(a_k \frac{\partial b_l}{\partial x_k} - b_k \frac{\partial a_l}{\partial x_k}\big)\big(\frac{\partial}{\partial x_l}\big)(f).$$

This shows that the commutator $[X, Y]$ is a smooth vector field on $\mathbb{R}^m$.

We shall now generalize to the manifold setting. This leads us first to the following notion of a topological vector bundle.

Definition 4.1. Let E and M be topological manifolds and $\pi : E \to M$ be a continuous surjective map. The triple (E, M, π) is called an n-dimensional **topological vector bundle** over M if

(i) for each $p \in M$ the fibre $E_p = \pi^{-1}(\{p\})$ is an n-dimensional vector space,
(ii) for each $p \in M$ there exists a **bundle chart** $(\pi^{-1}(U), \psi)$ consisting of the pre-image $\pi^{-1}(U)$ of an open neighbourhood U of p and a homeomorphism $\psi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ such that for all $q \in U$ the map $\psi_q = \psi|_{E_q} : E_q \to \{q\} \times \mathbb{R}^n$ is a vector space isomorphism.

A continuous map $\sigma : M \to E$ is called a **section** of the bundle (E, M, π) if $\pi \circ \sigma(p) = p$ for each $p \in M$.

Definition 4.2. A topological vector bundle (E, M, π) over M, of dimension n, is said to be **trivial** if there exists a global bundle chart $\psi : E \to M \times \mathbb{R}^n$.

Example 4.3. If n is a positive integer and M is a topological manifold then we have the n-dimensional vector bundle $(M \times \mathbb{R}^n, M, \pi)$ where

$$\pi : M \times \mathbb{R}^n \to M$$

is the projection map $\pi : (p, v) \mapsto p$. The identity map $\psi : M \times \mathbb{R}^n \to M \times \mathbb{R}^n$ is a global bundle chart so the bundle is trivial.

Definition 4.4. Let (E, M, π) be an n-dimensional topological vector bundle over M. A collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha) |\ \alpha \in I\}$$

of bundle charts is called a **bundle atlas** for (E, M, π) if $M = \cup_\alpha U_\alpha$ and for $\alpha, \beta \in I$ there exists a map $A_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbf{GL}_n(\mathbb{R})$ such that the corresponding continuous map

$$\psi_\beta \circ \psi_\alpha^{-1}|_{(U_\alpha \cap U_\beta)\times\mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \to (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha,\beta}(p))(v)).$$

The elements of $\{A_{\alpha,\beta}|\ \alpha, \beta \in I\}$ are called the **transition maps** of the bundle atlas $\mathcal{B}$.

Definition 4.5. Let E and M be **differentiable** manifolds and $\pi : E \to M$ be a **differentiable** map such that (E, M, π) is an n-dimensional topological vector bundle. A bundle atlas $\mathcal{B}$ for (E, M, π) is said to be differentiable if the corresponding transition maps are differentiable. A **differentiable vector bundle** is a topological vector bundle together with a maximal differentiable bundle atlas. By $C^\infty(E)$ we denote the set of all smooth sections of (E, M, π).

From now on we shall assume, when not stating otherwise, that all our vector bundles are smooth.

Definition 4.6. Let (E, M, π) be a vector bundle over a manifold M. Then we define the operations $+$ and $\cdot$ on the set $C^\infty(E)$ of smooth sections of (E, M, π) by

(i) $(v + w)_p = v_p + w_p$,
(ii) $(f \cdot v)_p = f(p) \cdot v_p$

for all $v, w \in C^\infty(E)$ and $f \in C^\infty(M)$. If U is an open subset of M then a set $\{v_1, \dots, v_n\}$ of smooth sections $v_1, \dots, v_n : U \to E$ on U is said to be a **local frame** for E if for each $p \in U$ the set $\{(v_1)_p, \dots, (v_n)_p\}$ is a basis for the vector space E_p.

According to Definition 2.18, the set of smooth real-valued functions on M is denoted by $C^\infty(M)$. With the above defined operations on $C^\infty(E)$ it becomes a module over $C^\infty(M)$ and in particular a vector space over the real numbers as the constant functions in $C^\infty(M)$.

Example 4.7. Let M^m be a differentiable manifold with maximal atlas $\hat{\mathcal{A}}$. Define the set TM by

$$TM = \{(p, v)|\ p \in M \ \text{ and } \ v \in T_pM\}$$

and let $\pi : TM \to M$ be the **projection map** satisfying

$$\pi : (p, v) \mapsto p.$$

Then the fibre $\pi^{-1}(\{p\})$ over a point $p \in M$ is the m-dimensional **tangent space** T_pM. The triple (TM, M, π) is called the **tangent bundle** of M.

We shall now equip TM with the structure of a smooth manifold. For every local chart $x : U \to \mathbb{R}^m$ from the maximal atlas $\hat{\mathcal{A}}$ of M we define a local chart

$$x^* : \pi^{-1}(U) \to \mathbb{R}^m \times \mathbb{R}^m$$

on the tangent bundle TM by the formula

$$x^* : (p, \sum_{k=1}^m v_k \big(\frac{\partial}{\partial x_k}\big)_p) \mapsto (x(p), (v_1, \dots, v_m)).$$

Proposition 3.19 shows that the map x^* is well-defined. The collection

$$\{(x^*)^{-1}(W) \subset TM |\ (U, x) \in \hat{\mathcal{A}} \ \text{ and } \ W \subset x(U) \times \mathbb{R}^m \text{ open}\}$$

is a basis for a topology $\mathcal{T}_{TM}$ on TM and $(\pi^{-1}(U), x^*)$ is a chart on the $2m$-dimensional **topological manifold** $(TM, \mathcal{T}_{TM})$.

If (U, x) and (V, y) are two charts in $\hat{\mathcal{A}}$ such that $p \in U \cap V$ then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \to \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(a, b) \mapsto (y \circ x^{-1}(a), \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(x^{-1}(a)) b_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(x^{-1}(a)) b_k),$$

see Exercise 4.1. Since we are assuming that $y \circ x^{-1}$ is differentiable it follows that $(y^*) \circ (x^*)^{-1}$ is also differentiable. This means that

$$\mathcal{A}^* = \{(\pi^{-1}(U), x^*) |\ (U, x) \in \hat{\mathcal{A}}\}$$

is a C^r-atlas on TM so $(TM, \widehat{\mathcal{A}^*})$ is a **differentiable manifold**. The surjective projection map $\pi : TM \to M$ is clearly differentiable.

We shall now describe how the tangent bundle (TM, M, π) can be given the structure of an m-dimensional differentiable vector bundle. For each point $p \in M$ the fibre $\pi^{-1}(\{p\})$ is the tangent space T_pM and hence an m-dimensional vector space. For a local chart $x : U \to \mathbb{R}^m$ in the maximal atlas $\hat{\mathcal{A}}$ of M we define $\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$ by

$$\bar{x} : (p, \sum_{k=1}^m v_k \big(\frac{\partial}{\partial x_k}\big)_p) \mapsto (p, (v_1, \dots, v_m)).$$

The restriction $\bar{x}_p = \bar{x}|_{T_pM} : T_pM \to \{p\} \times \mathbb{R}^m$ to the tangent space T_pM is given by

$$\bar{x}_p : \sum_{k=1}^m v_k \big(\frac{\partial}{\partial x_k}\big)_p \mapsto (v_1, \dots, v_m),$$

so it is clearly a vector space isomorphism. This implies that the map

$$\bar{x} : \pi^{-1}(U) \to U \times \mathbb{R}^m$$

is a local bundle chart. If (U, x) and (V, y) are two charts in $\hat{\mathcal{A}}$ such that $p \in U \cap V$ then the transition map

$$(\bar{y}) \circ (\bar{x})^{-1} : (U \cap V) \times \mathbb{R}^m \to (U \cap V) \times \mathbb{R}^m$$

is given by

$$(p, b) \mapsto (p, \sum_{k=1}^{m} \frac{\partial y_1}{\partial x_k}(p) b_k, \dots, \sum_{k=1}^{m} \frac{\partial y_m}{\partial x_k}(p) b_k).$$

This shows that

$$\mathcal{B} = \{(\pi^{-1}(U), \bar{x}) |\ (U, x) \in \hat{\mathcal{A}}\}$$

is a bundle atlas turning (TM, M, π) into an m-dimensional topological vector bundle. It immediately follows from above that (TM, M, π) together with the maximal bundle atlas $\hat{\mathcal{B}}$ defined by $\mathcal{B}$ is a **differentiable vector bundle**.

Definition 4.8. Let M be a differentiable manifold, then a section $X : M \to TM$ of the tangent bundle is called a **vector field**. The set of smooth vector fields $X : M \to TM$ is denoted by $C^\infty(TM)$.

Example 4.9. We have seen earlier that the 3-sphere S^3 in $\mathbb{H} \cong \mathbb{C}^2$ carries a group structure $\cdot$ given by

$$(z, w) \cdot (\alpha, \beta) = (z\alpha - w\bar{\beta}, z\beta + w\bar{\alpha}).$$

This makes $(S^3, \cdot)$ into a Lie group with neutral element $e = (1, 0)$. Put $v_1 = (i, 0)$, $v_2 = (0, 1)$ and $v_3 = (0, i)$ and for $k = 1, 2, 3$ define the curves $\gamma_k : \mathbb{R} \to S^3$ with

$$\gamma_k : t \mapsto \cos t \cdot (1, 0) + \sin t \cdot v_k.$$

Then $\gamma_k(0) = e$ and $\dot{\gamma}_k(0) = v_k$ so v_1, v_2, v_3 are elements of the tangent space $T_e S^3$. They are linearly independent so they generate $T_e S^3$. The group structure on S^3 can be used to extend vectors in $T_e S^3$ to vector fields on S^3 as follows. For $p \in S^3$ let $L_p : S^3 \to S^3$ be the left translation on S^3 by p given by $L_p : q \mapsto p \cdot q$. Then define the vector fields $X_1, X_2, X_3 \in C^\infty(TS^3)$ by

$$(X_k)_p = (dL_p)_e(v_k) = \frac{d}{dt}(L_p(\gamma_k(t)))|_{t=0}.$$

It is left as an exercise for the reader to show that at a point $p = (z, w) \in S^3$ the values of X_k at p is given by

$$(X_1)_p \quad = \quad (z, w) \cdot (i, 0) = (iz, -iw),$$

$$\begin{aligned}(X_2)_p &= (z,w)\cdot(0,1) = (-w,z),\\ (X_3)_p &= (z,w)\cdot(0,i) = (iw,iz).\end{aligned}$$

Our next aim is to introduce the Lie bracket on the set of vector fields $C^\infty(TM)$ on M.

Definition 4.10. Let M be a smooth manifold. For two vector fields $X, Y \in C^\infty(TM)$ we define the **Lie bracket** $[X,Y]_p : C^\infty(M) \to \mathbb{R}$ of X and Y at $p \in M$ by

$$[X,Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

The next result shows that the Lie bracket $[X,Y]_p$ actually is an element of the tangent space T_pM.

Proposition 4.11. *Let M be a smooth manifold, $X, Y \in C^\infty(TM)$ be vector fields on M, $f, g \in C^\infty(M)$ and $\lambda, \mu \in \mathbb{R}$. Then*

(i) $[X,Y]_p(\lambda\cdot f + \mu\cdot g) = \lambda\cdot[X,Y]_p(f) + \mu\cdot[X,Y]_p(g)$,
(ii) $[X,Y]_p(f\cdot g) = [X,Y]_p(f)\cdot g(p) + f(p)\cdot[X,Y]_p(g)$.

Proof.

$$\begin{aligned}&[X,Y]_p(\lambda f + \mu g)\\ =\;& X_p(Y(\lambda f+\mu g)) - Y_p(X(\lambda f+\mu g))\\ =\;& \lambda X_p(Y(f)) + \mu X_p(Y(g)) - \lambda Y_p(X(f)) - \mu Y_p(X(g))\\ =\;& \lambda[X,Y]_p(f) + \mu[X,Y]_p(g).\end{aligned}$$

$$\begin{aligned}&[X,Y]_p(f\cdot g)\\ =\;& X_p(Y(f\cdot g)) - Y_p(X(f\cdot g))\\ =\;& X_p(f\cdot Y(g) + g\cdot Y(f)) - Y_p(f\cdot X(g) + g\cdot X(f))\\ =\;& X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(g)Y_p(f) + g(p)X_p(Y(f))\\ & -Y_p(f)X_p(g) - f(p)Y_p(X(g)) - Y_p(g)X_p(f) - g(p)Y_p(X(f))\\ =\;& f(p)\{X_p(Y(g)) - Y_p(X(g))\} + g(p)\{X_p(Y(f)) - Y_p(X(f))\}\\ =\;& f(p)[X,Y]_p(g) + g(p)[X,Y]_p(f).\end{aligned}$$

□

Proposition 4.11 implies that if X, Y are smooth vector fields on M then the map $[X,Y] : M \to TM$ given by $[X,Y] : p \mapsto [X,Y]_p$ is a section of the tangent bundle. In Proposition 4.13 we shall prove that this section is smooth. For this we need the following technical lemma.

Lemma 4.12. *Let M^m be a smooth manifold and $X : M \to TM$ be a section of TM. Then the following conditions are equivalent*

(i) *the section X is smooth,*

(ii) *if* (U, x) *is a chart on* M *then the functions* $a_1, \dots, a_m : U \to \mathbb{R}$ *given by*

$$X|_U = \sum_{k=1}^{m} a_k \frac{\partial}{\partial x_k},$$

are smooth,

(iii) *if* $f : V \to \mathbb{R}$ *defined on an open subset* V *of* M *is smooth, then the function* $X(f) : V \to \mathbb{R}$ *with* $X(f)(p) = X_p(f)$ *is smooth.*

Proof. $(i) \Rightarrow (ii)$: The functions $a_k = \pi_{m+k} \circ x^* \circ X|_U : U \to TM \to x(U) \times \mathbb{R}^m \to \mathbb{R}$ are restrictions of compositions of smooth maps so therefore smooth.

$(ii) \Rightarrow (iii)$: Let (U, x) be a chart on M such that U is contained in V. By assumption the map

$$X(f|_U) = \sum_{i=1}^{m} a_i \frac{\partial f}{\partial x_i}$$

is smooth. This is true for each such chart (U, x) so the function $X(f)$ is smooth.

$(iii) \Rightarrow (i)$: Note that the smoothness of the section X is equivalent to $x^* \circ X|_U : U \to \mathbb{R}^{2m}$ being smooth for all charts (U, x) on M. On the other hand, this is equivalent to $x^*_k = \pi_k \circ x^* \circ X|_U : U \to \mathbb{R}$ being smooth for all $k = 1, 2, \dots, 2m$ and all charts (U, x) on M. It is trivial that the coordinates $x^*_k = x_k$ for $k = 1, \dots, m$ are smooth. But $x^*_{m+k} = a_k = X(x_k)$ for $k = 1, \dots, m$ hence also smooth by assumption. □

Proposition 4.13. *Let* M *be a manifold and* $X, Y \in C^\infty(TM)$ *be vector fields on* M*. Then the section* $[X, Y] : M \to TM$ *of the tangent bundle given by* $[X, Y] : p \mapsto [X, Y]_p$ *is smooth.*

Proof. Let $f : M \to \mathbb{R}$ be an arbitrary smooth function on M then $[X, Y](f) = X(Y(f)) - Y(X(f))$ is smooth so it follows from Lemma 4.12 that the section $[X, Y]$ is smooth. □

For later use we prove the following useful result.

Lemma 4.14. *Let* M *be a smooth manifold and* $[,]$ *be the Lie bracket on the tangent bundle* TM*. Then*

(i) $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$,

(ii) $[f \cdot X, Y] = f \cdot [X, Y] - Y(f) \cdot X$

for all $X, Y \in C^\infty(TM)$ *and* $f \in C^\infty(M)$.

Proof. If $g \in C^\infty(M)$, then

$$[X, f \cdot Y](g) \quad = \quad X(f \cdot Y(g)) - f \cdot Y(X(g))$$

$$\begin{aligned} &= X(f)\cdot Y(g) + f\cdot X(Y(g)) - f\cdot Y(X(g)) \\ &= (X(f)\cdot Y + f\cdot [X,Y])(g). \end{aligned}$$

This proves the first statement and the second follows from the skew-symmetry of the Lie bracket. □

Definition 4.15. A real vector space $(V, +, \cdot)$ equipped with an operation $[,] : V \times V \to V$ is said to be a real **Lie algebra** if the following relations hold

(i) $[\lambda X + \mu Y, Z] = \lambda[X,Z] + \mu[Y,Z]$,
(ii) $[X,Y] = -[Y,X]$,
(iii) $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$

for all $X, Y, Z \in V$ and $\lambda, \mu \in \mathbb{R}$. The equation (iii) is called the **Jacobi identity**.

Theorem 4.16. *Let M be a smooth manifold. The vector space $C^\infty(TM)$ of smooth vector fields on M equipped with the Lie bracket $[,] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$ is a real Lie algebra.*

Proof. See exercise 4.4. □

Definition 4.17. If $\phi : M \to N$ is a surjective map between differentiable manifolds then two vector fields $X \in C^\infty(TM)$, $\bar{X} \in C^\infty(TN)$ are said to be **ϕ-related** if $\bar{X}_{\phi(p)} = d\phi_p(X_p)$ for all $p \in M$. In that case we write $\bar{X} = d\phi(X)$.

Proposition 4.18. *Let $\phi : M \to N$ be a map between differentiable manifolds, $X, Y \in C^\infty(TM)$, $\bar{X}, \bar{Y} \in C^\infty(TN)$ such that $\bar{X} = d\phi(X)$ and $\bar{Y} = d\phi(Y)$. Then*

$$[\bar{X}, \bar{Y}] = d\phi([X,Y]).$$

Proof. Let $f : N \to \mathbb{R}$ be a smooth function, then

$$\begin{aligned} [\bar{X},\bar{Y}](f) &= d\phi(X)(d\phi(Y)(f)) - d\phi(Y)(d\phi(X)(f)) \\ &= X(d\phi(Y)(f)\circ\phi) - Y(d\phi(X)(f)\circ\phi) \\ &= X(Y(f\circ\phi)) - Y(X(f\circ\phi)) \\ &= [X,Y](f\circ\phi) \\ &= d\phi([X,Y])(f). \end{aligned}$$

□

Proposition 4.19. *Let $\phi : M \to N$ be a smooth bijective map between differentiable manifolds. If $X, Y \in C^\infty(TM)$ are vector fields on M, then*

(i) $d\phi(X) \in C^\infty(TN)$,

(ii) *the map* $d\phi : C^\infty(TM) \to C^\infty(TN)$ *is a Lie algebra homomorphism i.e.* $[d\phi(X), d\phi(Y)] = d\phi([X,Y])$.

Proof. The fact that the map ϕ is bijective implies that $d\phi(X)$ is a section of the tangent bundle. That $d\phi(X) \in C^\infty(TN)$ follows directly from the fact that

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p).$$

The last statement is a direct consequence of Proposition 4.18. □

Definition 4.20. Let M be a smooth manifold. Two vector fields $X, Y \in C^\infty(TM)$ are said to **commute** if $[X,Y] = 0$.

Proposition 4.21. *Let* M *be a differentiable manifold,* (U,x) *be local coordinates on* M *and*

$$\{\frac{\partial}{\partial x_k} \mid k = 1, 2, \dots, m\}$$

be the induced local frame for the tangent bundle TM*. Then the local frame fields commute i.e.*

$$[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}] = 0 \quad \text{for all} \quad k, l = 1, \dots, m.$$

Proof. The map $x : U \to x(U)$ is bijective and differentiable. The vector field $\partial/\partial x_k \in C^\infty(TU)$ is x-related to the coordinate vector field $\partial_{e_k} \in C^\infty(Tx(U))$. Then Proposition 4.19 implies that

$$dx([\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}]) = [\partial_{e_k}, \partial_{e_l}] = 0.$$

The last equation is an immediate consequence of the following well-known fact

$$[\partial_{e_k}, \partial_{e_l}](f) = \partial_{e_k}(\partial_{e_l}(f)) - \partial_{e_l}(\partial_{e_k}(f)) = 0$$

for all $f \in C^\infty(x(U))$. □

Definition 4.22. Let G be a Lie group with neutral element e. For $p \in G$ let $L_p : G \to G$ be the left translation by p with $L_p : q \mapsto pq$. A vector field $X \in C^\infty(TG)$ is said to be **left-invariant** if

$$X = dL_p(X) \quad \text{for all} \quad p \in G,$$

or equivalently,

$$X_{pq} = (dL_p)_q(X_q) \quad \text{for all} \quad p, q \in G.$$

The set of all left-invariant vector fields on G is called the **Lie algebra** of G and denoted by $\mathfrak{g}$.

The Lie algebras of the classical Lie groups introduced earlier are denoted by $\mathfrak{gl}_m(\mathbb{R})$, $\mathfrak{sl}_m(\mathbb{R})$, $\mathfrak{o}(m)$, $\mathfrak{so}(m)$, $\mathfrak{gl}_m(\mathbb{C})$, $\mathfrak{sl}_m(\mathbb{C})$, $\mathfrak{u}(m)$ and $\mathfrak{su}(m)$, respectively.

Note that if $X \in \mathfrak{g}$ is a left-invariant vector field on G then

$$X_p = (dL_p)_e(X_e)$$

so the value X_p of X at $p \in G$ is completely determined by the value X_e of X at e. This means that the map $\Phi : T_eG \to \mathfrak{g}$ given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

is a vector space isomorphism.

Proposition 4.23. *Let G be a Lie group and $\mathfrak{g}$ be the Lie algebra of G. Then $\mathfrak{g}$ is a Lie* **subalgebra** *of $C^\infty(TG)$ i.e. if $X, Y \in \mathfrak{g}$ then $[X, Y] \in \mathfrak{g}$,*

Proof. If $p \in G$ then the left translation $L_p : G \to G$ is a diffeomorphism so it follows from Proposition 4.19 that

$$dL_p([X, Y]) = [dL_p(X), dL_p(Y)] = [X, Y]$$

for all $X, Y \in \mathfrak{g}$. This proves that the Lie bracket $[X, Y]$ of two left-invariant vector fields X, Y is left-invariant and thereby that $\mathfrak{g}$ is a Lie subalgebra of $C^\infty(TG)$. □

The linear isomorphism $\Phi : T_eG \to \mathfrak{g}$ given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

induces a natural Lie bracket $[\ ,] : T_eG \times T_eG \to T_eG$ on the tangent space T_eG given by

$$[X_e, Y_e] = [X, Y]_e.$$

Proposition 4.24. *Let G be one of the classical Lie groups and T_eG be the tangent space of G at the neutral element e. Then the Lie bracket*

$$[\ ,] : T_eG \times T_eG \to T_eG$$

of T_eG is given by

$$[X_e, Y_e] = X_e \cdot Y_e - Y_e \cdot X_e$$

where $\cdot$ is the usual matrix multiplication.

Proof. We shall prove the result for the case when G is the real general linear group $\mathbf{GL}_m(\mathbb{R})$. For the other real classical Lie groups the result follows from the fact that they are all subgroups of $\mathbf{GL}_m(\mathbb{R})$. The same proof can be used for the complex cases.

Let $X, Y \in \mathfrak{gl}_m(\mathbb{R})$ be left-invariant vector fields on $\mathbf{GL}_m(\mathbb{R})$, $f : U \to \mathbb{R}$ be a function defined locally around the identity element $e \in$

$\mathbf{GL}_m(\mathbb{R})$ and p be an arbitrary point in U. Then the derivative $X_p(f)$ is given by

$$X_p(f) = \frac{d}{dt}(f(p \cdot \mathrm{Exp}(tX_e)))|_{t=0} = df_p(p \cdot X_e) = df_p(X_p).$$

The real general linear group $\mathbf{GL}_m(\mathbb{R})$ is an open subset of $\mathbb{R}^{m\times m}$ so we can use well-known rules from calculus and the second derivative $Y_e(X(f))$ is obtained as follows:

$$\begin{aligned} Y_e(X(f)) &= \frac{d}{dt}(X_{\mathrm{Exp}(tY_e)}(f))|_{t=0} \\ &= \frac{d}{dt}(df_{\mathrm{Exp}(tY_e)}(\mathrm{Exp}(tY_e) \cdot X_e))|_{t=0} \\ &= d^2 f_e(Y_e, X_e) + df_e(Y_e \cdot X_e). \end{aligned}$$

The Hessian $d^2 f_e$ of f is symmetric, hence

$$[X, Y]_e(f) = X_e(Y(f)) - Y_e(X(f)) = df_e(X_e \cdot Y_e - Y_e \cdot X_e).$$

□

Theorem 4.25. *Let G be a Lie group. Then the tangent bundle TG of G is trivial.*

Proof. Let $\{(X_1)_e, \dots, (X_m)_e\}$ be a basis for T_eG and extend each $(X_k)_e \in T_eG$ to the left-invariant vector field $X_k \in \mathfrak{g}$ with

$$(X_k)_p = (dL_p)_e((X_k)_e).$$

For each $p \in G$ the left translation $dL_p : G \to G$ is a diffeomorphism so the set $\{(X_1)_p, \dots, (X_m)_p\}$ is a basis for the tangent space T_pG. This implies that the map $\psi : TG \to G \times \mathbb{R}^m$ given by

$$\psi : (p, \sum_{k=1}^{m} v_k \cdot (X^k)_p) \mapsto (p, (v_1, \dots, v_m))$$

is well-defined. Furthermore it is a global bundle chart so the tangent bundle TG is trivial. □

Exercises

Exercise 4.1. Let $(M^m, \hat{\mathcal{A}})$ be a smooth manifold and (U, x), (V, y) be two charts in $\hat{\mathcal{A}}$ such that $U \cap V \neq \emptyset$. Let

$$f = y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^m$$

be the corresponding transition map. Show that the local frames

$$\{\frac{\partial}{\partial x_i} |\ i = 1, \dots, m\} \quad \text{and} \quad \{\frac{\partial}{\partial y_j} |\ j = 1, \dots, m\}$$

for TM on $U \cap V$ are related by

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^{m} \frac{\partial (f_j \circ x)}{\partial x_i} \cdot \frac{\partial}{\partial y_j}.$$

Exercise 4.2. Let m be a positive integer an $\mathbf{SO}(m)$ be the corresponding special orthogonal group.

(i) Find a basis for the tangent space $T_e\mathbf{SO}(m)$,
(ii) construct a non-vanishing vector field $Z \in C^\infty(T\mathbf{SO}(m))$,
(iii) determine all smooth vector fields on $\mathbf{SO}(2)$.

The Hairy Ball Theorem. Let m be a positive integer. Then there does not exist a continuous non-vanishing vector field $X \in C^0(TS^{2m})$ on the even dimensional sphere S^{2m}.

Exercise 4.3. Let m be a positive integer. Use the Hairy Ball Theorem to prove that the tangent bundles TS^{2m} of the even-dimensional spheres S^{2m} are not trivial. Construct a non-vanishing vector field $X \in C^\infty(TS^{2m+1})$ on the odd-dimensional sphere S^{2m+1}.

Exercise 4.4. Find a proof for Theorem 4.16.

CHAPTER 5

Riemannian Manifolds

In this chapter we introduce the notion of a Riemannian manifold (M, g). The metric g provides us with an inner product on each tangent space and can be used to measure angles and the lengths of curves in the manifold. This defines a distance function and turns the manifold into a metric space in a natural way. The Riemannian metric on a differentiable manifold is an important example of what is called a tensor field.

Let M be a smooth manifold, $C^\infty(M)$ denote the commutative ring of smooth functions on M and $C^\infty(TM)$ be the set of smooth vector fields on M forming a module over $C^\infty(M)$. Put

$$C_0^\infty(TM) = C^\infty(M)$$

and for each positive integer $r \in \mathbb{Z}^+$ let

$$C_r^\infty(TM) = C^\infty(TM) \otimes \cdots \otimes C^\infty(TM)$$

be the r-fold tensor product of $C^\infty(TM)$ over $C^\infty(M)$.

Definition 5.1. Let M be a differentiable manifold. A smooth **tensor field** A on M of type (r, s) is a map $A : C_r^\infty(TM) \to C_s^\infty(TM)$ which is multi-linear over $C^\infty(M)$ i.e. satisfying

$$\begin{aligned} & A(X_1 \otimes \cdots \otimes X_{k-1} \otimes (f \cdot Y + g \cdot Z) \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ = \ & f \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Y \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ & + g \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Z \otimes X_{k+1} \otimes \cdots \otimes X_r) \end{aligned}$$

for all $X_1, \dots, X_r, Y, Z \in C^\infty(TM)$, $f, g \in C^\infty(M)$ and $k = 1, \dots, r$. For the rest of this work we shall for $A(X_1 \otimes \cdots \otimes X_r)$ use the notation

$$A(X_1, \dots, X_r).$$

The next general result provides us with the most important property of a tensor field. It shows that the value of $A(X_1, \dots, X_r)$ at a point $p \in M$ only depends on the values of the vector fields $X_1, \dots, X_r$ at p and is independent of their values away from p.

Proposition 5.2. *Let $A : C^\infty_r(TM) \to C^\infty_s(TM)$ be a tensor field of type (r, s) and $p \in M$. Let $X_1, \dots, X_r$ and $Y_1, \dots, Y_r$ be smooth vector fields on M such that $(X_k)_p = (Y_k)_p$ for each $k = 1, \dots, r$. Then*

$$A(X_1, \dots, X_r)(p) = A(Y_1, \dots, Y_r)(p).$$

Proof. We shall prove the statement for $r = 1$, the rest follows by induction. Put $X = X_1$ and $Y = Y_1$ and let (U, x) be local coordinates on M. Choose a function $f \in C^\infty(M)$ such that $f(p) = 1$,

$$\text{support}(f) = \overline{\{p \in M|\ f(p) \neq 0\}}$$

is contained in U and define the vector fields $v_1, \dots, v_m \in C^\infty(TM)$ on M by

$$(v_k)_q = \begin{cases} f(q) \cdot (\frac{\partial}{\partial x_k})_q & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

Then there exist functions $\rho_k, \sigma_k \in C^\infty(M)$ such that

$$f \cdot X = \sum_{k=1}^m \rho_k \cdot v_k \quad \text{and} \quad f \cdot Y = \sum_{k=1}^m \sigma_k \cdot v_k.$$

This implies that

$$\begin{aligned} A(X)(p) &= f(p)A(X)(p) \\ &= (f \cdot A(X))(p) \\ &= A(f \cdot X)(p) \\ &= A(\sum_{k=1}^m \rho_k \cdot v_k)(p) \\ &= \sum_{k=1}^m (\rho_k \cdot A(v_k))(p) \\ &= \sum_{k=1}^m \rho_k(p) A(v_k)(p) \end{aligned}$$

and similarly

$$A(Y)(p) = \sum_{k=1}^m \sigma_k(p) A(v_k)(p).$$

The fact that $X_p = Y_p$ shows that $\rho_k(p) = \sigma_k(p)$ for all k. As a direct consequence we see that

$$A(X)(p) = A(Y)(p).$$

□

For a tensor A we shall by A_p denote the multi-linear restriction of A to the r-fold tensor product

$$T_pM \otimes \cdots \otimes T_pM$$

of the vector space T_pM over $\mathbb{R}$ given by

$$A_p : ((X_1)_p, \dots, (X_r)_p) \mapsto A(X_1, \dots, X_r)(p).$$

Definition 5.3. Let M be a smooth manifold. A **Riemannian metric** g on M is a tensor field

$$g : C_2^\infty(TM) \to C_0^\infty(TM)$$

such that for each $p \in M$ the restriction

$$g_p = g|_{T_pM \otimes T_pM} : T_pM \otimes T_pM \to \mathbb{R}$$

with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is an inner product on the tangent space T_pM. The pair (M, g) is called a **Riemannian manifold**. The study of Riemannian manifolds is called Riemannian Geometry. Geometric properties of (M, g) which only depend on the metric g are called **intrinsic** or metric properties.

Definition 5.4. Let $\gamma : I \to M$ be a C^1-curve in M. Then the **length** $L(\gamma)$ of γ is defined by

$$L(\gamma) = \int_I \sqrt{g(\dot\gamma(t), \dot\gamma(t))}dt.$$

Example 5.5. The standard inner product on the vector space $\mathbb{R}^m$ given by

$$\langle X, Y\rangle_{\mathbb{R}^m} = X^t \cdot Y = \sum_{k=1}^m X_k Y_k$$

defines a Riemannian metric on $\mathbb{R}^m$. The Riemannian manifold

$$E^m = (\mathbb{R}^m, \langle, \rangle_{\mathbb{R}^m})$$

is called the m-dimensional **Euclidean space**.

Example 5.6. For $m \in \mathbb{Z}^+$ equip the real vector space $\mathbb{R}^m$ with the Riemannian metric g given by

$$g_p(X, Y) = \frac{4}{(1 + |p|^2_{\mathbb{R}^m})^2}\langle X, Y\rangle_{\mathbb{R}^m}.$$

The Riemannian manifold $\Sigma^m = (\mathbb{R}^m, g)$ is called the m-dimensional **punctured round sphere**. Let $\gamma : \mathbb{R}^+ \to \Sigma^m$ be the curve with

$\gamma : t \mapsto (t, 0, \dots, 0)$. Then the length $L(\gamma)$ of γ can be determined as follows

$$L(\gamma) = 2\int_0^\infty \frac{\sqrt{\langle \dot\gamma, \dot\gamma\rangle}}{1+|\gamma|^2} dt = 2\int_0^\infty \frac{dt}{1+t^2} = 2[\arctan(t)]_0^\infty = \pi.$$

Example 5.7. Let $B_1^m(0)$ be the open unit ball in $\mathbb{R}^m$ given by

$$B_1^m(0) = \{p \in \mathbb{R}^m |\ |p|_{\mathbb{R}^m} < 1\}.$$

By the m-dimensional **hyperbolic ball** we mean $B_1^m(0)$ equipped with the Riemannian metric

$$g_p(X,Y) = \frac{4}{(1-|p|^2_{\mathbb{R}^m})^2}\langle X, Y\rangle_{\mathbb{R}^m}.$$

Let $\gamma : (0,1) \to B_1^m(0)$ be a curve given by $\gamma : t \mapsto (t, 0, \dots, 0)$. Then

$$L(\gamma) = 2\int_0^1 \frac{\sqrt{\langle \dot\gamma, \dot\gamma\rangle}}{1-|\gamma|^2} dt = 2\int_0^1 \frac{dt}{1-t^2} = [\log(\frac{1+t}{1-t})]_0^1 = \infty$$

As we shall now see a Riemannian manifold (M, g) has the structure of a metric space (M, d) in a natural way.

Proposition 5.8. *Let (M, g) be a Riemannian manifold. For two points $p, q \in M$ let C_{pq} denote the set of C^1-curves $\gamma : [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$ and define the function $d : M \times M \to \mathbb{R}_0^+$ by*

$$d(p,q) = \inf\{L(\gamma) |\ \gamma \in C_{pq}\}.$$

Then (M, d) is a **metric space** *i.e. for all $p, q, r \in M$ we have*

(i) $d(p,q) \geq 0$,
(ii) $d(p,q) = 0$ *if and only if* $p = q$,
(iii) $d(p,q) = d(q,p)$,
(iv) $d(p,q) \leq d(p,r) + d(r,q)$.

The topology on M induced by the metric d is identical to the one M carries as a topological manifold $(M, \mathcal{T})$, see Definition 2.1.

Proof. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998). □

A Riemannian metric on a differentiable manifold induces a Riemannian metric on any of its submanifolds as follows.

Definition 5.9. Let (N, h) be a Riemannian manifold and M be a submanifold of N. Then the smooth tensor field $g : C_2^\infty(TM) \to C_0^\infty(M)$ given by

$$g(X,Y) : p \mapsto h_p(X_p, Y_p).$$

is a Riemannian metric on M called the **induced metric** on M in (N, h).

Example 5.10. The Euclidean metric $\langle , \rangle_{\mathbb{R}^n}$ on $\mathbb{R}^n$ induces Riemannian metrics on the following submanifolds.

(i) the m-dimensional sphere $S^m \subset \mathbb{R}^n$, with $n = m + 1$,
(ii) the tangent bundle $TS^m \subset \mathbb{R}^n$, where $n = 2(m + 1)$,
(iii) the m-dimensional torus $T^m \subset \mathbb{R}^n$, with $n = 2m$,
(iv) the m-dimensional real projective space $\mathbb{R}P^m \subset \text{Sym}(\mathbb{R}^{m+1}) \cong \mathbb{R}^n$, where $n = (m + 2)(m + 1)/2$.

Example 5.11. The vector space $\mathbb{C}^{m\times m}$ of complex $m\times m$ matrices carries a natural Euclidean metric g given by

$$g(Z, W) = \text{Re}(\text{trace}(\bar{Z}^t W)) \quad \text{for all} \quad Z, W \in \mathbb{C}^{m\times m}.$$

This induces metrics on the submanifolds of $\mathbb{C}^{m\times m}$ such as $\mathbb{R}^{m\times m}$ and the classical Lie groups $\mathbf{GL}_m(\mathbb{R})$, $\mathbf{SL}_m(\mathbb{R})$, $\mathbf{O}(m)$, $\mathbf{SO}(m)$, $\mathbf{GL}_m(\mathbb{C})$, $\mathbf{SL}_m(\mathbb{C})$, $\mathbf{U}(m)$, $\mathbf{SU}(m)$.

Our next important step is to prove that every differentiable manifold M can be equipped with a Riemannian metric g. For this we need the following fact from topology.

Fact 5.12. *Every locally compact Hausdorff space with countable basis is paracompact.*

Corollary 5.13. *Let $(M, \mathcal{T})$ be a topological manifold. Let the collection $(U_\alpha)_{\alpha\in I}$ be an open covering of M such that for each $\alpha \in I$ the pair (U_α, ψ_α) is a chart on M. Then there exists*

(i) *a locally finite open refinement $(W_\beta)_{\beta\in J}$ such that for all $\beta \in J$, W_β is an open neighbourhood for a chart (W_β, ψ_β), and*
(ii) *a partition of unity $(f_\beta)_{\beta\in J}$ such that support$(f_\beta) \subset W_\beta$.*

Theorem 5.14. *Let $(M^m, \hat{\mathcal{A}})$ be a differentiable manifold. Then there exists a Riemannian metric g on M.*

Proof. For each point $p \in M$ let $(U_p, \phi_p) \in \hat{\mathcal{A}}$ be a chart such that $p \in U_p$. Then $(U_p)_{p\in M}$ is an open covering as in Corollary 5.13. Let $(W_\beta)_{\beta\in J}$ be a locally finite open refinement, (W_β, x^β) be charts on M and $(f_\beta)_{\beta\in J}$ be a partition of unity such that support(f_β) is contained in W_β. Let $\langle , \rangle_{\mathbb{R}^m}$ be the Euclidean metric on $\mathbb{R}^m$. Then for $\beta \in J$ define $g_\beta : C_2^\infty(TM) \to C_0^\infty(TM)$ by

$$g_\beta(\frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta})(p) = \begin{cases} f_\beta(p) \cdot \langle e_k, e_l\rangle_{\mathbb{R}^m} & \text{if } p \in W_\beta \\ 0 & \text{if } p \notin W_\beta \end{cases}$$

Note that at each point only finitely many of g_β are non-zero. This means that the well-defined tensor $g : C_2^\infty(TM) \to C_0^\infty(TM)$ given by

$$g = \sum_{\beta \in J} g_\beta$$

is a Riemannian metric on M. □

Definition 5.15. Let (M, g) and (N, h) be Riemannian manifolds. A map $\phi : (M, g) \to (N, h)$ is said to be **conformal** if there exists a function $\lambda : M \to \mathbb{R}$ such that

$$e^{\lambda(p)} g_p(X_p, Y_p) = h_{\phi(p)}(d\phi_p(X_p), d\phi_p(Y_p)),$$

for all $X, Y \in C^\infty(TM)$ and $p \in M$. The function e^λ is called the **conformal factor** of ϕ. A conformal map with $\lambda \equiv 0$ is said to be **isometric**. An isometric diffeomorphism is called an **isometry**.

Example 5.16. On the standard unit sphere S^m we have an action $\mathbf{O}(m+1) \times S^m \to S^m$ of the orthogonal group $\mathbf{O}(m+1)$ given by

$$(p, x) \mapsto p \cdot x$$

where $\cdot$ is the standard matrix multiplication. The following shows that the $\mathbf{O}(m+1)$-action on S^m is isometric

$$\langle pX, pY \rangle = X^t p^t p Y = X^t Y = \langle X, Y \rangle.$$

Example 5.17. Equip the orthogonal group $\mathbf{O}(m)$ as a submanifold of $\mathbb{R}^{m \times m}$ with the induced metric given by

$$\langle X, Y \rangle = \text{trace}(X^t Y).$$

For $p \in \mathbf{O}(m)$ the left translation $L_p : \mathbf{O}(m) \to \mathbf{O}(m)$ by p is given by $L_p : q \mapsto pq$. The tangent space $T_q\mathbf{O}(m)$ of $\mathbf{O}(m)$ at q is

$$T_q\mathbf{O}(m) = \{qX |\ X^t + X = 0\}$$

and the differential $(dL_p)_q : T_q\mathbf{O}(m) \to T_{pq}\mathbf{O}(m)$ of L_p is given by

$$(dL_p)_q : qX \mapsto pqX.$$

We then have

$$\begin{aligned}
\langle (dL_p)_q(qX), (dL_p)_q(qY) \rangle_{pq} &= \text{trace}((pqX)^t pqY) \\
&= \text{trace}(X^t q^t p^t pqY) \\
&= \text{trace}(qX)^t (qY). \\
&= \langle qX, qY \rangle_q.
\end{aligned}$$

This shows that the left translation $L_p : \mathbf{O}(m) \to \mathbf{O}(m)$ is an isometry for each $p \in \mathbf{O}(m)$.

Definition 5.18. Let G be a Lie group. A Riemannian metric g on G is said to be **left-invariant** if for each $p \in G$ the left translation $L_p : G \to G$ is an isometry.

As for the orthogonal group $\mathbf{O}(m)$ an inner product on the tangent space at the neutral element of any Lie group G can be transported via the left translations to obtain a left-invariant Riemannian metric on the group.

Proposition 5.19. *Let G be a Lie group and $\langle , \rangle_e$ be an inner product on the tangent space T_eG at the neutral element e. Then for each $p \in G$ the bilinear map $g_p(,) : T_pG \times T_pG \to \mathbb{R}$ with*

$$g_p(X_p, Y_p) = \langle dL_{p^{-1}}(X_p), dL_{p^{-1}}(Y_p)\rangle_e$$

is an inner product on the tangent space T_pG. The smooth tensor field $g : C_2^\infty(TG) \to C_0^\infty(G)$ given by

$$g : (X, Y) \mapsto (g(X, Y) : p \mapsto g_p(X_p, Y_p))$$

is a left-invariant Riemannian metric on G.

Proof. See Exercise 5.5. □

We shall now equip the real projective space $\mathbb{R}P^m$ with a Riemannian metric.

Example 5.20. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ and $\text{Sym}(\mathbb{R}^{m+1})$ be the linear space of symmetric real $(m+1)\times(m+1)$ matrices equipped with the metric g given by

$$g(X, Y) = \frac{1}{8}\text{trace}(X^t \cdot Y).$$

As in Example 3.24 we define a map $\phi : S^m \to \text{Sym}(\mathbb{R}^{m+1})$ by

$$\phi : p \mapsto (\rho_p : q \mapsto 2\langle q, p\rangle p - q).$$

Let $\alpha, \beta : \mathbb{R} \to S^n$ be two curves such that $\alpha(0) = p = \beta(0)$ and put $X = \dot\alpha(0)$, $Y = \dot\beta(0)$. Then for $\gamma \in \{\alpha, \beta\}$ we have

$$d\phi_p(\dot\gamma(0)) = (q \mapsto 2\langle q, \dot\gamma(0)\rangle p + 2\langle q, p\rangle\dot\gamma(0)).$$

If $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^{m+1}$, then

$$\begin{aligned} & g(d\phi_p(X), d\phi_p(Y)) \\ = \ & \frac{1}{8}\text{trace}(d\phi_p(X)^t \cdot d\phi_p(Y)) \\ = \ & \frac{1}{8}\sum_{q\in\mathcal{B}}\langle q, d\phi_p(X)^t \cdot d\phi_p(Y)q\rangle \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{8}\sum_{q\in\mathcal{B}}\langle d\phi_p(X)q, d\phi_p(Y)q\rangle \\
&= \frac{1}{2}\sum_{q\in\mathcal{B}}\langle \langle q,X\rangle p + \langle q,p\rangle X, \langle q,Y\rangle p + \langle q,p\rangle Y\rangle \\
&= \frac{1}{2}\sum_{q\in\mathcal{B}}\{\langle p,p\rangle\langle X,q\rangle\langle q,Y\rangle + \langle X,Y\rangle\langle p,q\rangle\langle p,q\rangle\} \\
&= \frac{1}{2}\{\langle X,Y\rangle + \langle X,Y\rangle\} \\
&= \langle X,Y\rangle.
\end{aligned}$$

This proves that the immersion ϕ is isometric. In Example 3.24 we have seen that the image $\phi(S^m)$ can be identified with the real projective space $\mathbb{R}P^m$. This inherits the induced metric from $\mathbb{R}^{(m+1)\times(m+1)}$ and the map $\phi : S^m \to \mathbb{R}P^m$ is what is called an isometric double cover of $\mathbb{R}P^m$.

Long before John Nash became famous in Hollywood he proved the next remarkable result in his paper *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20-63. It implies that every Riemannian manifold can be realized as a submanifold of a Euclidean space. The original proof of Nash was later simplified, see for example Matthias Gunther, *On the perturbation problem associated to isometric embeddings of Riemannian manifolds*, Annals of Global Analysis and Geometry **7** (1989), 69-77.

Deep Result 5.21. *For $3 \le r \le \infty$ let (M,g) be a Riemannian C^r-manifold. Then there exists an isometric C^r-embedding of (M,g) into a Euclidean space $\mathbb{R}^n$.*

We shall now see that parametrizations can be very useful tools for studying the intrinsic geometry of a Riemannian manifold (M,g). Let p be a point of M and $\hat\psi : U \to M$ be a local parametrization of M with $q \in U$ and $\hat\psi(q) = p$. The differential $d\hat\psi_q : T_q\mathbb{R}^m \to T_pM$ is bijective so there exist neighbourhoods U_q of q and U_p of p such that the restriction $\psi = \hat\psi|_{U_q} : U_q \to U_p$ is a diffeomorphism. On U_q we have the canonical frame $\{e_1,\dots,e_m\}$ for TU_q so $\{d\psi(e_1),\dots,d\psi(e_m)\}$ is a local frame for TM over U_p. We then define the pull-back metric $\tilde g = \psi^* g$ on U_q by

$$\tilde g(e_k,e_l) = g(d\psi(e_k), d\psi(e_l)).$$

Then $\psi : (U_q,\tilde g) \to (U_p,g)$ is an isometry so the intrinsic geometry of $(U_q,\tilde g)$ and that of (U_p,g) are exactly the same.

Example 5.22. Let G be one of the classical Lie groups and e be the neutral element of G. Let $\{X_1, \dots, X_m\}$ be a basis for the Lie algebra $\mathfrak{g}$ of G. For $p \in G$ define $\psi_p : \mathbb{R}^m \to G$ by

$$\psi_p : (t_1, \dots, t_m) \mapsto L_p(\prod_{k=1}^{m} \mathrm{Exp}(t_k X_k))$$

where $L_p : G \to G$ is the left translation given by $L_p(q) = pq$. Then

$$(d\psi_p)_0(e_k) = X_k(p)$$

for all k. This means that the differential $(d\psi_p)_0 : T_0\mathbb{R}^m \to T_pG$ is an isomorphism so there exist open neighbourhoods U_0 of 0 and U_p of p such that the restriction of ψ to U_0 is bijective onto its image U_p and hence a local parametrization of G around p.

We shall now study the normal bundle of a submanifold of a given Riemannian manifold. This is an important example of the notion of a vector bundle over a manifold.

Definition 5.23. Let (N, h) be a Riemannian manifold and M be a submanifold of N. For a point $p \in M$ we define the **normal space** N_pM of M at p by

$$N_pM = \{X \in T_pN |\ h_p(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

For all $p \in M$ we have the orthogonal decomposition

$$T_pN = T_pM \oplus N_pM.$$

The **normal bundle** of M in N is defined by

$$NM = \{(p, X) |\ p \in M,\quad X \in N_pM\}.$$

Example 5.24. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ equipped with its standard Euclidean metric $\langle , \rangle$. If $p \in S^m$ then the tangent space T_pS^m of S^m at p is

$$T_pS^m = \{X \in \mathbb{R}^{m+1} |\ \langle p, X\rangle = 0\}$$

so the normal space N_pS^m of S^m at p satisfies

$$N_pS^m = \{\lambda p \in \mathbb{R}^{m+1} |\ \lambda \in \mathbb{R}\}.$$

This shows that the normal bundle NS^m of S^m in $\mathbb{R}^{m+1}$ is given by

$$NS^m = \{(p, \lambda p) \in \mathbb{R}^{2m+2} |\ p \in S^m, \lambda \in \mathbb{R}\}.$$

Theorem 5.25. *Let (N^n, h) be a Riemannian manifold and M^m be a smooth submanifold of N. Then the normal bundle (NM, M, π) is a smooth $(n - m)$-dimensional vector bundle over M.*

Proof. See Exercise 5.7. □

We shall now determine the normal bundle $N\mathbf{O}(m)$ of the orthogonal group $\mathbf{O}(m)$ as a submanifold of $\mathbb{R}^{m\times m}$.

Example 5.26. The orthogonal group $\mathbf{O}(m)$ is a subset of the linear space $\mathbb{R}^{m\times m}$ equipped with the Riemannian metric

$$\langle X, Y\rangle = \text{trace}(X^t Y)$$

inducing a left-invariant metric on $\mathbf{O}(m)$. We have already seen that the tangent space $T_e\mathbf{O}(m)$ of $\mathbf{O}(m)$ at the neutral element e is

$$T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m\times m} |\ X^t + X = 0\}$$

and that the tangent bundle $T\mathbf{O}(m)$ of $\mathbf{O}(m)$ is given by

$$T\mathbf{O}(m) = \{(p, pX)|\ p \in \mathbf{O}(m),\ X \in T_e\mathbf{O}(m)\}.$$

The space $\mathbb{R}^{m\times m}$ of real $m \times m$ matrices has a linear decomposition

$$\mathbb{R}^{m\times m} = \text{Sym}(\mathbb{R}^m) \oplus T_e\mathbf{O}(m)$$

and every element $X \in \mathbb{R}^{m\times m}$ can be decomposed $X = X^\top + X^\perp$ in its symmetric and skew-symmetric parts given by

$$X^\top = (X - X^t)/2 \ \text{ and } \ X^\perp = (X + X^t)/2.$$

If $X \in T_e\mathbf{O}(m)$ and $Y \in \text{Sym}(\mathbb{R}^m)$ then

$$\begin{aligned}
\langle X, Y\rangle &= \text{trace}(X^t Y) \\
&= \text{trace}(Y^t X) \\
&= \text{trace}(X Y^t) \\
&= \text{trace}(-X^t Y) \\
&= -\langle X, Y\rangle.
\end{aligned}$$

This means that the normal bundle $N\mathbf{O}(m)$ of $\mathbf{O}(m)$ in $\mathbb{R}^{m\times m}$ is given by

$$N\mathbf{O}(m) = \{(p, pY)|\ p \in \mathbf{O}(m),\ Y \in \text{Sym}(\mathbb{R}^m)\}.$$

A given Riemannian metric g on M can be used to construct a family of natural metrics on the tangent bundle TM of M. The best known such examples are the Sasaki and Cheeger-Gromoll metrics. For a detailed survey on the geometry of tangent bundles equipped with these metrics we recommend the paper S. Gudmundsson, E. Kappos, *On the geometry of tangent bundles*, Expo. Math. **20** (2002), 1-41.

Exercises

Exercise 5.1. Let m be a positive integer and $\phi : \mathbb{R}^m \to \mathbb{C}^m$ be the standard parametrization of the m-dimensional torus T^m in $\mathbb{C}^m$ given by $\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$. Prove that ϕ is an isometric parametrization.

Exercise 5.2. Let m be a positive integer and

$$\pi_m : (S^m - \{(1, 0, \dots, 0)\}, \langle, \rangle_{\mathbb{R}^{m+1}}) \to (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \langle, \rangle_{\mathbb{R}^m})$$

be the **stereographic projection** given by

$$\pi_m : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0}(x_1, \dots, x_m).$$

Prove that π_m is an isometry.

Exercise 5.3. Let $B_1^2(0)$ be the open unit disk in the complex plane equipped with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Equip the upper half plane $\{z \in \mathbb{C} |\ \mathrm{Im}(z) > 0\}$ with the Riemannian metric

$$g(X, Y) = \frac{1}{\mathrm{Im}(z)^2} \langle X, Y \rangle_{\mathbb{R}^2}$$

and prove that the holomorphic function

$$\pi : B_1^2(0) \to \{z \in \mathbb{C} |\ \mathrm{Im}(z) > 0\}$$

given by

$$\pi : z \mapsto \frac{i + z}{1 + iz}$$

is an isometry.

Exercise 5.4. Equip the unitary group $\mathbf{U}(m)$ as a submanifold of $\mathbb{C}^{m \times m}$ with the induced metric given by

$$\langle Z, W \rangle = \mathrm{Re}(\mathrm{trace}(\bar{Z}^t W)).$$

Show that for each $p \in \mathbf{U}(m)$ the left translation $L_p : \mathbf{U}(m) \to \mathbf{U}(m)$ given by $L_p : q \mapsto pq$ is an isometry.

Exercise 5.5. Find a proof for Proposition 5.19.

Exercise 5.6. Let m be a positive integer and $\mathbf{GL}_m(\mathbb{R})$ be the corresponding real general linear group. Let g, h be two Riemannian metrics on $\mathbf{GL}_m(\mathbb{R})$ defined by

$$g_p(pZ, pW) = \text{trace}((pZ)^t pW), \quad h_p(pZ, pW) = \text{trace}(Z^t W).$$

Further let $\hat{g}, \hat{h}$ be the induced metrics on the special linear group $\mathbf{SL}_m(\mathbb{R})$ as a subset of $\mathbf{GL}_m(\mathbb{R})$.

(i) Which of the metrics $g, h, \hat{g}, \hat{h}$ are left-invariant?
(ii) Determine the normal space $N_e\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ in $\mathbf{GL}_m(\mathbb{R})$ with respect to g
(iii) Determine the normal bundle $N\mathbf{SL}_m(\mathbb{R})$ of $\mathbf{SL}_m(\mathbb{R})$ in $\mathbf{GL}_m(\mathbb{R})$ with respect to h.

Exercise 5.7. Find a proof for Theorem 5.25.

CHAPTER 6

The Levi-Civita Connection

In this chapter we introduce the Levi-Civita connection ∇ of a Riemannian manifold (M, g). This is the most important example of the general notion of a connection in a smooth vector bundle. We deduce an explicit formula for the Levi-Civita connection for Lie groups equipped with left-invariant metrics. We also give an example of a connection in the normal bundle of a submanifold of a Riemannian manifold and study its properties.

On the m-dimensional real vector space $\mathbb{R}^m$ we have the well-known differential operator

$$\partial : C^\infty(T\mathbb{R}^m) \times C^\infty(T\mathbb{R}^m) \to C^\infty(T\mathbb{R}^m)$$

mapping a pair of vector fields X, Y on $\mathbb{R}^m$ to the **directional derivative** $\partial_X Y$ of Y in the direction of X given by

$$(\partial_X Y)(x) = \lim_{t \to 0} \frac{Y(x + tX(x)) - Y(x)}{t}.$$

The most fundamental properties of the operator ∂ are expressed by the following. If $\lambda, \mu \in \mathbb{R}$, $f, g \in C^\infty(\mathbb{R}^m)$ and $X, Y, Z \in C^\infty(T\mathbb{R}^m)$ then

(i) $\partial_X(\lambda \cdot Y + \mu \cdot Z) = \lambda \cdot \partial_X Y + \mu \cdot \partial_X Z$,
(ii) $\partial_X(f \cdot Y) = X(f) \cdot Y + f \cdot \partial_X Y$,
(iii) $\partial_{(f \cdot X + g \cdot Y)} Z = f \cdot \partial_X Z + g \cdot \partial_Y Z$.

The next result shows that the differential operator ∂ is compatible with both the standard differentiable structure on $\mathbb{R}^m$ and its Euclidean metric.

Proposition 6.1. *Let the real vector space $\mathbb{R}^m$ be equipped with the standard Euclidean metric $\langle , \rangle$ and $X, Y, Z \in C^\infty(T\mathbb{R}^m)$ be smooth vector fields on $\mathbb{R}^m$. Then*

(iv) $\partial_X Y - \partial_Y X = [X, Y]$,
(v) $X(\langle Y, Z \rangle) = \langle \partial_X Y, Z \rangle + \langle Y, \partial_X Z \rangle$.

We shall now generalize the differential operator ∂ on the Euclidean space $\mathbb{R}^m$ to the so called Levi-Civita connection ∇ on a Riemannian manifold (M, g). First we introduce the concept of a connection in a smooth vector bundle.

Definition 6.2. Let (E, M, π) be a smooth vector bundle over M. A **connection** on (E, M, π) is a map $\hat{\nabla} : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$ such that

(i) $\hat{\nabla}_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \hat{\nabla}_X v + \mu \cdot \hat{\nabla}_X w$,
(ii) $\hat{\nabla}_X(f \cdot v) = X(f) \cdot v + f \cdot \hat{\nabla}_X v$,
(iii) $\hat{\nabla}_{(f \cdot X + g \cdot Y)} v = f \cdot \hat{\nabla}_X v + g \cdot \hat{\nabla}_Y v$.

for all $\lambda, \mu \in \mathbb{R}$, $X, Y \in C^\infty(TM)$, $v, w \in C^\infty(E)$ and $f, g \in C^\infty(M)$. A section $v \in C^\infty(E)$ of the vector bundle E is said to be **parallel** with respect to the connection $\hat{\nabla}$ if

$$\hat{\nabla}_X v = 0$$

for all vector fields $X \in C^\infty(TM)$.

Definition 6.3. Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Then we define the **torsion** $T : C_2^\infty(TM) \to C_1^\infty(TM)$ of $\hat{\nabla}$ by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

where $[,]$ is the Lie bracket on $C^\infty(TM)$. The connection $\hat{\nabla}$ is said to be **torsion-free** if its torsion T vanishes i.e.

$$[X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X$$

for all $X, Y \in C^\infty(TM)$.

Definition 6.4. Let (M, g) be a Riemannian manifold. Then a connection $\hat{\nabla}$ on the tangent bundle (TM, M, π) is said to be **metric** or compatible with the Riemannian metric g if

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z)$$

for all $X, Y, Z \in C^\infty(TM)$.

Let (M, g) be a Riemannian manifold and ∇ be a metric and torsion-free connection on its tangent bundle (TM, M, π). Then it is easily seen that the following equations hold

$$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, \nabla_X Z),$$

$$g(\nabla_X Y, Z) \quad = \quad g([X, Y], Z) + g(\nabla_Y X, Z)$$

$$
\begin{aligned}
&= g([X,Y],Z) + Y(g(X,Z)) - g(X,\nabla_Y Z),\\
0 &= -Z(g(X,Y)) + g(\nabla_Z X, Y) + g(X,\nabla_Z Y)\\
&= -Z(g(X,Y)) + g(\nabla_X Z + [Z,X], Y) + g(X, \nabla_Y Z - [Y,Z]).
\end{aligned}
$$

By adding these relations we yield

$$
\begin{aligned}
2\cdot g(\nabla_X Y, Z) &= \{X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y))\\
&\quad + g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z])\}.
\end{aligned}
$$

If $\{E_1,\dots,E_m\}$ is a local orthonormal frame for the tangent bundle then

$$\nabla_X Y = \sum_{k=1}^m g(\nabla_X Y, E_i)E_i.$$

As a direct consequence there exists **at most one** metric and torsion-free connection on the tangent bundle.

Definition 6.5. Let (M,g) be a Riemannian manifold then the map $\nabla : C^\infty(TM)\times C^\infty(TM)\to C^\infty(TM)$ given by

$$
\begin{aligned}
2\cdot g(\nabla_X Y, Z) &= \{X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y))\\
&\quad + g([Z,X],Y) + g([Z,Y],X) + g(Z,[X,Y])\}.
\end{aligned}
$$

is called the **Levi-Civita connection** on M.

Remark 6.6. It is very important to note that the Levi-Civita connection is an intrinsic object on (M,g) i.e. only depending on the differentiable structure of the manifold and its Riemannian metric.

Proposition 6.7. *Let (M,g) be a Riemannian manifold. Then the Levi-Civita connection ∇ is a connection on the tangent bundle TM of M.*

PROOF. It follows from Definition 3.6, Theorem 4.16 and the fact that g is a tensor field that

$$g(\nabla_X(\lambda\cdot Y_1 + \mu\cdot Y_2), Z) = \lambda\cdot g(\nabla_X Y_1, Z) + \mu\cdot g(\nabla_X Y_2, Z)$$

and

$$g(\nabla_{Y_1+Y_2} X, Z) = g(\nabla_{Y_1}X, Z) + g(\nabla_{Y_2}X, Z)$$

for all $\lambda,\mu\in\mathbb{R}$ and $X, Y_1, Y_2, Z \in C^\infty(TM)$. Furthermore we have for all $f\in C^\infty(M)$

$$
\begin{aligned}
&2\cdot g(\nabla_X fY, Z)\\
= \ &\{X(f\cdot g(Y,Z)) + f\cdot Y(g(X,Z)) - Z(f\cdot g(X,Y))\\
&+ f\cdot g([Z,X],Y) + g([Z,f\cdot Y],X) + g(Z,[X,f\cdot Y])\}
\end{aligned}
$$

$$\begin{aligned}
&= \{X(f)\cdot g(Y,Z) + f\cdot X(g(Y,Z)) + f\cdot Y(g(X,Z)) \\
&\quad -Z(f)\cdot g(X,Y) - f\cdot Z(g(X,Y)) + f\cdot g([Z,X],Y) \\
&\quad +g(Z(f)\cdot Y + f\cdot [Z,Y],X) + g(Z, X(f)\cdot Y + f\cdot [X,Y])\} \\
&= 2\cdot\{X(f)\cdot g(Y,Z) + f\cdot g(\nabla_X Y, Z)\} \\
&= 2\cdot g(X(f)\cdot Y + f\cdot \nabla_X Y, Z)
\end{aligned}$$

and

$$\begin{aligned}
&\quad 2\cdot g(\nabla_{f\cdot X} Y, Z) \\
&= \{f\cdot X(g(Y,Z)) + Y(f\cdot g(X,Z)) - Z(f\cdot g(X,Y)) \\
&\quad +g([Z,f\cdot X],Y) + f\cdot g([Z,Y],X) + g(Z,[f\cdot X,Y])\} \\
&= \{f\cdot X(g(Y,Z)) + Y(f)\cdot g(X,Z) + f\cdot Y(g(X,Z)) \\
&\quad -Z(f)\cdot g(X,Y) - f\cdot Z(g(X,Y)) \\
&\quad +g(Z(f)\cdot X,Y) + f\cdot g([Z,X],Y) \\
&\quad +f\cdot g([Z,Y],X) + f\cdot g(Z,[X,Y]) - g(Z, Y(f)\cdot X)\} \\
&= 2\cdot f\cdot g(\nabla_X Y, Z).
\end{aligned}$$

This proves that ∇ is a connection on the tangent bundle (TM, M, π). □

The next result is called the Fundamental Theorem of Riemannian geometry.

Theorem 6.8. *Let (M, g) be a Riemannian manifold. Then the Levi-Civita connection is the unique metric and torsion-free connection on the tangent bundle (TM, M, π).*

Proof. The difference $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$ equals twice the skew-symmetric part (w.r.t the pair (X, Y)) of the right hand side of the equation in Definition 6.5. This is the same as

$$= \frac{1}{2}\{g(Z,[X,Y]) - g(Z,[Y,X])\} = g(Z,[X,Y]).$$

This proves that the Levi-Civita connection is torsion-free.

The sum $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$ equals twice the symmetric part (w.r.t the pair (Y, Z)) on the right hand side of Definition 6.5. This is exactly

$$= \frac{1}{2}\{X(g(Y,Z)) + X(g(Z,Y))\} = X(g(Y,Z)).$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric g on M. □

A vector field $X \in C^\infty(TM)$ on (M, g) induces the **first order covariant derivative**

$$\nabla_X : C^\infty(TM) \to C^\infty(TM)$$

in the direction of X by

$$\nabla_X : Y \mapsto \nabla_X Y.$$

Definition 6.9. Let G be a Lie group. For a left-invariant vector field $Z \in \mathfrak{g}$ we define the map $\mathrm{ad}(Z) : \mathfrak{g} \to \mathfrak{g}$ by

$$\mathrm{ad}(Z) : X \mapsto [Z, X].$$

Proposition 6.10. *Let (G, g) be a Lie group equipped with a left-invariant metric. Then the Levi-Civita connection ∇ satisfies*

$$g(\nabla_X Y, Z) = \frac{1}{2}\{g([X, Y], Z) + g(ad(Z)(X), Y) + g(X, ad(Z)(Y))\}$$

for all $X, Y, Z \in \mathfrak{g}$. In particular, if for all $Z \in \mathfrak{g}$ the map $ad(Z)$ is skew symmetric with respect to g then

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Proof. See Exercise 6.2. □

Proposition 6.11. *Let G be one of the classical compact Lie groups $\mathbf{O}(m)$, $\mathbf{SO}(m)$, $\mathbf{U}(m)$ or $\mathbf{SU}(m)$ equipped with the left-invariant metric*

$$g(Z, W) = Re\ (trace\ (\bar{Z}^t W)).$$

Then for each $X \in \mathfrak{g}$ the operator $ad(X) : \mathfrak{g} \to \mathfrak{g}$ is skew symmetric.

Proof. See Exercise 6.3. □

Example 6.12. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Further let (U, x) be local coordinates on M and put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Then $\{X_1, \dots, X_m\}$ is a local frame of TM on U. For (U, x) we define the **Christoffel symbols** $\Gamma^k_{ij} : U \to \mathbb{R}$ of the connection ∇ with respect to (U, x) by

$$\sum_{k=1}^m \Gamma^k_{ij} X_k = \nabla_{X_i} X_j.$$

On the subset $x(U)$ of $\mathbb{R}^m$ we define the metric $\tilde{g}$ by

$$\tilde{g}(e_i, e_j) = g_{ij} = g(X_i, X_j).$$

The differential dx is bijective so Proposition 4.19 implies that

$$dx([X_i, X_j]) = [dx(X_i), dx(X_j)] = [\partial_{e_i}, \partial_{e_j}] = 0$$

and hence $[X_i, X_j] = 0$. From the definition of the Levi-Civita connection we now get

$$\begin{aligned}\sum_{k=1}^{m}\Gamma_{ij}^{k}g_{kl} &= \langle\sum_{k=1}^{m}\Gamma_{ij}^{k}X_k, X_l\rangle \\ &= \langle\nabla_{X_i}X_j, X_l\rangle \\ &= \frac{1}{2}\{X_i\langle X_j, X_l\rangle + X_j\langle X_l, X_i\rangle - X_l\langle X_i, X_j\rangle\} \\ &= \frac{1}{2}\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\}.\end{aligned}$$

If $g^{kl} = (g^{-1})_{kl}$ then

$$\Gamma_{ij}^{k} = \frac{1}{2}\sum_{l=1}^{m}g^{kl}\{\frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l}\}.$$

Definition 6.13. Let N be a smooth manifold, M be a submanifold of N and $\tilde{X} \in C^\infty(TM)$ be a vector field on M. Let U be an open subset of N such that $U \cap M \neq \emptyset$. A **local extension** of $\tilde{X}$ to U is a vector field $X \in C^\infty(TU)$ such that $\tilde{X}_p = X_p$ for all $p \in U \cap M$. If $U = N$ then X is called a **global extension**.

Fact 6.14. *Let (N, h) be a Riemannian manifold and M be a submanifold of N. Then vector fields $\tilde{X} \in C^\infty(TM)$ and $\tilde{Y} \in C^\infty(NM)$ have global extensions $X, Y \in C^\infty(TN)$.*

Let (N, h) be a Riemannian manifold and M be a submanifold equipped with the induced metric g. Let $Z \in C^\infty(TN)$ be a vector field on N and $\tilde{Z} = Z|_M : M \to TN$ be the restriction of Z to M. Note that $\tilde{Z}$ is not necessarily an element of $C^\infty(TM)$ i.e. a vector field on the submanifold M. For each $p \in M$ the tangent vector $\tilde{Z}_p \in T_pN$ can be decomposed

$$\tilde{Z}_p = \tilde{Z}_p^\top + \tilde{Z}_p^\perp$$

in a unique way into its tangential part $(\tilde{Z}_p)^\top \in T_pM$ and its normal part $(\tilde{Z}_p)^\perp \in N_pM$. For this we write $\tilde{Z} = \tilde{Z}^\top + \tilde{Z}^\perp$.

Let $\tilde{X}, \tilde{Y} \in C^\infty(TM)$ be vector fields on M and $X, Y \in C^\infty(TN)$ be their extensions to N. If $p \in M$ then $(\nabla_X Y)_p$ only depends on the value $X_p = \tilde{X}_p$ and the value of Y along some curve $\gamma : (-\epsilon, \epsilon) \to N$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = X_p = \tilde{X}_p$. For this see Remark 7.3. Since $X_p \in T_pM$ we may choose the curve γ such that the image $\gamma((-\epsilon, \epsilon))$ is contained in M. Then $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$ for $t \in (-\epsilon, \epsilon)$. This means that $(\nabla_X Y)_p$ only depends on $\tilde{X}_p$ and the value of $\tilde{Y}$ along γ, hence independent of the way $\tilde{X}$ and $\tilde{Y}$ are extended. This shows that the following maps $\tilde{\nabla}$ and B are well defined.

Definition 6.15. Let (N, h) be a Riemannian manifold and M be a submanifold equipped with the induced metric g. Then we define two operators

$$\tilde{\nabla} : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

and

$$B : C^\infty(TM) \times C^\infty(TM) \to C^\infty(NM)$$

by

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\top \quad \text{and} \quad B(\tilde{X}, \tilde{Y}) = (\nabla_X Y)^\perp,$$

where $X, Y \in C^\infty(TN)$ are any extensions of $\tilde{X}, \tilde{Y}$.

The operator B is called the **second fundamental form** of M in (N, h). It is symmetric and hence tensorial in both its arguments, see Exercise 6.7.

Theorem 6.16. *Let (N, h) be a Riemannian manifold and M be a submanifold of N with the induced metric g. Then the operator $\tilde{\nabla}$ is the Levi-Civita connection of the submanifold (M, g).*

Proof. See Exercise 6.8. □

The Levi-Civita connection on (N, h) induces a metric connection $\bar{\nabla}$ on the normal bundle NM of M in N as follows.

Proposition 6.17. *Let (N, h) be a Riemannian manifold and M be a submanifold with the induced metric g. Let $X, Y \in C^\infty(TN)$ be vector fields extending $\tilde{X} \in C^\infty(TM)$ and $\tilde{Y} \in C^\infty(NM)$. Then the map $\bar{\nabla} : C^\infty(TM) \times C^\infty(NM) \to C^\infty(NM)$ given by*

$$\bar{\nabla}_{\tilde{X}}\tilde{Y} = (\nabla_X Y)^\perp$$

is a well-defined connection on the normal bundle NM satisfying

$$\tilde{X}(h(\tilde{Y}, \tilde{Z})) = h(\bar{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) + h(\tilde{Y}, \bar{\nabla}_{\tilde{X}}\tilde{Z})$$

for all $\tilde{X} \in C^\infty(TM)$ and $\tilde{Y}, \tilde{Z} \in C^\infty(NM)$.

Proof. See Exercise 6.9. □

Exercises

Exercise 6.1. Let M be a smooth manifold and $\hat{\nabla}$ be a connection on the tangent bundle (TM, M, π). Prove that the torsion $T : C_2^\infty(TM) \to C_1^\infty(TM)$ of $\hat{\nabla}$ is a tensor field of type $(2, 1)$.

Exercise 6.2. Find a proof for Proposition 6.10.

Exercise 6.3. Find a proof for Proposition 6.11.

Exercise 6.4. Let Sol^3 be the 3-dimensional subgroup of $\mathbf{SL}_3(\mathbb{R})$ given by

$$Sol^3 = \{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid p = (x, y, z) \in \mathbb{R}^3 \}.$$

Let $X, Y, Z \in \mathfrak{g}$ be left-invariant vector fields on Sol^3 such that

$$X_e = \frac{\partial}{\partial x}|_{p=0}, \quad Y_e = \frac{\partial}{\partial y}|_{p=0} \text{ and } Z_e = \frac{\partial}{\partial z}|_{p=0}.$$

Show that

$$[X, Y] = 0, \quad [Z, X] = X \text{ and } [Z, Y] = -Y.$$

Let g be a left-invariant Riemannian metric on G such that $\{X, Y, Z\}$ is an orthonormal basis for the Lie algebra $\mathfrak{g}$. Calculate the vector fields

$$\nabla_X Y, \quad \nabla_Y X, \quad \nabla_X Z, \quad \nabla_Z X, \quad \nabla_Y Z \text{ and } \nabla_Z Y.$$

Exercise 6.5. Let $\mathbf{SO}(m)$ be the special orthogonal group equipped with the metric

$$\langle X, Y \rangle = \frac{1}{2} \text{trace}(X^t Y).$$

Prove that $\langle , \rangle$ is left-invariant and that for any left-invariant vector fields $X, Y \in \mathfrak{so}(m)$ we have

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

Let A, B, C be elements of the Lie algebra $\mathfrak{so}(3)$ with

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prove that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{so}(3)$ and calculate

$$\nabla_A B, \quad \nabla_B C \text{ and } \nabla_C A.$$

Exercise 6.6. Let $\mathbf{SL}_2(\mathbb{R})$ be the real special linear group equipped with the metric

$$\langle X, Y\rangle_p = \frac{1}{2}\,\mathrm{trace}((p^{-1}X)^t(p^{-1}Y)).$$

Let A, B, C be elements of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with

$$A_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad C_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Prove that $\{A, B, C\}$ is an orthonormal basis for $\mathfrak{sl}_2(\mathbb{R})$ and calculate

$$\nabla_A B, \quad \nabla_B C \quad \text{and} \quad \nabla_C A.$$

Exercise 6.7. Let (N, h) be a Riemannian manifold with Levi-Civita connection ∇ and (M, g) be a submanifold with the induced metric. Prove that the second fundamental form B of M in N is symmetric and tensorial in both its arguments.

Exercise 6.8. Find a proof for Theorem 6.16.

Exercise 6.9. Find a proof for Proposition 6.17.

CHAPTER 7

Geodesics

In this chapter we introduce the notion of a geodesic on a Riemannian manifold (M, g). This is a solution to a second order non-linear system of ordinary differential equations. We show that geodesics are solutions to two different variational problems. They are critical points to the so called energy functional and furthermore locally shortest paths between their endpoints.

Definition 7.1. Let M be a smooth manifold and (TM, M, π) be its tangent bundle. A **vector field** X **along a curve** $\gamma : I \to M$ is a curve $X : I \to TM$ such that $\pi \circ X = \gamma$. By $C^\infty_\gamma(TM)$ we denote the set of all smooth vector fields along γ. For $X, Y \in C^\infty_\gamma(TM)$ and $f \in C^\infty(I)$ we define the operations $+$ and $\cdot$ by

(i) $(X + Y)(t) = X(t) + Y(t)$,
(ii) $(f \cdot X)(t) = f(t) \cdot X(t)$.

This turns $(C^\infty_\gamma(TM), +, \cdot)$ into a module over $C^\infty(I)$ and a real vector space over the constant functions in particular. For a given smooth curve $\gamma : I \to M$ in M the smooth vector field $X : I \to TM$ with $X : t \mapsto (\gamma(t), \dot\gamma(t))$ is called the **tangent field** along γ.

The next result gives a rule for differentiating a vector field along a given curve and shows how this is related to the Levi-Civita connection.

Proposition 7.2. *Let (M, g) be a smooth Riemannian manifold and $\gamma : I \to M$ be a curve in M. Then there exists a unique operator*

$$\frac{D}{dt} : C^\infty_\gamma(TM) \to C^\infty_\gamma(TM)$$

such that for all $\lambda, \mu \in \mathbb{R}$ and $f \in C^\infty(I)$,

(i) *$D(\lambda \cdot X + \mu \cdot Y)/dt = \lambda \cdot (DX/dt) + \mu \cdot (DY/dt)$,*
(ii) *$D(f \cdot Y)/dt = df/dt \cdot Y + f \cdot (DY/dt)$, and*
(iii) *for each $t_0 \in I$ there exists an open subinterval J_0 of I such that $t_0 \in J_0$ and if $X \in C^\infty(TM)$ is a vector field with $X_{\gamma(t)} = Y(t)$ for all $t \in J_0$ then*

$$\Big(\frac{DY}{dt}\Big)(t_0) = (\nabla_{\dot\gamma} X)_{\gamma(t_0)}.$$

PROOF. Let us first prove the uniqueness, so for the moment we assume that such an operator exists. For a point $t_0 \in I$ choose a chart (U, x) on M and open subinterval $J \subset I$ such that $t_0 \in J$, $\gamma(J) \subset U$ and put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Then any vector field Y along the restriction of γ to J can be written in the form

$$Y(t) = \sum_{j=1}^{m} \alpha_j(t)\big(X_j\big)_{\gamma(t)}$$

for some functions $\alpha_j \in C^\infty(J)$. The second condition means that

$$(1) \qquad \big(\frac{DY}{dt}\big)(t) = \sum_{j=1}^{m} \alpha_j(t)\big(\frac{DX_j}{dt}\big)_{\gamma(t)} + \sum_{k=1}^{m} \dot{\alpha}_k(t)\big(X_k\big)_{\gamma(t)}.$$

Let $x \circ \gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$ then

$$\dot{\gamma}(t) = \sum_{i=1}^{m} \dot{\gamma}_i(t)\big(X_i\big)_{\gamma(t)}$$

and the third condition for D/dt implies that

$$(2) \qquad \big(\frac{DX_j}{dt}\big)_{\gamma(t)} = (\nabla_{\dot{\gamma}} X_j)_{\gamma(t)} = \sum_{i=1}^{m} \dot{\gamma}_i(t)(\nabla_{X_i} X_j)_{\gamma(t)}.$$

Together equations (1) and (2) give

$$(3) \qquad \big(\frac{DY}{dt}\big)(t) = \sum_{k=1}^{m}\Big(\dot{\alpha}_k(t) + \sum_{i,j=1}^{m} \Gamma_{ij}^k(\gamma(t))\dot{\gamma}_i(t)\alpha_j(t)\Big)\big(X_k\big)_{\gamma(t)}.$$

This shows that the operator D/dt is uniquely determined.

It is easily seen that if we use equation (3) for defining an operator D/dt then it satisfies the necessary conditions of Proposition 7.2. This proves the existence of the operator D/dt. □

Remark 7.3. It follows from the fact that the Levi-Civita connection is tensorial in its first argument i.e.

$$\nabla_{f \cdot Z} X = f \cdot \nabla_Z X$$

and the equation

$$(\nabla_{\dot{\gamma}} X)_{\gamma(t_0)} = \big(\frac{DY}{dt}\big)(t_0)$$

in Proposition 7.2 that the value $(\nabla_Z X)_p$ of $\nabla_Z X$ at p only depends on the value of Z_p of Z at p and the values of Y along some curve γ satisfying $\gamma(0) = p$ and $\dot{\gamma}(0) = Z_p$. This allows us to use the notation $\nabla_{\dot{\gamma}} Y$ for DY/dt.

The Levi-Civita connection can now be used to define the notions of parallel vector fields and geodesics on Riemannian manifolds. We will show that they are solutions to ordinary differential equations.

Definition 7.4. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^1-curve. A vector field X along γ is said to be **parallel** if

$$\nabla_{\dot\gamma} X = 0.$$

A C^2-curve $\gamma : I \to M$ on M is said to be a **geodesic** if its tangent field $\dot\gamma$ is parallel along γ i.e.

$$\nabla_{\dot\gamma} \dot\gamma = 0.$$

The next result shows that for given initial values at a point $p \in M$ we get a parallel vector field globally defined along any curve through that point.

Theorem 7.5. *Let (M, g) be a Riemannian manifold and $I = (a, b)$ be an open interval on the real line $\mathbb{R}$. Further let $\gamma : [a, b] \to M$ be a continuous curve which is C^1 on I, $t_0 \in I$ and $X_0 \in T_{\gamma(t_0)}M$. Then there exists a unique parallel vector field Y along γ such that $X_0 = Y(t_0)$.*

PROOF. Let (U, x) be a chart on M such that $\gamma(t_0) \in U$ and put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Let J be an open subset of I such that the image $\gamma(J)$ is contained in U. Then the tangent of the restriction of γ to J can be written as

$$\dot\gamma(t) = \sum_{i=1}^{m} \dot\gamma_i(t) \big(X_i\big)_{\gamma(t)}.$$

Similarly, let Y be a vector field along γ represented by

$$Y(t) = \sum_{j=1}^{m} \alpha_j(t) \big(X_j\big)_{\gamma(t)}.$$

Then

$$\begin{aligned} \big(\nabla_{\dot\gamma} Y\big)(t) &= \sum_{j=1}^{m} \{\dot\alpha_j(t) \big(X_j\big)_{\gamma(t)} + \alpha_j(t) \big(\nabla_{\dot\gamma} X_j\big)_{\gamma(t)}\} \\ &= \sum_{k=1}^{m} \{\dot\alpha_k(t) + \sum_{i,j=1}^{m} \alpha_j(t) \dot\gamma_i(t) \Gamma_{ij}^k(\gamma(t))\} \big(X_k\big)_{\gamma(t)}. \end{aligned}$$

This implies that the vector field Y is parallel i.e. $\nabla_{\dot\gamma} Y \equiv 0$ if and only if the following first order **linear** system of ordinary differential equations is satisfied

$$\dot{\alpha}_k(t) + \sum_{i,j=1}^{m} \alpha_j(t)\dot{\gamma}_i(t)\Gamma_{ij}^k(\gamma(t)) = 0$$

for all $k = 1, \dots, m$. It follows from Fact 7.6 that to each initial value $\alpha(t_0) = (v_1, \dots, v_m) \in \mathbb{R}^m$ with

$$Y_0 = \sum_{k=1}^{m} v_k \big(X_k\big)_{\gamma(t_0)}$$

there exists a unique solution $\alpha = (\alpha_1, \dots, \alpha_m)$ to the above system. This gives us the unique parallel vector field Y

$$Y(t) = \sum_{k=1}^{m} \alpha_k(t)\big(X_k\big)_{\gamma(t)}$$

along J. Since the Christoffel symbols are bounded along the compact set $[a, b]$ is is clear that the parallel vector field can be extended to the whole of $I = (a, b)$. □

The following result is the well-known theorem of Picard-Lindelöf.

Fact 7.6. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t, x) - f(t, y)| \leq L \cdot |x - y|$$

for all $(t, x), (t, y) \in U$. If $(t_0, x_0) \in U$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

Lemma 7.7. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a smooth curve and X, Y be parallel vector fields along γ. Then the function $g(X, Y) : I \to \mathbb{R}$ given by $t \mapsto g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)})$ is constant. In particular, if γ is a geodesic then $g(\dot{\gamma}, \dot{\gamma})$ is constant along γ.*

Proof. Using the fact that the Levi-Civita connection is metric we obtain

$$\frac{d}{dt}(g(X, Y)) = g(\nabla_{\dot{\gamma}} X, Y) + g(X, \nabla_{\dot{\gamma}} Y) = 0.$$

This proves that the function $g(X, Y)$ is constant along γ. □

The following result on parallel vector fields is a useful tool in Riemannian geometry. It turns out to be very useful in Chapter 9.

Proposition 7.8. *Let (M, g) be a Riemannian manifold, $p \in M$ and $\{v_1, \dots, v_m\}$ be an orthonormal basis for the tangent space T_pM. Let $\gamma : I \to M$ be a smooth curve such that $\gamma(0) = p$ and $X_1, \dots, X_m$ be parallel vector fields along γ such that $X_k(0) = v_k$ for $k = 1, 2, \dots, m$. Then the set $\{X_1(t), \dots, X_m(t)\}$ is a orthonormal basis for the tangent space $T_{\gamma(t)}M$ for all $t \in I$.*

PROOF. This is a direct consequence of Lemma 7.7. □

Geodesics are of great importance in Riemannian geometry. For those we have the following fundamental uniquness and existence result.

Theorem 7.9. *Let (M, g) be a Riemannian manifold. If $p \in M$ and $v \in T_pM$ then there exists an open interval $I = (-\epsilon, \epsilon)$ and a unique geodesic $\gamma : I \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$.*

PROOF. Let (U, x) be a chart on M such that $p \in U$ and put $X_i = \partial/\partial x_i \in C^\infty(TU)$. Let J be an open subset of I such that the image $\gamma(J)$ is contained in U. Then the tangent of the restriction of γ to J can be written as

$$\dot\gamma(t) = \sum_{i=1}^m \dot\gamma_i(t) \big(X_i\big)_{\gamma(t)}.$$

By differentiation we then obtain

$$\begin{aligned}
\nabla_{\dot\gamma}\dot\gamma &= \sum_{j=1}^m \nabla_{\dot\gamma}\big(\dot\gamma_j(t)\big(X_j\big)_{\gamma(t)}\big) \\
&= \sum_{j=1}^m \{\ddot\gamma_j(t)\big(X_j\big)_{\gamma(t)} + \sum_{i=1}^m \dot\gamma_i(t)\dot\gamma_j(t)\big(\nabla_{X_i}X_j\big)_{\gamma(t)}\} \\
&= \sum_{k=1}^m \{\ddot\gamma_k(t) + \sum_{i,j=1}^m \dot\gamma_i(t)\dot\gamma_j(t)\Gamma_{ij}^k(\gamma(t))\}\big(X_k\big)_{\gamma(t)}.
\end{aligned}$$

Hence the curve γ is a geodesic if and only if

$$\ddot\gamma_k(t) + \sum_{i,j=1}^m \dot\gamma_i(t)\dot\gamma_j(t)\Gamma_{ij}^k(\gamma(t)) = 0$$

for all $k = 1, \dots, m$. It follows from Fact 7.10 that for initial values $q = x(p)$ and $w = (dx)_p(v)$ there exists an open interval $(-\epsilon, \epsilon)$ and a unique solution $(\gamma_1, \dots, \gamma_m)$ satisfying the initial conditions

$$(\gamma_1(0), \dots, \gamma_m(0)) = q \quad \text{and} \quad (\dot\gamma_1(0), \dots, \dot\gamma_m(0)) = w.$$

□

The following result is a second order consequence of the well-known theorem of Picard-Lindelöf.

Fact 7.10. *Let $f : U \to \mathbb{R}^n$ be a continuous map defined on an open subset U of $\mathbb{R} \times \mathbb{R}^n$ and $L \in \mathbb{R}^+$ such that*

$$|f(t,x) - f(t,y)| \le L \cdot |x - y|$$

for all $(t,x), (t,y) \in U$. If $(t_0, x_0) \in U$ and $x_1 \in \mathbb{R}^n$ then there exists a unique local solution $x : I \to \mathbb{R}^n$ to the following initial value problem

$$x''(t) = f(t, x(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

Remark 7.11. The Levi-Civita connection ∇ on a given Riemannian manifold (M, g) is an inner object i.e. completely determined by the differentiable structure on M and the Riemannian metric g, see Remark 6.6. Hence the same applies for the condition

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

for a given curve $\gamma : I \to M$. This means that the image of a geodesic under a local isometry is again a geodesic.

Example 7.12. Let $E^m = (\mathbb{R}^m, \langle , \rangle_{\mathbb{R}^m})$ be the Euclidean space. For the trivial chart $\text{id}_{\mathbb{R}^m} : \mathbb{R}^m \to \mathbb{R}^m$ the metric on E^m is given by $g_{ij} = \delta_{ij}$. As a direct consequence of Example 6.12 we see that

$$\Gamma_{ij}^k = 0 \ \text{ for all } \ i, j, k = 1, \dots, m.$$

Hence $\gamma : I \to \mathbb{R}^m$ is a geodesic if and only if $\ddot{\gamma}(t) = 0$. For $p \in \mathbb{R}^m$ and $v \in T_p\mathbb{R}^m \cong \mathbb{R}^m$ define

$$\gamma_{(p,v)} : \mathbb{R} \to \mathbb{R}^m \ \text{ by } \ \gamma_{(p,v)}(t) = p + t \cdot v.$$

Then $\gamma_{(p,v)}(0) = p$, $\dot{\gamma}_{(p,v)}(0) = v$ and $\ddot{\gamma}_{(p,v)} = 0$. It now follows from Theorem 7.9 that the geodesics in E^m are the straight lines.

Definition 7.13. A geodesic $\gamma : I \to (M, g)$ in a Riemannian manifold is said to be **maximal** if it cannot be extended to a geodesic defined on an interval J strictly containing I. The manifold (M, g) is said to be **complete** if for each point $(p, v) \in TM$ there exists a geodesic $\gamma : \mathbb{R} \to M$ defined on the whole of $\mathbb{R}$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

Proposition 7.14. *Let (N, h) be a Riemannian manifold with Levi-Civita connection ∇ and M be a submanifold equipped with the induced metric g. A curve $\gamma : I \to M$ is a geodesic in M if and only if*

$$(\nabla_{\dot{\gamma}} \dot{\gamma})^\top = 0.$$

Proof. Following Theorem 6.16 the Levi-Civita connection $\tilde{\nabla}$ on (M, g) satisfies

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top}.$$

□

Example 7.15. Let E^{m+1} be the $(m+1)$-dimensional Euclidean space and S^m be the unit sphere in E^{m+1} with the induced metric. At a point $p \in S^m$ the normal space N_pS^m of S^m in E^{m+1} is simply the line generated by p. If $\gamma : I \to S^m$ is a curve on the sphere, then

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top} = (\partial_{\dot{\gamma}}\dot{\gamma})^{\top} = \ddot{\gamma}^{\top} = \ddot{\gamma} - \ddot{\gamma}^{\perp} = \ddot{\gamma} - \langle \ddot{\gamma}, \gamma \rangle \gamma.$$

This shows that γ is a geodesic on the sphere S^m if and only if

$$\ddot{\gamma} = \langle \ddot{\gamma}, \gamma \rangle \gamma. \tag{4}$$

For a point $(p, X) \in TS^m$ define the curve $\gamma = \gamma_{(p,X)} : \mathbb{R} \to S^m$ by

$$\gamma : t \mapsto \begin{cases} p & \text{if } X = 0 \\ \cos(|X|t) \cdot p + \sin(|X|t) \cdot X/|X| & \text{if } X \neq 0. \end{cases}$$

Then one easily checks that $\gamma(0) = p$, $\dot{\gamma}(0) = X$ and that γ satisfies the geodesic equation (4). This shows that the non-constant geodesics on S^m are precisely the great circles and the sphere is complete.

Example 7.16. Let $\mathrm{Sym}(\mathbb{R}^{m+1})$ be equipped with the metric

$$\langle A, B \rangle = \frac{1}{8}\,\mathrm{trace}(A^t B).$$

Then we know that the map $\phi : S^m \to \mathrm{Sym}(\mathbb{R}^{m+1})$ with

$$\phi : p \mapsto (2pp^t - e)$$

is an isometric immersion and that the image $\phi(S^m)$ is isometric to the m-dimensional real projective space $\mathbb{R}P^m$. This means that the geodesics on $\mathbb{R}P^m$ are exactly the images of geodesics on S^m. This shows that the real projective spaces are complete.

Definition 7.17. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^r-curve on M. A **variation** of γ is a C^r-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that for all $s \in I$, $\Phi_0(s) = \Phi(0, s) = \gamma(s)$. If the interval is compact i.e. of the form $I = [a, b]$, then the variation Φ is called **proper** if for all $t \in (-\epsilon, \epsilon)$, $\Phi_t(a) = \gamma(a)$ and $\Phi_t(b) = \gamma(b)$.

Definition 7.18. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a C^2-curve on M. For every compact interval $[a, b] \subset I$ we define the energy functional $E_{[a,b]}$ by

$$E_{[a,b]}(\gamma) = \frac{1}{2}\int_a^b g(\dot\gamma(t), \dot\gamma(t))dt.$$

A C^2-curve $\gamma : I \to M$ is called **a critical point** for the energy functional if every proper variation Φ of $\gamma|_{[a,b]}$ satisfies

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

We shall now prove that geodesics can be characterized as the critical points of the energy functional.

Theorem 7.19. *A C^2-curve $\gamma : I = [a, b] \to M$ is a critical point for the energy functional if and only if it is a geodesic.*

PROOF. For a C^2-map $\Phi : (-\epsilon, \epsilon) \times I \to M$, $\Phi : (t, s) \mapsto \Phi(t, s)$ we define the vector fields $X = d\Phi(\partial/\partial s)$ and $Y = d\Phi(\partial/\partial t)$ along Φ. The following shows that the vector fields X and Y commute.

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y] \\ &= [d\Phi(\partial/\partial s), d\Phi(\partial/\partial t)] \\ &= d\Phi([\partial/\partial s, \partial/\partial t]) \\ &= 0, \end{aligned}$$

since $[\partial/\partial s, \partial/\partial t] = 0$. We now assume that Φ is a proper variation of γ. Then

$$\begin{aligned} \frac{d}{dt}(E_{[a,b]}(\Phi_t)) &= \frac{1}{2}\frac{d}{dt}(\int_a^b g(X, X)ds) \\ &= \frac{1}{2}\int_a^b \frac{d}{dt}(g(X, X))ds \\ &= \int_a^b g(\nabla_Y X, X)ds \\ &= \int_a^b g(\nabla_X Y, X)ds \\ &= \int_a^b (\frac{d}{ds}(g(Y, X)) - g(Y, \nabla_X X))ds \\ &= [g(Y, X)]_a^b - \int_a^b g(Y, \nabla_X X)ds. \end{aligned}$$

The variation is proper, so $Y(t,a) = Y(t,b) = 0$. Furthermore $X(0,s) = \partial\Phi/\partial s(0,s) = \dot\gamma(s)$, so

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = -\int_a^b g(Y(0,s),(\nabla_{\dot\gamma}\dot\gamma)(s))ds.$$

The last integral vanishes for every proper variation Φ of γ if and only if $\nabla_{\dot\gamma}\dot\gamma = 0$. □

A geodesic $\gamma : I \to (M,g)$ is a special case of what is called a **harmonic map** $\phi : (M,g) \to (N,h)$ between Riemannian manifolds. Other examples are conformal immersions $\psi : (M^2,g) \to (N,h)$ which parametrize the so called minimal surfaces in (N,h). For a reference on harmonic maps see H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs **132**, AMS (1993).

Let (M^m,g) be an m-dimensional Riemannian manifold, $p \in M$ and

$$S_p^{m-1} = \{v \in T_pM|\ g_p(v,v) = 1\}$$

be the unit sphere in the tangent space T_pM at p. Then every point $w \in T_pM \setminus \{0\}$ can be written as $w = r_w \cdot v_w$, where $r_w = |w|$ and $v_w = w/|w| \in S_p^{m-1}$. For $v \in S_p^{m-1}$ let $\gamma_v : (-\alpha_v, \beta_v) \to M$ be the maximal geodesic such that $\alpha_v, \beta_v \in \mathbb{R}^+ \cup \{\infty\}$, $\gamma_v(0) = p$ and $\dot\gamma_v(0) = v$. It can be shown that the real number

$$\epsilon_p = \inf\{\alpha_v, \beta_v|\ v \in S_p^{m-1}\}$$

is positive so the open ball

$$B_{\epsilon_p}^m(0) = \{v \in T_pM|\ g_p(v,v) < \epsilon_p^2\}$$

is non-empty. The **exponential map** $\exp_p : B_{\epsilon_p}^m(0) \to M$ at p is defined by

$$\exp_p : w \mapsto \begin{cases} p & \text{if } w = 0 \\ \gamma_{v_w}(r_w) & \text{if } w \neq 0. \end{cases}$$

Note that for $v \in S_p^{m-1}$ the line segment $\lambda_v : (-\epsilon_p, \epsilon_p) \to T_pM$ with $\lambda_v : t \mapsto t \cdot v$ is mapped onto the geodesic γ_v i.e. locally we have $\gamma_v = \exp_p \circ \lambda_v$. One can prove that the map $\exp_p$ is smooth and it follows from its definition that the differential

$$d(\exp_p)_0 : T_pM \to T_pM$$

is the identity map for the tangent space T_pM. Then the inverse mapping theorem tells us that there exists an $r_p \in \mathbb{R}^+$ such that if $U_p = B_{r_p}^m(0)$ and $V_p = \exp_p(U_p)$ then $\exp_p|_{U_p} : U_p \to V_p$ is a diffeomorphism parametrizing the open subset V_p of M.

The next result shows that the geodesics are locally the shortest paths between their endpoints.

Theorem 7.20. *Let (M, g) be a Riemannian manifold. Then the geodesics are locally the shortest paths between their end points.*

PROOF. Let $p \in M$, $U = B_r^m(0)$ in T_pM and $V = \exp_p(U)$ be such that the restriction

$$\phi = \exp_p|_U : U \to V$$

of the exponential map at p is a diffeomorphism. We define a metric $\tilde{g}$ on U such that for each $X, Y \in C^\infty(TU)$ we have

$$\tilde{g}(X, Y) = g(d\phi(X), d\phi(Y)).$$

This turns $\phi : (U, \tilde{g}) \to (V, g)$ into an isometry. It then follows from the construction of the exponential map, that the geodesics in $(U, \tilde{g})$ through the point $0 = \phi^{-1}(p)$ are exactly the lines $\lambda_v : t \mapsto t \cdot v$ where $v \in T_pM$.

Now let $q \in B_r^m(0) \setminus \{0\}$ and $\lambda_q : [0, 1] \to B_r^m(0)$ be the curve $\lambda_q : t \mapsto t \cdot q$. Further let $\sigma : [0, 1] \to U$ be any C^1-curve such that $\sigma(0) = 0$ and $\sigma(1) = q$. Along the curve σ we define the vector field X with $X : t \mapsto \sigma(t)$ and the tangent field $\dot{\sigma} : t \to \dot{\sigma}(t)$ to σ. Then the radial component $\dot{\sigma}_{\text{rad}}$ of $\dot{\sigma}$ is the orthogonal projection of $\dot{\sigma}$ onto the line generated by X i.e.

$$\dot{\sigma}_{\text{rad}} : t \mapsto \frac{\tilde{g}(\dot{\sigma}(t), X(t))}{\tilde{g}(X(t), X(t))} X(t).$$

Then it is easily checked that

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{|\tilde{g}(\dot{\sigma}(t), X(t))|}{|X(t)|}$$

and

$$\frac{d}{dt}|X(t)| = \frac{d}{dt}\sqrt{\tilde{g}(X(t), X(t))} = \frac{\tilde{g}(\dot{\sigma}(t), X(t))}{|X(t)|}.$$

Combining these two relations we yield

$$|\dot{\sigma}_{\text{rad}}(t)| \geq \frac{d}{dt}|X(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\dot{\sigma}(t)| dt \\ &\geq \int_0^1 |\dot{\sigma}_{\text{rad}}(t)| dt \end{aligned}$$

$$\begin{aligned} &\geq \int_0^1 \frac{d}{dt}|X(t)|dt \\ &= |X(1)| - |X(0)| \\ &= |q| \\ &= L(\lambda_q). \end{aligned}$$

This proves that in fact γ is the shortest path connecting p and q. □

Definition 7.21. Let (N, h) be a Riemannian manifold and M be a submanifold with the induced metric g. Then the **mean curvature vector field** of M in N is the smooth section $H : M \to NM$ of the normal bundle NM given by

$$H = \frac{1}{m} \text{ trace } B = \frac{1}{m} \sum_{k=1}^{m} B(X_k, X_k).$$

Here B is the second fundamental form of M in N and $\{X_1, \dots, X_m\}$ is any local orthonormal frame for the tangent bundle TM of M. The submanifold M is said to be **minimal** in N if $H \equiv 0$ and **totally geodesic** in N if $B \equiv 0$.

Proposition 7.22. *Let (N, h) be a Riemannian manifold and M be a submanifold equipped with the induced metric g. Then the following conditions are equivalent:*

(i) *M is totally geodesic in N*
(ii) *if $\gamma : I \to M$ is a curve, then the following conditions are equivalent*
  (a) *$\gamma : I \to M$ is a geodesic in M,*
  (b) *$\gamma : I \to M$ is a geodesic in N.*

Proof. The result is a direct consequence of the following decomposition formula

$$\nabla_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top} + (\nabla_{\dot{\gamma}}\dot{\gamma})^{\perp} = \tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} + B(\dot{\gamma}, \dot{\gamma}).$$

□

Proposition 7.23. *Let (N, h) be a Riemannian manifold and M be a complete submanifold of N. For a point (p, v) of the tangent bundle TM let $\gamma_{(p,v)} : I \to N$ be the maximal geodesic in N with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Then M is totally geodesic in (N, h) if and only if $\gamma_{(p,v)}(I) \subset M$ for all $(p, v) \in TM$.*

Proof. See Exercise 7.3. □

Corollary 7.24. *Let (N, h) be a Riemannian manifold, $p \in N$ and V be an m-dimensional linear subspace of the tangent space T_pN of N at p. Then there exists (locally) at most one totally geodesic submanifold M of (N, h) such that $T_pM = V$.*

PROOF. See Exercise 7.4. $\square$

Proposition 7.25. *Let (N, h) be a Riemannian manifold and M be a submanifold of N which is the fix point set of an isometry $\phi : N \to N$. Then M is totally geodesic in N.*

PROOF. Let $p \in M$, $v \in T_pM$ and $c : J \to M$ be a curve such that $c(0) = p$ and $\dot{c}(0) = v$. Since M is the fix point set of ϕ we have $\phi(p) = p$ and $d\phi_p(v) = v$. Further let $\gamma : I \to N$ be the maximal geodesic in N with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. The map $\phi : N \to N$ is an isometry so the curve $\phi \circ \gamma : I \to N$ is also a geodesic. The uniqueness result of Theorem 7.9, $\phi(\gamma(0)) = \gamma(0)$ and $d\phi(\dot{\gamma}(0)) = \dot{\gamma}(0)$ then imply that $\phi(\gamma) = \gamma$. Hence the image of the geodesic $\gamma : I \to N$ is contained in M, so following Proposition 7.23 the submanifold M is totally geodesic in N. $\square$

Corollary 7.26. *Let $m < n$ be positive integers. Then the m-dimensional sphere*

$$S^m = \{(x, 0) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} |\ |x|^2 = 1\}$$

is a totally geodesic submanifold of

$$S^n = \{(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} |\ |x|^2 + |y|^2 = 1\}.$$

PROOF. The statement is a direct consequence of the fact that S^m is the fixpoint set of the isometry $\phi : S^n \to S^n$ of S^n with $(x, y) \mapsto (x, -y)$. $\square$

Corollary 7.27. *Let $m < n$ be positive integers. Let H^n be the n-dimensional hyperbolic space modelled on the upper half space $\mathbb{R}^+ \times \mathbb{R}^{n-1}$ equipped with the Riemannian metric*

$$g(X, Y) = \frac{1}{x_1^2} \langle X, Y \rangle_{\mathbb{R}^n},$$

where $x = (x_1, \dots, x_n) \in H^n$. Then the m-dimensional hyperbolic space

$$H^m = \{(x, 0) \in H^n |\ x \in \mathbb{R}^m\}$$

is totally geodesic in H^n.

PROOF. See Exercise 7.6. $\square$

Exercises

Exercise 7.1. The result of Exercise 5.3 shows that the two dimensional hyperbolic disc H^2 introduced in Example 5.7 is isometric to the upper half plane $M = (\{(x, y) \in \mathbb{R}^2 |\ y \in \mathbb{R}^+\}$ equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{y^2}\langle X, Y\rangle_{\mathbb{R}^2}.$$

Use your local library to find all geodesics in (M, g).

Exercise 7.2. Let n be a positive integer and $\mathbf{O}(n)$ be the orthogonal group equipped with the standard left-invariant metric

$$g(A, B) = \text{trace}(A^t B).$$

Prove that a C^2-curve $\gamma : (-\epsilon, \epsilon) \to \mathbf{O}(n)$ is a geodesic if and only if

$$\gamma^t \cdot \ddot{\gamma} = \ddot{\gamma}^t \cdot \gamma.$$

Exercise 7.3. Find a proof for Proposition 7.23.

Exercise 7.4. Find a proof for Corollary 7.24.

Exercise 7.5. For the real parameter $\theta \in (0, \pi/2)$ define the 2-dimensional torus T^2_θ by

$$T^2_\theta = \{(\cos\theta e^{i\alpha}, \sin\theta e^{i\beta}) \in S^3 |\ \alpha, \beta \in \mathbb{R}\}.$$

Determine for which $\theta \in (0, \pi/2)$ the torus T^2_θ is a minimal submanifold of the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 |\ |z_1|^2 + |z_2|^2 = 1\}.$$

Exercise 7.6. Find a proof for Corollary 7.27.

Exercise 7.7. Determine the totally geodesic submanifolds of the m-dimensional real projective space $\mathbb{R}P^m$.

Exercise 7.8. Let the orthogonal group $\mathbf{O}(n)$ be equipped with the left-invariant metric

$$g(A, B) = \text{trace}(A^t B)$$

and let K be a Lie subgroup of $\mathbf{O}(n)$. Prove that K is totally geodesic in $\mathbf{O}(n)$.

CHAPTER 8

The Riemann Curvature Tensor

In this chapter we introduce the Riemann curvature tensor and the notion of sectional curvature of a Riemannian manifold. These generalize the Gaussian curvature playing a central role in classical differential geometry.

We prove that the Euclidean spaces, the standard spheres and the hyperbolic spaces all have constant sectional curvature. We determine the Riemannian curvature tensor for manifolds of constant sectional curvature and also for an important class of Lie groups. We then derive the important Gauss equation comparing the sectional curvatures of a submanifold and that of its ambient space.

Let (M, g) be a Riemannian manifold and ∇ be its Levi-Civita connection. Then to each vector field $X \in C^\infty(TM)$ we have the **first order covariant derivative**

$$\nabla_X : C^\infty(TM) \to C^\infty(TM)$$

in the direction of X satisfying

$$\nabla_X : Z \mapsto \nabla_X Z.$$

We shall now generalize this and introduce the covariant derivative of tensor fields of type $(r, 0)$ or $(r, 1)$.

As motivation, let us assume that A is a tensor field of type $(2, 1)$. If we differentiate $A(Y, Z)$ in the direction of X applying the naive "product rule"

$$\nabla_X(A(Y, Z)) = (\nabla_X A)(Y, Z) + A(\nabla_X Y, Z) + A(Y, \nabla_X Z)$$

we get

$$(\nabla_X A)(Y, Z) = \nabla_X(A(Y, Z)) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

where $\nabla_X A$ is the "covariant derivative" of the tensor field A in the direction of X. This naive idea turns out to be very useful and leads to the following formal definition.

Definition 8.1. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For a tensor field $A : C^\infty_r(TM) \to C^\infty_0(TM)$ of type $(r, 0)$ we define its **covariant derivative**

$$\nabla A : C^\infty_{r+1}(TM) \to C^\infty_0(TM)$$

by

$$\nabla A : (X, X_1, \dots, X_r) \mapsto (\nabla_X A)(X_1, \dots, X_r) =$$

$$X(A(X_1, \dots, X_r)) - \sum_{k=1}^{r} A(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r).$$

A tensor field A of type $(r, 0)$ is said to be **parallel** if $\nabla A \equiv 0$.

The following result can be seen as, yet another, compatibility of the Levi-Civita connection ∇ of (M, g) with the Riemannian metric g.

Proposition 8.2. *Let (M, g) be a Riemannian manifold. Then the metric g is a parallel tensor field of type $(2, 0)$.*

Proof. See Exercise 8.1. □

Let (M, g) be a Riemannian manifold. Then its Levi-Civita connection ∇ is tensorial in its first argument i.e. if $X, Y \in C^\infty(TM)$ and $f, g \in C^\infty(M)$ then

$$\nabla_{(fX + gY)} Z = f\nabla_X Z + g\nabla_Y Z.$$

This means that a vector field $Z \in C^\infty(TM)$ on M induces a **natural tensor field** $\mathcal{Z} : C^\infty_1(TM) \to C^\infty_1(TM)$ of type $(1, 1)$ given by

$$\mathcal{Z} : X \mapsto \nabla_X Z.$$

Definition 8.3. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. For a tensor field $B : C^\infty_r(TM) \to C^\infty_1(TM)$ of type $(r, 1)$ we define its **covariant derivative**

$$\nabla B : C^\infty_{r+1}(TM) \to C^\infty_1(TM)$$

by

$$\nabla B : (X, X_1, \dots, X_r) \mapsto (\nabla_X B)(X_1, \dots, X_r) =$$

$$\nabla_X(B(X_1, \dots, X_r)) - \sum_{k=1}^{r} B(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r).$$

A tensor field B of type $(r, 1)$ is said to be **parallel** if $\nabla B \equiv 0$.

Definition 8.4. Let $X, Y \in C^\infty(TM)$ be two vector fields on the Riemannian manifold (M, g) with Levi-Civita connection ∇. Then the **second order covariant derivative**

$$\nabla^2_{X,Y} : C^\infty(TM) \to C^\infty(TM)$$

is defined by

$$\nabla^2_{X,Y} : Z \mapsto (\nabla_X \mathcal{Z})(Y),$$

where $\mathcal{Z}$ is the natural $(1,1)$-tensor field induced by $Z \in C^\infty(TM)$.

As a direct consequence of Definitions 8.3 and 8.4 we see that if $X, Y, Z \in C^\infty(TM)$ then the second order covariant derivative $\nabla^2_{X,Y}$ satisfies

$$\nabla^2_{X,Y} Z = \nabla_X(\mathcal{Z}(Y)) - \mathcal{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

Definition 8.5. Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then we define its Riemann curvature operator

$$R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$$

as twice the skew-symmetric part of the second covariant derivative ∇^2 i.e.

$$R(X, Y)Z = \nabla^2_{X,Y} Z - \nabla^2_{Y,X} Z.$$

The next remarkable result shows that the curvature operator is a tensor field.

Theorem 8.6. *Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. Then the* **curvature** $R : C_3^\infty(TM) \to C_1^\infty(TM)$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is a **tensor field** *on M of type $(3, 1)$.*

Proof. See Exercise 8.2. □

The reader should note that the Riemann curvature tensor R is an intrinsic object since it only depends on the intrinsic Levi-Civita connection ∇. The following result shows that the curvature tensor has many nice properties of symmetry.

Proposition 8.7. *Let (M, g) be a Riemannian manifold. For vector fields $X, Y, Z, W \in C^\infty(TM)$ on M we then have*

(i) $R(X, Y)Z = -R(Y, X)Z$,
(ii) $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$,
(iii) $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$,
(iv) $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$,
(v) $6 \cdot R(X, Y)Z = R(X, Y + Z)(Y + Z) - R(X, Y - Z)(Y - Z) + R(X + Z, Y)(X + Z) - R(X - Z, Y)(X - Z)$.

Proof. See Exercise 8.3. □

Part (iii) of Proposition 8.7 is the so called first **Bianchi identity**. The second Bianchi identity is a similar result concerning the covariant derivative ∇R of the curvature tensor. This will not be treated here.

Let (M, g) be a Riemannian manifold and $p \in M$. Then a **section** V at p is a 2-dimensional subspace of the tangent space T_pM. The set

$$G_2(T_pM) = \{V | \ V \text{ is a section of } T_pM\}$$

of sections is called the **Grassmannian** of 2-planes at p.

Lemma 8.8. *Let (M, g) be a Riemannian manifold, $p \in M$ and $X, Y, Z, W \in T_pM$ be tangent vectors at p such that the two sections span$_{\mathbb{R}}\{X, Y\}$ and span$_{\mathbb{R}}\{Z, W\}$ are identical. Then*

$$\frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2} = \frac{g(R(Z,W)W,Z)}{|Z|^2|W|^2 - g(Z,W)^2}.$$

PROOF. See Exercise 8.4. □

This leads to the following natural definition of the sectional curvature.

Definition 8.9. Let (M, g) be a Riemannian manifold and $p \in M$ Then the function $K_p : G_2(T_pM) \to \mathbb{R}$ given by

$$K_p : \text{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X,Y)Y,X)}{|X|^2|Y|^2 - g(X,Y)^2}$$

is called the **sectional curvature** at p. We often write $K(X, Y)$ for $K(\text{span}_{\mathbb{R}}\{X, Y\})$.

Definition 8.10. Let (M, g) be a Riemannian manifold, $p \in M$ and $K_p : G_2(T_pM) \to \mathbb{R}$ be the sectional curvature at p. Then we define the functions $\delta, \Delta : M \to \mathbb{R}$ by

$$\delta : p \mapsto \min_{V \in G_2(T_pM)} K_p(V) \quad \text{and} \quad \Delta : p \mapsto \max_{V \in G_2(T_pM)} K_p(V).$$

The Riemannian manifold (M, g) is said to be

(i) of **positive curvature** if $\delta(p) \geq 0$ for all p,
(ii) of **strictly positive curvature** if $\delta(p) > 0$ for all p,
(iii) of **negative curvature** if $\Delta(p) \leq 0$ for all p,
(iv) of **strictly negative curvature** if $\Delta(p) < 0$ for all p,
(v) of **constant curvature** if $\delta = \Delta$ is constant,
(vi) **flat** if $\delta \equiv \Delta \equiv 0$.

The next result shows how the curvature tensor can be expressed in terms of local coordinates.

Proposition 8.11. *Let (M, g) be a Riemannian manifold and let (U, x) be local coordinates on M. For $i, j, k, l = 1, \dots, m$ put*

$$X_i = \frac{\partial}{\partial x_i}, \quad g_{ij} = g(X_i, X_j) \quad \text{and} \quad R_{ijkl} = g(R(X_i, X_j)X_k, X_l).$$

Then

$$R_{ijkl} = \sum_{s=1}^{m} g_{sl}\Big(\frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^{m}\{\Gamma_{jk}^r \cdot \Gamma_{ir}^s - \Gamma_{ik}^r \cdot \Gamma_{jr}^s\}\Big),$$

where Γ_{ij}^k are the Christoffel symbols of the Levi-Civita connection ∇ of (M, g) with respect to (U, x).

Proof. Using the fact that $[X_i, X_j] = 0$ we obtain

$$\begin{aligned}
R(X_i, X_j)X_k &= \nabla_{X_i}\nabla_{X_j}X_k - \nabla_{X_j}\nabla_{X_i}X_k \\
&= \sum_{s=1}^{m}\{\nabla_{X_i}(\Gamma_{jk}^s \cdot X_s) - \nabla_{X_j}(\Gamma_{ik}^s \cdot X_s)\} \\
&= \sum_{s=1}^{m}\Big(\frac{\partial \Gamma_{jk}^s}{\partial x_i} \cdot X_s + \sum_{r=1}^{m}\Gamma_{jk}^s\Gamma_{is}^r X_r - \frac{\partial \Gamma_{ik}^s}{\partial x_j} \cdot X_s - \sum_{r=1}^{m}\Gamma_{ik}^s\Gamma_{js}^r X_r\Big) \\
&= \sum_{s=1}^{m}\Big(\frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^{m}\{\Gamma_{jk}^r\Gamma_{ir}^s - \Gamma_{ik}^r\Gamma_{jr}^s\}\Big)X_s.
\end{aligned}$$

□

For the m-dimensional vector space $\mathbb{R}^m$ equipped with the Euclidean metric $\langle , \rangle_{\mathbb{R}^m}$ the set $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}$ is a global frame for the tangent bundle $T\mathbb{R}^m$. In this situation we have $g_{ij} = \delta_{ij}$, so $\Gamma_{ij}^k \equiv 0$ by Example 6.12. This implies that $R \equiv 0$ so E^m is flat.

Example 8.12. The standard sphere S^m has constant sectional curvature $+1$ (see Exercises 8.7 and 8.8) and the hyperbolic space H^m has constant sectional curvature -1 (see Exercise 8.9).

Our next aim is a formula for the curvature tensor for manifolds of constant sectional curvature. This we present in Corollary 8.16. First we need some preparations.

Lemma 8.13. *Let (M, g) be a Riemannian manifold, $p \in M$ and $Y \in T_pM$. Then the map $\tilde{Y} : T_pM \to T_pM$ given by*

$$\tilde{Y} : X \mapsto R(X, Y)Y$$

is a symmetric endomorphism of the tangent space T_pM.

PROOF. For $Z \in T_pM$ we have

$$\begin{aligned} g(\tilde{Y}(X), Z) &= g(R(X,Y)Y, Z) = g(R(Y,Z)X, Y) \\ &= g(R(Z,Y)Y, X) = g(X, \tilde{Y}(Z)). \end{aligned}$$

□

For a tangent vector $Y \in T_pM$ with $|Y| = 1$ let $\mathcal{N}(Y)$ be the normal space to Y

$$\mathcal{N}(Y) = \{X \in T_pM |\ g(X,Y) = 0\}.$$

The fact that $\tilde{Y}(Y) = 0$ and Lemma 8.13 ensure the existence of an orthonormal basis of eigenvectors $X_1, \dots, X_{m-1}$ for the restriction of the symmetric endomorphism $\tilde{Y}$ to $\mathcal{N}(Y)$. Without loss of generality, we can assume that the corresponding eigenvalues satisfy

$$\lambda_1(p) \le \cdots \le \lambda_{m-1}(p).$$

If $X \in \mathcal{N}(Y)$, $|X| = 1$ and $\tilde{Y}(X) = \lambda X$ then

$$K_p(X,Y) = g(R(X,Y)Y, X) = g(\tilde{Y}(X), X) = \lambda.$$

This means that the eigenvalues satisfy

$$\delta(p) \le \lambda_1(p) \le \cdots \le \lambda_{m-1}(p) \le \Delta(p).$$

Definition 8.14. Let (M, g) be a Riemannian manifold. Then define the smooth tensor field $R_1 : C_3^\infty(TM) \to C_1^\infty(TM)$ of type $(3, 1)$ by

$$R_1(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

Proposition 8.15. *Let (M, g) be a smooth Riemannian manifold and X, Y, Z be vector fields on M. Then*

(i) $|R(X,Y)Y - \frac{\delta+\Delta}{2} R_1(X,Y)Y| \le \frac{1}{2}(\Delta - \delta)|X||Y|^2$
(ii) $|R(X,Y)Z - \frac{\delta+\Delta}{2} R_1(X,Y)Z| \le \frac{2}{3}(\Delta - \delta)|X||Y||Z|$

PROOF. Without loss of generality we can assume that $|X| = |Y| = |Z| = 1$. If $X = X^\perp + X^\top$ with $X^\perp \perp Y$ and $X^\top$ is a multiple of Y then $R(X,Y)Z = R(X^\perp, Y)Z$ and $|X^\perp| \le |X|$ so we can also assume that $X \perp Y$. Then $R_1(X,Y)Y = \langle Y, Y\rangle X - \langle X, Y\rangle Y = X$.

The first statement follows from the fact that the symmetric endomorphism of T_pM with

$$X \mapsto \{R(X,Y)Y - \frac{\Delta + \delta}{2} \cdot X\}$$

restricted to $\mathcal{N}(Y)$ has eigenvalues in the interval $[\frac{\delta-\Delta}{2}, \frac{\Delta-\delta}{2}]$.

It is easily checked that the operator R_1 satisfies the conditions of Proposition 8.7 and hence $D = R - \frac{\Delta+\delta}{2} \cdot R_1$ as well. This implies that

$$\begin{aligned} 6 \cdot D(X,Y)Z &= D(X,Y+Z)(Y+Z) - D(X,Y-Z)(Y-Z) \\ &+ D(X+Z,Y)(X+Z) - D(X-Z,Y)(X-Z). \end{aligned}$$

The second statement then follows from

$$\begin{aligned} 6|D(X,Y)Z| &\leq \frac{1}{2}(\Delta-\delta)\{|X|(|Y+Z|^2 + |Y-Z|^2) \\ &\qquad +|Y|(|X+Z|^2 + |X-Z|^2)\} \\ &= \frac{1}{2}(\Delta-\delta)\{2|X|(|Y|^2+|Z|^2) + 2|Y|(|X|^2+|Z|^2)\} \\ &= 4(\Delta-\delta). \end{aligned}$$

□

As a direct consequence of Proposition 8.15 we have the following useful result.

Corollary 8.16. *Let (M,g) be a Riemannian manifold of constant curvature κ. Then the curvature tensor R is given by*

$$R(X,Y)Z = \kappa \cdot (g(Y,Z)X - g(X,Z)Y).$$

Proof. The result follows directly from $\kappa = \delta = \Delta$. □

Proposition 8.17. *Let (G,g) be a Lie group equipped with a left-invariant metric such that for all $X \in \mathfrak{g}$ the endomorphism*

$$ad(X) : \mathfrak{g} \to \mathfrak{g}$$

is skew-symmetric with respect to g. Then for any left-invariant vector fields $X, Y, Z \in \mathfrak{g}$ the curvature tensor R is given by

$$R(X,Y)Z = -\frac{1}{4}\ [[X,Y],Z].$$

Proof. See Exercise 8.6. □

We shall now prove the important **Gauss equation** comparing the curvature tensors of a submanifold and its ambient space in terms of the second fundamental form.

Theorem 8.18. *Let (N,h) be a Riemannian manifold and M be a submanifold of N equipped with the induced metric g. Let $X,Y,Z,W \in C^\infty(TN)$ be vector fields extending $\tilde{X},\tilde{Y},\tilde{Z},\tilde{W} \in C^\infty(TM)$. Then*

$$\begin{aligned} & g(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) - h(R(X,Y)Z,W) \\ = \ & h(B(\tilde{Y},\tilde{Z}),B(\tilde{X},\tilde{W})) - h(B(\tilde{X},\tilde{Z}),B(\tilde{Y},\tilde{W})). \end{aligned}$$

PROOF. Using the definitions of the curvature tensors R, $\tilde{R}$, the Levi-Civita connection $\tilde{\nabla}$ and the second fundamental form of $\tilde{M}$ in M we obtain

$$\begin{aligned}
& g(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) \\
&= g(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}) \\
&= h((\nabla_X(\nabla_Y Z - B(Y,Z)))^\top - (\nabla_Y(\nabla_X Z - B(X,Z)))^\top, W) \\
&\quad -h((\nabla_{[X,Y]}Z - B([X,Y],Z))^\top, W) \\
&= h(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W) \\
&\quad -h(\nabla_X(B(Y,Z)),W) + \nabla_Y(B(X,Z)),W) \\
&= h(R(X,Y)Z,W) \\
&\quad h((B(Y,Z)),\nabla_X W) - h(B(X,Z)),\nabla_Y W) \\
&= h(R(X,Y)Z,W) \\
&\quad +h(B(Y,Z),B(X,W)) - h(B(X,Z),B(Y,W)).
\end{aligned}$$

□

We shall now apply the Gauss equation to the classical situation of a regular surface Σ as a submanifold of the 3-dimensional Euclidean space $\mathbb{R}^3$. Let $\{\tilde{X},\tilde{Y}\}$ be a local orthonormal frame for the tangent bundle $T\Sigma$ of Σ around a point $p\in\Sigma$ and $\tilde{N}$ be the local Gauss map with $\tilde{N} = \tilde{X}\times\tilde{Y}$. If X, Y, N are local extensions of $\tilde{X},\tilde{Y},\tilde{N}$ then the second fundamental form B of Σ in $\mathbb{R}^3$ satisfies

$$\begin{aligned}
B(\tilde{X},\tilde{Y}) &= <\partial_X Y, N> N \\
&= -<Y,\partial_X N> N \\
&= -<Y, dN(X)> N \\
&= <Y, S_p(X)> N,
\end{aligned}$$

where $S_p : T_p\Sigma \to T_p\Sigma$ is the shape operator at p. If we now apply the fact that $\mathbb{R}^3$ is flat, the Gauss equation tells us that the sectional curvature $K(\tilde{X},\tilde{Y})$ of Σ at p satisfies

$$\begin{aligned}
K(\tilde{X},\tilde{Y}) &= <\tilde{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}> \\
&= <B(\tilde{Y},\tilde{Y}),B(\tilde{X},\tilde{X})> - <B(\tilde{X},\tilde{Y}),B(\tilde{Y},\tilde{X})> \\
&= \det S_p.
\end{aligned}$$

In other word, the sectional curvature $K(\tilde{X},\tilde{Y})$ is the determinant of the shape operator S_p at p i.e. the classical Gaussian curvature.

As a direct consequence of the Gauss equation we have the following useful result, see for example Exercises 8.8 and 8.9.

Corollary 8.19. *Let (N, h) be a Riemannian manifold and (M, g) be a totally geodesic submanifold of N. Let $X, Y, Z, W \in C^\infty(TN)$ be vector fields extending $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(TM)$. Then*

$$g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) = h(R(X, Y)Z, W).$$

We conclude this chapter by defining the Ricci and scalar curvatures of a Riemannian manifold. These are obtained by taking traces over the curvature tensor and play an important role in Riemannian geometry.

Definition 8.20. Let (M, g) be a Riemannian manifold, then

(i) the **Ricci operator** $r : C_1^\infty(TM) \to C_1^\infty(M)$ is defined by

$$r(X) = \sum_{i=1}^m R(X, e_i)e_i,$$

(ii) the **Ricci curvature** $Ric : C_2^\infty(TM) \to C_0^\infty(TM)$ by

$$Ric(X, Y) = \sum_{i=1}^m g(R(X, e_i)e_i, Y), \quad \text{and}$$

(iii) the **scalar curvature** $s \in C^\infty(M)$ by

$$s = \sum_{j=1}^m Ric(e_j, e_j) = \sum_{j=1}^m \sum_{i=1}^m g(R(e_i, e_j)e_j, e_i).$$

Here $\{e_1, \dots, e_m\}$ is any local orthonormal frame for the tangent bundle.

Corollary 8.21. *Let (M, g) be a Riemannian manifold of constant sectional curvature κ. Then the following holds*

$$s = m \cdot (m-1) \cdot \kappa.$$

PROOF. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis, then Corollary 8.16 implies that

$$\begin{aligned} Ric(e_j, e_j) &= \sum_{i=1}^m g(R(e_j, e_i)e_i, e_j) \\ &= \sum_{i=1}^m g(\kappa(g(e_i, e_i)e_j - g(e_j, e_i)e_i), e_j) \\ &= \kappa(\sum_{i=1}^m g(e_i, e_i)g(e_j, e_j) - \sum_{i=1}^m g(e_i, e_j)g(e_i, e_j)) \end{aligned}$$

$$= \kappa(\sum_{i=1}^{m} 1 - \sum_{i=1}^{m} \delta_{ij}) = (m-1) \cdot \kappa.$$

To obtain the formula for the scalar curvature s we only have to multiply the constant Ricci curvature $Ric(e_j, e_j)$ by m. □

For further reading on different notions of curvature we recommend the interesting book, Wolfgang Kühnel, Differential Geometry: Curves - Surfaces - Manifolds, AMS (2002).

Exercises

Exercise 8.1. Let (M, g) be a Riemannian manifold. Prove that the tensor field g of type $(2, 0)$ is parallel with respect to the Levi-Civita connection.

Exercise 8.2. Let (M, g) be a Riemannian manifold. Prove that the curvature R is a tensor field of type $(3, 1)$.

Exercise 8.3. Find a proof for Proposition 8.7.

Exercise 8.4. Find a proof for Lemma 8.8.

Exercise 8.5. Let $\mathbb{R}^m$ and $\mathbb{C}^m$ be equipped with their standard Euclidean metric g given by

$$g(z, w) = \text{Re} \sum_{k=1}^{m} z_k \bar{w}_k$$

and let

$$T^m = \{z \in \mathbb{C}^m |\ |z_1| = ... = |z_m| = 1\}$$

be the m-dimensional torus in $\mathbb{C}^m$ with the induced metric. Find an isometric immersion $\phi : \mathbb{R}^m \to T^m$, determine all geodesics on T^m and prove that the torus is flat.

Exercise 8.6. Find a proof for Proposition 8.17.

Exercise 8.7. Let the Lie group $S^3 \cong \mathbf{SU}(2)$ be equipped with the metric

$$g(Z, W) = \frac{1}{2} \text{Re}(\text{trace}(\bar{Z}^t W)).$$

(i) Find an orthonormal basis for $T_e\mathbf{SU}(2)$.
(ii) Prove that $(\mathbf{SU}(2), g)$ has constant sectional curvature $+1$.

Exercise 8.8. Let S^m be the unit sphere in $\mathbb{R}^{m+1}$ equipped with the standard Euclidean metric $\langle, \rangle_{\mathbb{R}^{m+1}}$. Use the results of Corollaries 7.26, 8.19 and Exercise 8.7 to prove that $(S^m, \langle, \rangle_{\mathbb{R}^{m+1}})$ has constant sectional curvature $+1$.

Exercise 8.9. Let H^m be the m-dimensional hyperbolic space modelled on the upper half space $\mathbb{R}^+ \times \mathbb{R}^{m-1}$ equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{x_1^2} \langle X, Y \rangle_{\mathbb{R}^m},$$

where $x = (x_1, \dots, x_m) \in H^m$. For $k = 1, \dots, m$ let the vector fields $X_k \in C^\infty(TH^m)$ be given by

$$(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$$

and define the operation $*$ on H^m by

$$(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x).$$

Prove that

(i) $(H^m, *)$ is a Lie group,
(ii) the vector fields $X_1, \dots, X_m$ are left-invariant,
(iii) $[X_k, X_l] = 0$ and $[X_1, X_k] = X_k$ for $k, l = 2, ..., m$,
(iv) the metric g is left-invariant,
(v) (H^m, g) has constant curvature -1.

Compare with Exercises 6.4 and 7.1.

CHAPTER 9

Curvature and Local Geometry

This chapter is devoted to the study of the local geometry of Riemannian manifolds and how this is controlled by the curvature tensor. For this we introduce the notion of a Jacobi field which is a useful tool in differential geometry. With this at hand we yield a fundamental comparison result describing the curvature dependence of local distances.

Let (M, g) be a Riemannian manifold. By a smooth **1-parameter family of geodesics** we mean a C^∞-map

$$\Phi : (-\epsilon, \epsilon) \times I \to M$$

such that the curve $\gamma_t : I \to M$ given by $\gamma_t : s \mapsto \Phi(t, s)$ is a geodesic for all $t \in (-\epsilon, \epsilon)$. The variable $t \in (-\epsilon, \epsilon)$ is called the **family parameter** of Φ.

Proposition 9.1. *Let (M, g) be a Riemannian manifold and $\Phi : (-\epsilon, \epsilon) \times I \to M$ be a 1-parameter family of geodesics. Then for each $t \in (-\epsilon, \epsilon)$ the vector field $J_t : I \to C^\infty(TM)$ along γ_t given by*

$$J_t(s) = \frac{\partial \Phi}{\partial t}(t, s)$$

satisfies the second order ordinary differential equation

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t)\dot{\gamma}_t = 0.$$

Proof. Along Φ we put $X(t, s) = \partial\Phi/\partial s$ and $J(t, s) = \partial\Phi/\partial t$. The fact that $[\partial/\partial t, \partial/\partial s] = 0$ implies that

$$[J, X] = [d\Phi(\partial/\partial t), d\Phi(\partial/\partial s)] = d\Phi([\partial/\partial t, \partial/\partial s]) = 0.$$

Since Φ is a family of geodesics we have $\nabla_X X = 0$ and the definition of the curvature tensor then gives

$$\begin{aligned} R(J, X)X &= \nabla_J \nabla_X X - \nabla_X \nabla_J X - \nabla_{[J, X]} X \\ &= -\nabla_X \nabla_J X \\ &= -\nabla_X \nabla_X J. \end{aligned}$$

Hence for each $t \in (-\epsilon, \epsilon)$ we have

$$\nabla_{\dot{\gamma}_t}\nabla_{\dot{\gamma}_t}J_t + R(J_t, \dot{\gamma}_t)\dot{\gamma}_t = 0.$$

□

Definition 9.2. Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and $X = \dot{\gamma}$. A C^2 vector field J along γ is called a **Jacobi field** if

$$\nabla_X\nabla_X J + R(J, X)X = 0 \tag{5}$$

along γ. We denote the space of all Jacobi fields along γ by $\mathcal{J}_\gamma(TM)$.

We shall now give an example of a 1-parameter family of geodesics in the $(m+1)$-dimensional Euclidean space E^{m+1}.

Example 9.3. Let $c, n : \mathbb{R} \to E^{m+1}$ be smooth curves such that the image $n(\mathbb{R})$ of n is contained in the unit sphere S^m. If we define a map $\Phi : \mathbb{R} \times \mathbb{R} \to E^{m+1}$ by

$$\Phi : (t, s) \mapsto c(t) + s \cdot n(t)$$

then for each $t \in \mathbb{R}$ the curve $\gamma_t : s \mapsto \Phi(t, s)$ is a straight line and hence a geodesic in E^{m+1}. By differentiating with respect to the family parameter t we yield the Jacobi field $J \in \mathcal{J}_{\gamma_0}(TE^{m+1})$ along γ_0 with

$$J(s) = \frac{d}{dt}\Phi(t, s)|_{t=0} = \dot{c}(0) + s \cdot \dot{n}(0).$$

The Jacobi equation (5) on a Riemannian manifold is linear in J. This means that the space of Jacobi fields $\mathcal{J}_\gamma(TM)$ along γ is a vector space. We are now interested in determining the dimension of this space

Proposition 9.4. *Let $\gamma : I \to M$ be a geodesic, $0 \in I$, $p = \gamma(0)$ and $X = \dot{\gamma}$ along γ. If $v, w \in T_pM$ are two tangent vectors at p then there exists a unique Jacobi field J along γ, such that $J_p = v$ and $(\nabla_X J)_p = w$.*

Proof. Let $\{X_1, \dots, X_m\}$ be an orthonormal frame of parallel vector fields along γ, see Proposition 7.8. If J is a vector field along γ, then

$$J = \sum_{i=1}^m a_i X_i$$

where $a_i = g(J, X_i)$ are smooth functions on I. The vector fields $X_1, \dots, X_m$ are parallel so

$$\nabla_X J = \sum_{i=1}^m \dot{a}_i X_i \quad \text{and} \quad \nabla_X\nabla_X J = \sum_{i=1}^m \ddot{a}_i X_i.$$

For the curvature tensor we have

$$R(X_i, X)X = \sum_{k=1}^{m} b_i^k X_k,$$

where $b_i^k = g(R(X_i, X)X, X_k)$ are smooth functions on I depending on the geometry of (M, g). This means that $R(J, X)X$ is given by

$$R(J, X)X = \sum_{i,k=1}^{m} a_i b_i^k X_k.$$

and that J is a Jacobi field if and only if

$$\sum_{i=1}^{m} (\ddot{a}_i + \sum_{k=1}^{m} a_k b_k^i) X_i = 0.$$

This is equivalent to the second order system

$$\ddot{a}_i + \sum_{k=1}^{m} a_k b_k^i = 0 \quad \text{for all } i = 1, 2, \dots, m$$

of linear ordinary differential equations in $a = (a_1, \dots, a_m)$. A global solution will always exist and is uniquely determined by $a(0)$ and $\dot{a}(0)$. This implies that J exists globally and is uniquely determined by the initial conditions

$$J(0) = v \quad \text{and} \quad (\nabla_X J)(0) = w.$$

□

Corollary 9.5. *Let (M, g) be an m-dimensional Riemannian manifold and $\gamma : I \to M$ be a geodesic in M. Then the vector space $\mathcal{J}_\gamma(TM)$ of all Jacobi fields along γ has the dimension $2m$.*

The following Lemma shows that when proving results about Jacobi fields along a geodesic γ we can always assume, without loss of generality, that $|\dot{\gamma}| = 1$.

Lemma 9.6. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. If $\lambda \in \mathbb{R}^*$ and $\sigma : \lambda I \to I$ is given by $\sigma : t \mapsto t/\lambda$, then $\gamma \circ \sigma : \lambda I \to M$ is a geodesic and $J \circ \sigma$ is a Jacobi field along $\gamma \circ \sigma$.*

Proof. See Exercise 9.1. □

Next we determine the Jacobi fields which are tangential to a given geodesic.

Proposition 9.7. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic with $|\dot\gamma| = 1$ and J be a Jacobi field along γ. Let $J^\top$ be the tangential part of J given by*

$$J^\top = g(J, \dot\gamma)\dot\gamma \quad \text{and} \quad J^\perp = J - J^\top$$

be its normal part. Then $J^\top$ and $J^\perp$ are Jacobi fields along γ and there exist $a, b \in \mathbb{R}$ such that $J^\top(s) = (as + b)\dot\gamma(s)$ for all $s \in I$.

Proof. We now have

$$\begin{aligned}
\nabla_{\dot\gamma}\nabla_{\dot\gamma}J^\top + R(J^\top, \dot\gamma)\dot\gamma &= \nabla_{\dot\gamma}\nabla_{\dot\gamma}(g(J, \dot\gamma)\dot\gamma) + R(g(J, \dot\gamma)\dot\gamma, \dot\gamma)\dot\gamma \\
&= g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}J, \dot\gamma)\dot\gamma \\
&= -g(R(J, \dot\gamma)\dot\gamma, \dot\gamma)\dot\gamma \\
&= 0.
\end{aligned}$$

This shows that the tangential part $J^\top$ of J is a Jacobi field. The fact that $\mathcal{J}_\gamma(TM)$ is a vector space implies that the normal part $J^\perp = J - J^\top$ of J also is a Jacobi field.

By differentiating $g(J, \dot\gamma)$ twice along γ we obtain

$$\frac{d^2}{ds^2}(g(J, \dot\gamma)) = g(\nabla_{\dot\gamma}\nabla_{\dot\gamma}J, \dot\gamma) = -g(R(J, \dot\gamma)\dot\gamma, \dot\gamma) = 0$$

so $g(J, \dot\gamma(s)) = (as + b)$ for some $a, b \in \mathbb{R}$. □

Corollary 9.8. *Let (M, g) be a Riemannian manifold, $\gamma : I \to M$ be a geodesic and J be a Jacobi field along γ. If*

$$g(J(t_0), \dot\gamma(t_0)) = 0 \quad \text{and} \quad g((\nabla_{\dot\gamma}J)(t_0), \dot\gamma(t_0)) = 0$$

for some $t_0 \in I$, then $g(J(t), \dot\gamma(t)) = 0$ for all $t \in I$.

Proof. This is a direct consequence of the fact that the function $g(J, \dot\gamma)$ satisfies the second order ordinary differential equation $\ddot f = 0$ and the initial conditions $f(0) = 0$ and $\dot f(0) = 0$. □

Our next aim is to show that if the Riemannian manifold (M, g) has constant sectional curvature then we can solve the Jacobi equation

$$\nabla_X\nabla_X J + R(J, X)X = 0$$

along any given geodesic $\gamma : I \to M$. For this we introduce the following notation. For a real number $\kappa \in \mathbb{R}$ we define the $c_\kappa, s_\kappa : \mathbb{R} \to \mathbb{R}$ by

$$c_\kappa(s) = \begin{cases} cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0, \\ 1 & \text{if } \kappa = 0, \\ cos(\sqrt{\kappa}s) & \text{if } \kappa > 0. \end{cases}$$

and

$$s_\kappa(s) = \begin{cases} sinh(\sqrt{|\kappa|}s)/\sqrt{|\kappa|} & \text{if } \kappa < 0, \\ s & \text{if } \kappa = 0, \\ sin(\sqrt{\kappa}s)/\sqrt{\kappa} & \text{if } \kappa > 0. \end{cases}$$

It is a well known fact that the unique solution to the initial value problem

$$\ddot{f} + \kappa \cdot f = 0, \quad f(0) = a \text{ and } \dot{f}(0) = b$$

is the function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(s) = ac_\kappa(s) + bs_\kappa(s)$.

Example 9.9. Let $\mathbb{C}$ be the complex plane with the standard Euclidean metric $\langle , \rangle_{\mathbb{R}^2}$ of constant sectional curvature $\kappa = 0$. The rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto se^{it}$. Along the geodesic $\gamma_0 : s \mapsto s$ we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = is$$

with $|J_0(s)| = |s| = |s_\kappa(s)|$.

Example 9.10. Let S^2 be the unit sphere in the standard Euclidean 3-space $\mathbb{C} \times \mathbb{R}$ with the induced metric of constant sectional curvature $\kappa = +1$. Rotations about the $\mathbb{R}$-axis produce a 1-parameter family of geodesics $\Phi_t : s \mapsto (sin(s)e^{it}, cos(s))$. Along the geodesic $\gamma_0 : s \mapsto (sin(s), cos(s))$ we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = (isin(s), 0)$$

with $|J_0(s)|^2 = sin^2(s) = |s_\kappa(s)|^2$.

Example 9.11. Let $B_1^2(0)$ be the open unit disk in the complex plane with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \langle , \rangle_{\mathbb{R}^2}$$

of constant sectional curvature $\kappa = -1$. Rotations about the origin produce a 1-parameter family of geodesics $\Phi_t : s \mapsto tanh(s/2)e^{it}$. Along the geodesic $\gamma_0 : s \mapsto tanh(s/2)$ we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = i \cdot tanh(s/2)$$

with

$$|J_0(s)|^2 = \frac{4 \cdot tanh^2(s/2)}{(1 - tanh^2(s/2))^2} = sinh^2(s) = |s_\kappa(s)|^2.$$

Example 9.12. Let (M, g) be a Riemannian manifold of constant sectional curvature κ and $\gamma : I \to M$ be a geodesic with $|X| = 1$ where $X = \dot{\gamma}$. Further let $P_1, P_2, \dots, P_{m-1}$ be parallel vector fields along γ such that $g(P_i, P_j) = \delta_{ij}$ and $g(P_i, X) = 0$. Any vector field J along γ may now be written as

$$J(s) = \sum_{i=1}^{m-1} f_i(s) P_i(s) + f_m(s) X(s).$$

This means that J is a Jacobi field if and only if

$$\begin{aligned}
\sum_{i=1}^{m-1} \ddot{f}_i(s) P_i(s) + \ddot{f}_m(s) X(s) &= \nabla_X \nabla_X J \\
&= -R(J, X)X \\
&= -R(J^\perp, X)X \\
&= -\kappa(g(X, X)J^\perp - g(J^\perp, X)X) \\
&= -\kappa J^\perp \\
&= -\kappa \sum_{i=1}^{m-1} f_i(s) P_i(s).
\end{aligned}$$

This is equivalent to the following system of ordinary differential equations

$$\ddot{f}_m(s) = 0 \ \text{ and } \ \ddot{f}_i(s) + \kappa f_i(s) = 0 \ \text{ for all } i = 1, 2, \dots, m-1. \tag{6}$$

It is clear that for the initial values

$$\begin{aligned}
J(s_0) &= \sum_{i=1}^{m-1} v_i P_i(s_0) + v_m X(s_0), \\
(\nabla_X J)(s_0) &= \sum_{i=1}^{m-1} w_i P_i(s_0) + w_m X(s_0)
\end{aligned}$$

or equivalently

$$f_i(s_0) = v_i \ \text{ and } \ \dot{f}_i(s_0) = w_i \ \text{ for all } \ i = 1, 2, \dots, m$$

we have a unique and explicit solution to the system (6) on the whole of I. They are given by

$$f_m(s) = v_m + s w_m \ \text{ and } \ f_i(s) = v_i c_\kappa(s) + w_i s_\kappa(s)$$

for all $i = 1, 2, \dots, m-1$. It should be noted that if $g(J, X) = 0$ and $J(0) = 0$ then

$$|(\nabla_X J)(s)| = |(\nabla_X J)(0)| s_\kappa(s).$$

In the next example we give a complete description of the Jacobi fields along a geodesic on the 2-dimensional sphere.

Example 9.13. Let S^2 be the unit sphere in the standard Euclidean 3-space $\mathbb{C} \times \mathbb{R}$ with the induced metric of constant curvature $\kappa = +1$ and $\gamma : \mathbb{R} \to S^2$ be the geodesic given by $\gamma : s \mapsto (e^{is}, 0)$. Then $\dot{\gamma}(s) = (ie^{is}, 0)$ so it follows from Proposition (9.7) that all Jacobi fields tangential to γ are given by

$$J^T_{(a,b)}(s) = (as + b)(ie^{is}, 0) \quad \text{for some } a, b \in \mathbb{R}.$$

The vector field $P : \mathbb{R} \to TS^2$ given by $s \mapsto ((e^{is}, 0), (0, 1))$ satisfies $\langle P, \dot{\gamma} \rangle = 0$ and $|P| = 1$. The sphere S^2 is 2-dimensional and $\dot{\gamma}$ is parallel along γ so P must be parallel. This implies that all the Jacobi fields orthogonal to $\dot{\gamma}$ are given by

$$J^N_{(a,b)}(s) = (0, a\cos s + b\sin s) \;\; \text{for some } a, b \in \mathbb{R}.$$

In more general situations, where we do not have constant curvature the exponential map can be used to produce Jacobi fields as follows. Let (M, g) be a complete Riemannian manifold, $p \in M$ and $v, w \in T_pM$. Then $s \mapsto s(v+tw)$ defines a 1-parameter family of lines in the tangent space T_pM which all pass through the origin $0 \in T_pM$. Remember that the exponential map

$$\exp_p|_{B^m_{\varepsilon_p}(0)} : B^m_{\varepsilon_p}(0) \to \exp_p(B^m_{\varepsilon_p}(0))$$

maps lines in T_pM through the origin onto geodesics on M. Hence the map

$$\Phi_t : s \mapsto \exp_p(s(v + tw))$$

is a 1-parameter family of geodesics through $p \in M$, as long as $s(v+tw)$ is an element of $B^m_{\varepsilon_p}(0)$. This means that

$$J(s) = \frac{\partial \Phi_t}{\partial t}(t, s)|_{t=0} = d(\exp_p)_{s(v+tw)}(sw)|_{t=0} = d(\exp_p)_{sv}(sw)$$

is a Jacobi field along the geodesic $\gamma : s \mapsto \Phi_0(s)$ with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Here

$$d(\exp_p)_{s(v+tw)} : T_{s(v+tw)}T_pM \to T_{\exp_p(s(v+tw))}M$$

is the linear tangent map of the exponential map $\exp_p$ at $s(v + tw)$. Now differentiating with respect to the parameter s gives

$$(\nabla_X J)(0) = \frac{d}{ds}(d(\exp_p)_{sv}(sw))|_{s=0} = d(\exp_p)_0(w) = w.$$

The above calculations show that

$$J(0) = 0 \;\; \text{and} \;\; (\nabla_X J)(0) = w.$$

The following technical result is needed for the proof of the main Theorem 9.15 at the end of this chapter.

Lemma 9.14. *Let (M, g) be a Riemannian manifold with sectional curvature uniformly bounded above by Δ and $\gamma : [0, \alpha] \to M$ be a geodesic on M with $|X| = 1$ where $X = \dot{\gamma}$. Further let $J : [0, \alpha] \to TM$ be a Jacobi field along γ such that $g(J, X) = 0$ and $|J| \neq 0$ on $(0, \alpha)$. Then*

(i) $d^2(|J|)/ds^2 + \Delta \cdot |J| \geq 0$,

(ii) *if $f : [0, \alpha] \to \mathbb{R}$ is a C^2-function such that*
  (a) $\ddot{f} + \Delta \cdot f = 0$ *and* $f > 0$ *on* $(0, \alpha)$,
  (b) $f(0) = |J(0)|$, *and*
  (c) $\dot{f}(0) = |\nabla_X J(0)|$,
  then $f(s) \leq |J(s)|$ *on* $(0, \alpha)$,

(iii) *if* $J(0) = 0$, *then* $|\nabla_X J(0)| \cdot s_\Delta(s) \leq |J(s)|$ *for all* $s \in (0, \alpha)$.

Proof. (i) Using the facts that $|X| = 1$ and $\langle X, J \rangle = 0$ we obtain

$$\begin{aligned}
\frac{d^2}{ds^2}(|J|) &= \frac{d^2}{ds^2}\sqrt{g(J,J)} = \frac{d}{ds}(\frac{g(\nabla_X J, J)}{|J|}) \\
&= \frac{g(\nabla_X \nabla_X J, J)}{|J|} + \frac{|\nabla_X J|^2 |J|^2 - g(\nabla_X J, J)^2}{|J|^3} \\
&\geq \frac{g(\nabla_X \nabla_X J, J)}{|J|} \\
&= -\frac{g(R(J,X)X, J)}{|J|} \\
&= -K(X, J) \cdot |J| \\
&\geq -\Delta \cdot |J|.
\end{aligned}$$

(ii) Define the function $h : [0, \alpha) \to \mathbb{R}$ by

$$h(s) = \begin{cases} \frac{|J(s)|}{f(s)} & \text{if } s \in (0, \alpha), \\ \lim_{s \to 0} \frac{|J(s)|}{f(s)} = 1 & \text{if } s = 0. \end{cases}$$

Then

$$\begin{aligned}
\dot{h}(s) &= \frac{1}{f^2(s)}(\frac{d}{ds}(|J(s)|)f(s) - |J(s)|\dot{f}(s)) \\
&= \frac{1}{f^2(s)}\int_0^s \frac{d}{dt}(\frac{d}{dt}(|J(t)|)f(t) - |J(t)|\dot{f}(t))dt \\
&= \frac{1}{f^2(s)}\int_0^s (\frac{d^2}{dt^2}(|J(t)|)f(t) - |J(t)|\ddot{f}(t))dt
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{f^2(s)}\int_0^s f(t)(\frac{d^2}{dt^2}(|J(t)|) + \Delta \cdot |J(t)|)dt \\ &\geq 0. \end{aligned}$$

This implies that $\dot{h}(s) \geq 0$ so $f(s) \leq |J(s)|$ for all $s \in (0, \alpha)$.

(iii) The function $f(s) = |(\nabla_X J)(0)| \cdot s_\Delta(s)$ satisfies the differential equation

$$\ddot{f}(s) + \Delta f(s) = 0$$

and the initial conditions $f(0) = |J(0)| = 0$, $\dot{f}(0) = |(\nabla_X J)(0)|$ so it follows from (ii) that $|(\nabla_X J)(0)| \cdot s_\Delta(s) = f(s) \leq |J(s)|$. □

Let (M, g) be a Riemannian manifold of sectional curvature which is uniformly bounded above, i.e. there exists a $\Delta \in \mathbb{R}$ such that $K_p(V) \leq \Delta$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_Δ, g_Δ) be another Riemannian manifold which is complete and of constant sectional curvature $K \equiv \Delta$. Let $p \in M$, $p_\Delta \in M_\Delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\Delta}M_\Delta$.

Let U be an open neighbourhood of $\mathbb{R}^m$ around 0 such that the exponential maps $(\exp)_p$ and $(\exp)_{p_\Delta}$ are diffeomorphisms from U onto their images $(\exp)_p(\text{U})$ and $(\exp)_{p_\Delta}(U)$, respectively. Let (r, p, q) be a geodesic triangle i.e. a triangle with sides which are shortest paths between their endpoints. Furthermore let $c : [a, b] \to M$ be the geodesic connecting r and q and $v : [a, b] \to T_pM$ be the curve defined by $c(t) = (\exp)_p(v(t))$. Put $c_\Delta(t) = (\exp)_{p_\Delta}(v(t))$ for $t \in [a, b]$ and then it directly follows that $c(a) = r$ and $c(b) = q$. Finally put $r_\Delta = c_\Delta(a)$ and $q_\Delta = c_\Delta(b)$.

Theorem 9.15. *For the above situation the following inequality for the distance function d is satisfied*

$$d(q_\Delta, r_\Delta) \leq d(q, r).$$

Proof. Define a 1-parameter family $s \mapsto s \cdot v(t)$ of straight lines in T_pM through p. Then

$$\Phi_t : s \mapsto (\exp)_p(s \cdot v(t)) \quad \text{and} \quad \Phi_t^\Delta : s \mapsto (\exp)_{p_\Delta}(s \cdot v(t))$$

are 1-parameter families of geodesics through $p \in M$, and $p_\Delta \in M_\Delta$, respectively. Hence

$$J_t = \partial\Phi_t/\partial t \quad \text{and} \quad J_t^\Delta = \partial\Phi_t^\Delta/\partial t$$

are Jacobi fields satisfying the initial conditions

$$J_t(0) = J_t^\Delta(0) = 0 \quad \text{and} \quad (\nabla_X J_t)(0) = (\nabla_X J_t^\Delta)(0) = \dot{v}(t).$$

Using Lemma 9.14 we now obtain

$$|\dot{c}_\Delta(t)| \quad = \quad |J_t^\Delta(1)|$$

$$\begin{aligned} &= |(\nabla_X J_t^{\Delta})(0)| \cdot s_{\Delta}(1) \\ &= |(\nabla_X J_t)(0)| \cdot s_{\Delta}(1) \\ &\leq |J_t(1)| \\ &= |\dot{c}(t)| \end{aligned}$$

The curve c is the shortest path between r and q so we have

$$d(r_{\Delta}, q_{\Delta}) \leq L(c_{\Delta}) \leq L(c) = d(r, q).$$

□

We now add the assumption that the sectional curvature of the manifold (M, g) is uniformly bounded below i.e. there exists a $\delta \in \mathbb{R}$ such that $\delta \leq K_p(V)$ for all $V \in G_2(T_pM)$ and $p \in M$. Let (M_δ, g_δ) be a complete Riemannian manifold of constant sectional curvature δ. Let $p \in M$ and $p_\delta \in M_\delta$ and identify $T_pM \cong \mathbb{R}^m \cong T_{p_\delta}M_\delta$. Then a similar construction as above gives two pairs of points $q, r \in M$ and $q_\delta, r_\delta \in M_\delta$ and shows that

$$d(q, r) \leq d(q_\delta, r_\delta).$$

Combining these two results we obtain **locally**

$$d(q_{\Delta}, r_{\Delta}) \leq d(q, r) \leq d(q_\delta, r_\delta).$$

Exercises

Exercise 9.1. Find a proof for Lemma 9.6.

Exercise 9.2. Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a geodesic such that $X = \dot{\gamma} \neq 0$. Further let J be a non-vanishing Jacobi field along γ with $g(X, J) = 0$. Prove that if $g(J, J)$ is constant along γ then (M, g) does not have strictly negative curvature.