THE CLASSIFICATION OF HARMONIC MORPHISMS TO EUCLIDEAN SPACE

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1. Introduction

Harmonic morphism is a smooth map between Riemannian manifolds which pulls back germs of harmonic functions to germs of harmonic functions. It may be charactrized as harmonic maps which are horizontally weakly conformal [5,9]. One task of studying harmonic morphism is constructing concrete examples; Another one is classification of all harmonic morphisms between all special manifolds (in particularly, between connected open set of space forms), see for example [6] and [10]. In particularly, we have

Theorem 1.1. [10] *Let $\phi : R^m \to N^n (n \geq 3)$ be a harmonic morphism with totally geodesic fibers. Then $N = R^n$ and ϕ is an orthogonal projection $P : R^m \to R^n$, followed by a homothety.*

Theorem 1.2. [6] *Let U be an open and connected subset of R^m and $\phi : U \to R^n$ be a horizontally homothetic harmonic morphism, with totally geodesic fibers. Then ϕ is the restriction of an orthogonal projection $P : R^m \to R^n$, followed by a homothety.*

Using the method of submanifold geometry, we can prove bove theorem is also correct if the condition, fibers are totally geodesic, is removed, i.e., we can prove the following

Theorem 1.3. *Let U be an open and connected subset of R^m and $\phi : U \to R^n$ be a horizontally homothetic harmonic morphism. Then ϕ is the restriction of an orthogonal projection $P : R^m \to R^n$, followed by a homothety.*

As a Corollary, we can easily prove the following

Corollary 1.4. [8]*Let U be an open and connected subset of C^m and $\phi : U \to C^n$ be a holomorphic map and be not a constant map.*
(1) *When $n = 1$, ϕ is a harmonic morphism.*
(2) *When $n \geq 2$, ϕ is harmonic morphism if and only if ϕ is the restriction of an orthogonal projection $P : C^m \to C^n$, followed by a homothety.*

2. Preliminaries

Let $\phi : M^m \to N^n$ be a smooth map and $x \in M$. Let $v_x = ker(d\phi_x)$ and $H_x = v_x^{\perp}$ be the vertical space and horizontal space at point x respectively.For any $X \in T_xM$, denote vX and HX the critical and horizontal component of X respectively. If $C_\phi = \{x \in M \mid d\phi_x = 0\}$ and $M^* = M \backslash C_\phi$, then ϕ is said to be horizontally conformal if and only if there exists a continuous function $\lambda : M^* \to R^+$ such that $\lambda^2 < X, Y >=< d\phi(X), d\phi(Y) >$ for all $X, Y \in H_x$ and $x \in M^*$. We see that $\lambda^2 =| d\phi |^2 /n$ is smooth on M. A horizontally conformal map ϕ is said to be horizontally homothetic if $grad(\lambda^2) \equiv 0$, i.e., $grad\lambda^2$ is vertical. It was proven in [4] that a horizontally homothetic harmonic morphism is submersion. We need the following

Lemma 2.1. [1]*Let $\phi : M \to N$ be a horizontally conformal submersion and $dimN \geq 3$. Then two of the following conditions imply the other:*
(1) *ϕ is harmonic;*
(2) *ϕ has minimal fibers;*
(3) *ϕ is horizontally homothetic, i.e., $grad(\lambda^2) \equiv 0$.*

3. SUBMANIFOLD GEOMETRY OF FIBERS

Lemma 3.1. *Let $\phi : M \to N$ be a horizontally conformal submersion and let W be a vertical vector field and X be a basic vector field. If λ_ϕ is invariant along fibers, then $[W, X]$ is a vertical vector field.*

Proof. Because X be a basic vector field, we can find a vector field $\bar{X}$ on N such that $d\phi(X) = \bar{X}$. When λ_ϕ is invariant along fibers, $\lambda_\phi^{-1}X$ is ϕ_*-related to the $\frac{1}{\lambda_\phi \circ \phi^{-1}}\bar{X}$ and $W(\lambda_\phi) = 0$. Thus we have

$$\phi_*[W, \lambda_\phi^{-1}X] = [0, \frac{1}{\lambda_\phi \circ \phi^{-1}}\bar{X}] = 0,$$

which says that $[W, \lambda_\phi^{-1}X]$ is a vertical vector field. But $[W, \lambda_\phi^{-1}X] = \lambda_\phi^{-1}[W, X] + W(\lambda_\phi)X = \lambda_\phi^{-1}[W, X]$, so $[W, X]$ is vertical. □

Lemma 3.2. *Let $\phi : M \to N$ be a horizontally conformal submersion. If its horizontal distribution is integrable and λ_ϕ is invariant along fibers, then any leaf of horizontial distribution is totally geodesic.*

Proof. Let X and Y be two horizontial vector fields. By lemma 1.5 in [7], we have

$$v\nabla_X Y = 1/2\{v[X, Y] - \lambda_\phi^2 < X, Y > grad_v(\lambda_\phi^{-2})\}.$$

So by assumption of lemma, we can get $v\nabla_X Y = 0$, i.e., leaves of horizontial distribution are totally geodesic. □

Next we genelarize the notion of $\phi-$parallelism in Riemannian submersion [11] to in horizontally conformal submersion. Let $L = \phi^{-1}(y)$ be a fiber of horizontally conformal submertion $\phi : M \to N$, then $H |_L$ is just the normal bundle $v(L)$ of L in M. A section X of $v(L)$ is called *ϕ-parallel* if there is a fixed vector $\bar{X}$ in T_yN such that $d\phi_x(X) = \bar{X}$ for any $x \in L$. There is another standard parallelism on $v(L)$ obtained from the Riemannian structure M, i.e., $\nabla^\perp$-parallelism, where $\nabla^\perp$ is the induced normal connection. We can prove

Proposition 3.3. *Let $\phi : M \to N$ be a horizontally conformal submersion. If λ_ϕ is invariant along fibers, then its horizontal distribution is integrable if and only if every ϕ-parallel vector field on fiber is a ∇-parallel normal field.*

Proof. When λ_ϕ is invariant along fibers, unit basic vector fields of M when restricted to L is ϕ-parallel vector field. By lemma 3.1 and 3.2, we can prove the proposition as did in theorem 2.3 [11]. □

Theorem 3.4. *Let $\phi : M^m \to N^n$ be a harmonic morphism. If λ_ϕ is invariant along fibers and the horizental distribution is integral, then any connected component L of regular fibers has the following properties:*
(1) *Normal bundle of L is flat;*
(2) *L has parallel mean curvature.*

Proof. (1)Let C_ϕ be the critical set of ϕ and $M^* = M \backslash C_\phi$, then $\phi : M^* \to N$ is a horizontally conformal submersion. From proposition 3.3,we know that normal bundle of L is flat .

(2)Let η be the mean curvature vector field of L in M. Let $\{e_i\}_{1 \le i \le m-n}$ be a local orthonormal frame field on L and $\{e_\alpha\}_{m-n+1 \le \alpha \le m}$ be a parallel normal frame field of L in M. From the result of Eells and Lemaire[4], we know that

$$(n-2)H\nabla \log e(\phi) + 2(m-n)\eta = 0.$$

So $\eta = \frac{n-2}{2(m-n)} \Sigma_\alpha (\nabla_{e_\alpha} \log e(\phi)) e_\alpha$. On the other hand, we have

$$\nabla_{e_i} \nabla_{e_\alpha} \log e(\phi) = \nabla_{e_\alpha} \nabla_{e_i} \log e(\phi) + [e_i, e_\alpha] \log e(\phi) = 0,$$

which is from the fact that $e(\phi) = n/2\lambda_\phi^2$ is invariant along fibers and $[e_i, e_\alpha]$ is vertical by lemma 3.1. So $\nabla_{e_\alpha} \log e(\phi)$ is constant on fibers. Because e_α is ∇-parallel, L has parallel mean curvature vector field. □

4. Proof of the main results

Proof. (Theorem 1.3) By Proposition 2.6 in [7], we know that λ is constant and horizontal distribution is integrable. By theorem 3.4, we know that normal bundle of connected component L of any fiber is flat. So we can choose a local orthonormal frame field $\{e_i\}_{1 \le i \le m-n}$ on L and a parallel normal frame field $\{e_\alpha\}_{m-n+1 \le \alpha \le m}$ of L in M such that e_α is $\nabla^\perp$-parallel and e_i is principal direction. Let $\{\omega^A\}_{1 \le A \le m}$ be a dual frame fields and ω_B^A be connection forms. So we have $\omega_i^\alpha = \lambda_i^\alpha \omega_i$.

For any fixed $p \in L$, choose open neibourhood V of p in L such that there is a small constant t_0. When $x \in V$ and $0 \le t_\alpha < t_0$ for any $m-n+1 \le \alpha \le m$, $x^* = x + \Sigma_\alpha t_\alpha e_\alpha \in U$. Define $U(p) = \{x^* = x + \Sigma_\alpha t_\alpha e_\alpha \mid x \in V, 0 \le t_\alpha < t_0\}$, then $U(p)$ is a open neighborhood of p in U. Consider the map

$$x^* : V \to R^m$$
$$x^* = x + t_\alpha e_\alpha$$

for $\le t_\alpha < t_0$. So

$$dx^* = dx + t_\alpha de_\alpha = \Sigma_i (1 - t_\alpha \lambda_i^\alpha) \omega^i e_i. \tag{4.1}$$

Thus we may choose the following local frame on $V^* = x^*(V) \subset U$: $e_\alpha^* = e_\alpha$, $e_i^* = e_i$, $\omega^{i*} = (1 - t_\alpha \lambda_i^\alpha)\omega^i$. Then

$$\omega_\beta^{\alpha *} = de_\alpha^* \cdot e_\beta^* = de_\alpha \cdot e_\beta = \omega_\beta^\alpha = 0,$$
$$\omega_\alpha^{i*} = de_i^* \cdot e_\alpha^* = de_i \cdot e_\alpha = \lambda_i^\alpha \omega_i = \frac{\lambda_i^\alpha}{1 - t_\alpha \lambda_i^\alpha} \omega^{i*}.$$

So on V^* principal curvatures are

$$\lambda_i^\alpha(x^*) = \frac{\lambda_i^\alpha(x)}{1 - t_\alpha \lambda_i^\alpha(x)}. \tag{4.2}$$

But by proposition 2.6 in [7] and lemma 3.2, we know that leaves of the horizontal distribution are flat and totally geodesic. So from (4.1), we know that $V^* = x^*(V)$

is an open neighborhood of a fiber for small t_α. So by lemma 2.1, V^* is minimal, i.e., its mean cuevature

$$\Sigma_i \frac{\lambda_i^\alpha(x)}{1 - t_\alpha \lambda_i^\alpha(x)} = 0.$$

Now taking derivation for t_α, we have

$$\Sigma_i \frac{(\lambda_i^\alpha(x))^2}{(1 - t_\alpha \lambda_i^\alpha(x))^2} = 0.$$

So $\lambda_i^\alpha(x) = 0$ for any $x \in V$ and for $m - n + 1 \leq \alpha \leq m$, $1 \leq i \leq m - n$. Thus fiber L is totally geodesic. □

Proof. (Corollary 1.4) (1) has been proven in [5].

(2)Certainly, When ϕ is the restriction of an orthogonal projection $P : C^m :\rightarrow C^n$ followed by a homothety, ϕ is harmonic morphism.

Now let $\phi : U \rightarrow C^n$ be a holomorphic map. Then for any $q \in C^n$, $\phi^{-1}(q)$ is complex submanifold of U and thus is Kaehler submanifold. So $\phi^{-1}(q)$ is a minimal submanifold. By lemma 2.1, we know that ϕ is horizontally homothetic. So we can easily prove corollary by using theorem 1.3. □

Remark 4.1. *Corollary 1.4 as problem is first raised by J.C.Wood[8].*

Corollary 4.2. *Let U be an open and connected subset of R^{2m} and J be an almost Hermitian structure. Let $\phi : U \rightarrow C^n (n > 1)$ be a holomorphic harmonic morphism with respect to J. If J is invariant along fibers, then ϕ is the restriction of an orthogonal projection $P : R^{2m} \rightarrow C^n$, followed by a homothety.*

Proof. For any $q \in C^n$, $\phi^{-1}(q)$ is complex submanifold of U with respect to the complex structure J. Because we assume that J is invariant along fiber $\phi^{-1}(q)$, so $\phi^{-1}(q)$ is superminimal in U[3]. But J is an almost Hermitian structure, we can get $\phi^{-1}(q)$ is a minimal submanifold. Now the next proof is the same as corollary 1.4. □

Remark 4.3. *When $n = 1$, P.Baird and J.C.Wood have constructed non-trival examples which satisfy conditions of corollary 4.2.*

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