On the Geometry of Certain Harmonic Two-Spheres in Complex Grassmannians.

SIGMUNDUR GUDMUNDSSON (*)

Sunto. – In questo articolo ci si interessa alla geometria di \( \tilde{S}(S^2) \), essendo \( \tilde{S}: S^2 \to G_{k,n}(\mathbb{C}) \) mappe armoniche costruite a partire dalla successione di Gauss di una funzione olomorfa \( f: S^2 \to \mathbb{C}P^{n-1} \). Si calcola la metrica indotta da \( ds_2^2 \) sfruttando l'immersione naturale \( G_{k,n}(\mathbb{C}) \to \text{Sym}(\mathbb{C}^n) \). Si dimostra poi una proprietà di simmetria della metrica \( ds_2^2 \), e quindi anche della \( ds_4^2 \). Ciò consente di fornire esempi in cui si ha \( \tilde{S}(S^2) \subset G_{k,n} \), con \( k > 1 \) e con curvatura costante, ed altri con simmetrie di rotazione e di riflessione.

0. – Introduction.

The last years have been very fruitful for the research on harmonic maps between Riemannian manifolds. J. Eells and L. Lemaire have written three very useful survey articles on this subject [7], [8], [9]. One of the problems being dealt with, is the classification of harmonic maps from the two-sphere into homogeneous spaces. Answers have been given in some cases, for example if the homogeneous space is a complex Grassmannian. This case is of special interest for this paper, and the development can be traced in [6], [3], [10], [11], [15], [1], [5], [4], [16]. For an explicit construction of all harmonic maps in this case, see [17] and [18]. The geometry of harmonic spheres in complex projective spaces has been studied in [2], where some very nice results have been achieved. All the harmonic two-spheres with constant curvature are classified. They are up to a holomorphic isometry of \( \mathbb{C}P^{n-1} \), elements of the Gauß-sequence of the so called Veronese-map.

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1. Preliminaries.

For $k, n \in \mathbb{N}$ with $0 < k < n$, the complex Grassmannians $G_{k,n}(\mathbb{C})$ are defined by

$$G_{k,n}(\mathbb{C}) := \{ V \subset \mathbb{C}^n \mid V \text{ is a } k\text{-dimensional subspace of } \mathbb{C}^n \}.$$  

Let $\mathbb{C}^{n \times k}$ be the vector space of complex $n \times k$ matrices with the Euclidean scalar product $\langle A, B \rangle := \text{Re} \, \text{trace } A^t B$, and

$$GL_{k,n}(\mathbb{C}) := \{ A \in \mathbb{C}^{n \times k} \mid \det A^t A \neq 0 \} = \{ A \in \mathbb{C}^{n \times k} \mid \text{rank } A = k \}.$$  

We call $\cong: GL_{k,n}(\mathbb{C}) \to G_{k,n}(\mathbb{C})$ with

$$\cong: A = (a_1, \ldots, a_k) \mapsto \text{span} \{a_1, \ldots, a_k\}$$

the projection of $GL_{k,n}(\mathbb{C})$ onto $G_{k,n}(\mathbb{C})$ and $A \in GL_{k,n}(\mathbb{C})$ homogeneous coordinates for $\cong(A) \in G_{k,n}(\mathbb{C})$. If

$$U_{k,n} := \{ A \in GL_{k,n}(\mathbb{C}) \mid A^t A = I \} \quad \text{and} \quad \cong: U_{k,n} \to G_{k,n}(\mathbb{C})$$

the restriction of $\cong$ to $U_{k,n}$, then we get the following commutative diagram.

$$\begin{array}{ccc}
GL_{k,n}(\mathbb{C}) & \xrightarrow{\cong} & U_{k,n} \\
\uparrow & \Downarrow \cong & \downarrow \pi \\
G_{k,n}(\mathbb{C}) & \xrightarrow{\pi} & G_{k,n}(\mathbb{C})
\end{array}$$

where $\Omega: GL_{k,n}(\mathbb{C}) \to U_{k,n}$ is the Gram-Schmidt orthonormalisation process for the row vectors of $A \in GL_{k,n}(\mathbb{C})$. The topology on $G_{k,n}(\mathbb{C})$ is the quotient topology defined by $\cong$ and its Fubini-Study metric is uniquely defined, by demanding that $\cong$ is a Riemannian submersion.

Every holomorphic map $\tilde{\phi}: S^2 \to CP^{n-1}$ can be given by homogeneous coordinates $[f_1, \ldots, f_n]$, where $f_i$ are polynomials in $z \in \mathbb{C}$. For a map $\phi: \mathbb{C} \to \mathbb{C}^n - \{0\}$ with isolated singularities we define $G'(\phi), G''(\phi): \mathbb{C} \to \mathbb{C}^n$ by

\begin{align*}
(1) \quad G'(\phi) &= \frac{\partial \phi}{\partial z} - \frac{\langle \partial \phi / \partial z, \phi \rangle}{\langle \phi, \phi \rangle} \cdot \phi, \\
(2) \quad G''(\phi) &= \frac{\partial \phi}{\partial \bar{z}} - \frac{\langle \partial \phi / \partial \bar{z}, \phi \rangle}{\langle \phi, \phi \rangle} \cdot \phi,
\end{align*}
where $\langle , \rangle$ is the usual hermitian scalar product of $\mathbb{C}^n$. $G^r(\phi)$ (respectively $G^r(\phi)$) is called the $\partial'^r$-Gauß-map (respectively $\partial'^r$-Gauß-map) of $\phi$. We further define by induction $G^0(\phi) := \phi$, $G^i(\phi) := G^r(G^{i-1}(\phi))$ and $G^{-i}(\phi) := G^r(G^{-i+1}(\phi))$, for $i \geq 1$. It is obvious from equations (1) and (2) that if $f$ are polynomial homogeneous coordinates of a holomorphic map $\tilde{f}: S^2 \rightarrow \mathbb{C}P^{n-1}$ then the coordinates of $G^r(f)$ are rational functions in $z, \bar{z} \in \mathbb{C}$. If $G^r(f)$ is non zero, then $(\tilde{f}, r)$ defines a map $G^r(\tilde{f}): S^2 \rightarrow \mathbb{C}P^{n-1}$ with homogeneous coordinates $G^r(f)$. These maps are harmonic (see [5] and [4]).

**Definition 1.1.** A map $\tilde{\phi}: S^2 \rightarrow \mathbb{C}P^{n-1}$ is called full if

$$\tilde{\phi}(S^2) \subset \mathbb{C}P^{m-1} \subset \mathbb{C}P^{n-1} \Rightarrow m = n,$$

i.e. $\tilde{\phi}(S^2)$ does not lie in a proper projective subspace of $\mathbb{C}P^{n-1}$.

Harmonic maps $\tilde{\phi}: S^2 \rightarrow \mathbb{C}P^{n-1}$ have been classified by the following theorem [10] (see also [6], [12] and [3]).

**Theorem 1.2.** If

$$H := \{ (\tilde{f}, r)|\tilde{f}: S^2 \rightarrow \mathbb{C}P^{n-1}, \text{ full and holomorphic, and } r \in \mathbb{N}_0 \text{ with } 0 \leq r \leq n-1 \}$$

and

$$K := \{ \tilde{\phi}: S^2 \rightarrow \mathbb{C}P^{n-1} | \tilde{\phi} \text{ full and harmonic} \},$$

then $g: H \rightarrow K$ given by $g: (\tilde{f}, r) \mapsto G^r(\tilde{f})$ is a bijection.

Every harmonic map is therefore an iterative $\partial'^r$-Gauß-map of a holomorphic map.

**Definition 1.3.** If $\tilde{\phi}: S^2 \rightarrow \mathbb{C}P^{n-1}$ is a full holomorphic map, then we call $\{ \tilde{f}, G^1(\tilde{f}), …, G^{n-1}(\tilde{f}) \}$ the Gauß-sequence of $\tilde{f}$.

It can be shown that for the homogeneous coordinates of the elements of the Gauß-sequence of $\tilde{f}$, we have $G^r(\tilde{f}) \perp G^s(\tilde{f}) \forall r \neq s$.

If $\tilde{\phi}: S^2 \rightarrow \mathbb{C}P^{n-1}$ is a full holomorphic map, and we define $\phi: \mathbb{C} \rightarrow GL_{h, n}(\mathbb{C})$ by $\phi: z \mapsto [G^{s_1}(\phi)(z), ..., G^{s_k}(\phi)(z)]$, with $s_i \in \{0, ..., n-1\}$ and $s_i \neq s_j$ if $i \neq j$, then $\phi$ defines a harmonic map $\tilde{\phi}: S^2 \rightarrow G_{h, n}(\mathbb{C})$ [11], [15], [13]. We denote the set of all such maps by $h(\tilde{\phi})$. $\tilde{\phi}$ is holomorphic if and only if $\{s_1, ..., s_k\} = \{0, ..., k-1\}$ and antiholomorphic if and only if $\{s_1, ..., s_k\} = \{n-k, ..., n-1\}$.

The main task of this paper is to describe the geometry of
\( \tilde{\varphi}(S^2) \subset G_{k,n}(C) \) where \( \tilde{\varphi} \in h(\tilde{f}) \). We remind the reader of the fact that every non-constant harmonic map \( \tilde{\varphi}: S^2 \to N \) into a Riemannian manifold is weakly conformal.

2. – Metrics and curvatures.

We now want to calculate the pullback metric \( ds^2_{\tilde{\varphi}} \) of \( S^2 \) via \( \tilde{\varphi} \). This has been done in [2] for the special case of \( k = 1 \). For \( k > 1 \) the calculations in \( G_{k,n}(C) \) become much more complicated, so we use an isometric embedding of \( G_{k,n}(C) \) into \( \text{Sym}(C^n) \cong \mathbb{R}^{n^2} \), to make things easier.

**Definition 2.1.** – Let \( L(C^n) \) be the set of complex linear endomorphisms of \( C^n \), with the real scalar product

\[
\langle A, B \rangle_L := \frac{1}{2} \cdot \Re \text{trace} (A \cdot \overline{B}^t) = \frac{1}{2} \cdot \Re \sum_{q \in \text{ONB}} \langle A(q), B(q) \rangle,
\]

where \( \text{ONB} \) is an orthonormal basis of \( C^n \).

The set of symmetric, complex linear endomorphisms

\[
\text{Sym}(C^n) := \{ A \in L(C^n) \mid \langle A \cdot x, y \rangle = \langle x, A \cdot y \rangle \ \forall x, y \in C^n \},
\]

\[
= \{ A \in L(C^n) \mid A = \overline{A}^t \},
\]

is a \( n^2 \)-dimensional real subspace of \( L(C^n) \). We define \( \tilde{E}: U_{k,n} \to \text{Sym}(C^n) \) by

\[
\tilde{E}: A = (a_1, \ldots, a_k) \mapsto \pi_A,
\]

where \( \pi_A \) is the orthogonal projection onto the span \( \{a_1, \ldots, a_k\} \).

\( \pi_A \) is then defined by

\[
\pi_A(q) = \sum_{i=1}^k \langle q, a_i \rangle a_i \quad \forall q \in C^n.
\]

If \( A, B \in U_{k,n} \) lie on the same fibre of \( \pi \), then \( \pi_A = \pi_B \), so \( \tilde{E} \) induces a mapping \( \tilde{E}: G_{k,n}(C) \to \text{Sym}(C^n) \). If \( \tilde{\varphi}: S^2 \to G_{k,n}(C) \) is harmonic and \( \varphi: W \to GL_{k,n}(C) \) with \( \varphi = [\varphi_1, \ldots, \varphi_k] \) are local homogeneous coordinates of \( \varphi \), then we get the following commutative diagram

\[
\begin{array}{cccccc}
W & \xrightarrow{\varphi} & GL_{k,n}(C) & \xrightarrow{\alpha} & U_{k,n} & \xrightarrow{\tilde{E}} & \text{Sym}(C^n) \\
\downarrow & & \downarrow \pi & & \downarrow \pi & & \\
S^2 & \xrightarrow{\tilde{\varphi}} & G_{k,n}(C) & \xrightarrow{\pi} & G_{k,n}(C) & \xrightarrow{E} & \text{Sym}(C^n)
\end{array}
\]
LEMMA 2.2.

i) $E$ is an isometric embedding,

ii) $E(G_{k,n}(C))$ maps $G_{k,n}(C)$ into a sphere of $\text{Sym}(C)$ with radius $\sqrt{k/2}$.

PROOF. – See [14] and [13]. □

For $r \in \{-1, 0, ..., n-2\}$ let $F_r, H_r \in L(C^n)$ be given by

$$F_{-1} := H_{-1} := 0,$$

$$F_r(q) := \frac{\langle q, G^r \rangle}{\langle G^r, G^r \rangle} \cdot G^{r+1} \quad \text{and} \quad H_r(q) := \frac{\langle q, G^{r+1} \rangle}{\langle G^r, G^r \rangle} \cdot G^r,$$

$\forall q \in C^n$ and $r \in \{0, ..., n-2\}$, with $G^r := G^r(f)$. Then $F_r + H_r, i(F_r - H_r) \in \text{Sym}(C^n)$, and simple calculations show that

$$\langle F_r, F_p \rangle_L = \langle H_r, H_p \rangle_L = \frac{1}{2} \cdot \varepsilon_{rp} \cdot \frac{|G^{r+1}|^2}{|G^r|^2}$$

and

$$\langle F_r, H_p \rangle_L = 0, \quad \forall r, p \in \{0, 1, ..., n-2\}.$$

$B := \{F_r + H_r, i(F_r - H_r) \mid r \in \{0, 1, ..., n-2\}\}$ is therefore an orthogonal basis for a $2(n-1)$-dimensional subspace of $\text{Sym}(C^n)$, i.e. of $T_0 E(CP^{n-1}) \subset T_0 \text{Sym}(C^n)$ with $\bar{\phi} = E \circ \bar{\phi}$. For $\sigma = \{s_1, ..., s_k\} \subset \{0, 1, ..., n-1\}$, where $s_i \neq s_j$ for $i \neq j$, and $\phi_i = G^{s_i}$, let

$$\phi_i^* := \frac{\langle q, G^{s_i} \rangle}{\langle G^{s_i}, G^{s_i} \rangle} \cdot G^{s_i} \in \text{Sym}(C^n), \quad \forall i \in \{1, ..., k\}.$$

For $\phi^* : W \rightarrow \text{Sym}(C^n)$ defined by $\phi^* = \bar{E} \circ \Omega \circ \phi$, we obviously have

$$\phi^* = \sum_{i=1}^k \phi_i^*$$

and therefore

$$\frac{\partial \phi^*}{\partial z} = \sum_{i=1}^k \frac{\partial \phi_i^*}{\partial z} \quad \text{and} \quad \frac{\partial \phi^*}{\partial \bar{z}} = \sum_{i=1}^k \frac{\partial \phi_i^*}{\partial \bar{z}}.$$
For \( r \in \{-1, 0, 1, \ldots, n - 1\} \) let
\[
\lambda^2_r := \begin{cases} 0 & \text{for } r \in \{-1, n - 1\}, \\ \frac{|G^{r+1}|^2}{|G^r|^2} & \text{otherwise}. \end{cases}
\]

**Lemma 2.3.** – For the operators \( G', G'' \), and \( r \in \{1, 2, \ldots, n\} \) we have
\[
G'' \circ G' (G^{-1}) = G'' (G^r) = -\lambda^2_{r-1} \cdot G^{-1}.
\]

**Lemma 2.4.** – If \( r \in \{0, 1, \ldots, n-1\} \) then
\[
\Delta \log |G^r|^2 = 4 \cdot (\lambda^2_r - \lambda^2_{r-1}).
\]

Lemmas 2.3 and 2.4 are stated in [2] without proofs, for proofs see [13].

By simple calculations and the use of Lemma 2.3 we get
\[
\frac{\partial \phi^*}{\partial z} = F_s - F_{s-1} \quad \text{and} \quad \frac{\partial \phi^*}{\partial \bar{z}} = H_s - H_{s-1}.
\]

Since \( \tilde{\phi} : S^2 \to G_{k,n}(\mathbb{C}) \) is harmonic and hence weakly conformal, the induced metric \( ds^2_{\tilde{\phi}} \) is of the form
\[
ds^2_{\tilde{\phi}} = \lambda^2_{\tilde{\phi}} \, dz \otimes d\bar{z}
\]
and by the classical theory of surfaces
\[
\lambda^2_{\tilde{\phi}} = \left\| \frac{\partial \phi^*}{\partial x} \right\|^2_L = \left\| \frac{\partial \phi^*}{\partial y} \right\|^2_L = \left\| \frac{\partial \phi^*}{\partial z} \right\|^2_L = \left\| \frac{\partial \phi^*}{\partial \bar{z}} \right\|^2_L = 2 \left\| \frac{\partial \phi^*}{\partial z} \right\|^2_L = 2 \left\| \phi^* \right\|^2_L = 2 \left\| \phi^* \right\|^2_L = \left\| \sum_{i=1}^{k} F_s - F_{s-1} \right\|^2_L = 2 \left\| \sum_{i=1}^{k} F_s - F_{s-1} \right\|^2_L = 2 \left[ \sum_{i=1}^{k} \left( \|F_s\|^2_L^2 + \|F_s - F_{s-1}\|^2_L \right) - 2 \sum_{i,j=1}^{k} \left\langle F_s, F_{s-1} \right\rangle \right] = \sum_{i=1}^{k} \lambda^2_{s_i} + \lambda^2_{s_{i-1}} - 2 \sum_{i,j=1}^{k} \phi_{s_i, s_{i-1}} \lambda^2_{s_i}.\]
If we look at the Gauß-sequence of the holomorphic \( \bar{f} \) as a connected chain

\[
\bar{f} \leftrightarrow G^1(\bar{f}) \ldots \leftrightarrow G^{n-1}(\bar{f}) ,
\]
choose \( \sigma = \{ s_1, \ldots, s_k \} \subset \{ 0, \ldots, n-1 \} \) with \( s_i \neq s_j \) if \( i \neq j \), and break the links corresponding to the complement of \( \sigma \), then the chain falls apart into \( m \) pieces,

\[
K_t : G^{i_t+1}(\bar{f}) \leftrightarrow \ldots \leftrightarrow G^{i_t+n}(\bar{f}) \quad \text{for } t = 1, \ldots, m.
\]

We call these pieces the connected components of the chain corresponding to \( \bar{\varphi} \in h(\bar{f}) \) defined by \( \varphi = [G^{s_t}(f), \ldots, G^{s_r}(f)] \).

If \( G^{h_t}(\bar{f}), G^{s_t}(\bar{f}) \) belong to different connected components, then \( \delta_{i_t, s_{t-1}} \) vanishes and from the last formula we get,

\[
\lambda^2_\varphi = \sum_{t=1}^m \lambda^2_{K_t},
\]

with

\[
\lambda^2_{K_t} = \sum_{i=1}^{r_t} (\lambda^2_{i_t+i} + \lambda^2_{i_t+i-1}) - 2 \sum_{i,j=1}^{r_t} \delta_{(i_t+i, i_t+j-1)} \lambda^2_{i_t+i} =
\]

\[
= \sum_{i=1}^{r_t} \lambda^2_{i_t+i} + \sum_{i=0}^{r_t-1} \lambda^2_{i_t+i} - 2 \sum_{i=1}^{r_t-1} \lambda^2_{i_t+i} = \lambda^2_{i_t} + \lambda^2_{i_t+n_t},
\]

so

\[
(3) \quad \lambda^2_\varphi = \sum_{t=1}^m (\lambda^2_{i_t} + \lambda^2_{i_t+n_t}).
\]

With this formula it is easily seen that \( \lambda^2_\varphi = \lambda^2_\varphi \), but this is already known since

\( \perp : G_{k,n}^n(C) \to G_{n-k,n}^n(C) \) defined by \( \perp : V \mapsto V^\perp \),

is an isometry. This serves as a check for our calculations. For \( k = 1 \) we get \( \lambda^2_{\varphi^+} = \lambda^2_{\varphi^-} + \lambda^2_{\varphi^{-1}} \), as stated in [2]. If \( \bar{\varphi} : S^2 \to G_{k,n}(C) \) is holomorphic i.e. \( m = 1 \), \( l_i = -1 \), the metric is very simple \( \lambda^2_\varphi = \lambda^2_{\varphi^{-1}} \).

Since the metric is conformal with conformal factor \( \lambda^2_\varphi \), we have the following formula for the curvature

\[
K_\varphi = -\frac{\Delta \log \lambda^2_\varphi}{2\lambda^2_\varphi}.
\]
In the case that $\tilde{\varphi}$ is holomorphic, we get by $\lambda_2^2 = \lambda_{n-1}^2$ and the use of Lemma 2.4

$$K_\varphi = 4 - 2 \cdot \frac{\lambda_{n-1}^2 + \lambda_{n-2}^2}{\lambda_{n-1}^2}.$$ 

3. – Symmetries.

We are now interested in symmetries $p: S^2 \to S^2$ w.r.t. the pullback metric $ds^2_{\varphi}$. It is easy to see that if $p$ is a conformal map, then the two following conditions are equivalent.

i) $p$ is an isometry w.r.t. $ds^2_{\varphi}$,

ii) for every local coordinate $z: W \to \mathbb{C}$ we have

$$|\det dp(z)|^2 \cdot \lambda^2_{\varphi} \circ p(z) = \lambda^2_{\varphi}(z).$$

If $U \in U(n)$ and $U^*: G_{k,n}(\mathbb{C}) \to G_{k,n}(\mathbb{C})$ is defined by

$$U^*: \text{span}\{a_1, \ldots, a_k\} \mapsto \text{span}\{U(a_1), \ldots, U(a_k)\},$$

then $U^*$ is an isometry. If $p: S^2 \to S^2$ is a map such that the following diagram commutes

$$\begin{array}{ccc}
S^2 & \xrightarrow{p} & S^2 \\
\tilde{\varphi} \downarrow & & \downarrow \tilde{\varphi} \\
G_{k,n}(\mathbb{C}) & \xrightarrow{U^*} & G_{k,n}(\mathbb{C})
\end{array},$$

then is $p$ obviously an isometry w.r.t. $ds^2_{\varphi}$. For $k = 1$, $\tilde{\varphi} = \tilde{f}: S^2 \to \mathbb{C}P^{n-1}$ holomorphic and the above diagram there exists a open neighbourhood $W = S^2 \setminus \{x_1, \ldots, x_l\}$ and homogeneous coordinates $f: W \to \mathbb{C}^n \setminus \{0\}$, such that

$$f(p(z)) = q(z) \cdot U \cdot f(z) \quad \text{and} \quad q(z) \neq 0 \ \forall z \in W.$$

This equation is then true for all except finite number of points of $S^2$. As an isometry $p$ can not have branched points, so we have $p'(z) \neq 0 \ \forall z \in W$.

**Lemma 3.1.** – For the above situation and $z \in W$ we have:

(4)  \[ G^r(p(z)) = \frac{q(z)}{(p'(z))^r} \cdot U \cdot G^r(z) \quad \forall r \in \{0, 1, \ldots, n-1\}, \]
where $G^r$ are homogeneous coordinates for the elements of the Gauß-sequence of $\tilde{f}$.

**Proof.**

i) For $r = 0$ the statement is simply the assumption.

ii) We now assume that the statement is true for $r$ and show that it is then also true for $r + 1$. If we define

$$\alpha(z) := \frac{q(z)}{[p'(z)]^{r+1}}$$

and

$$\beta(z) := \frac{q'(z)[p'(z)]^r - q(z)[p'(z)^r]'}{[p'(z)]^{2r+1}},$$

then differentiating (4) gives

$$\frac{\partial G^r}{\partial z}(p(z)) = \beta(z) \cdot U \cdot G^r(z) + \alpha(z) \cdot U \cdot \frac{\partial G^r}{\partial z}(z).$$

By the Definition of $G^{r+1}(p(z))$ we have

$$G^{r+1}(p(z)) = \frac{\partial G^r}{\partial z}(p(z)) - \frac{\left< \frac{\partial G^r}{\partial z}(p(z)), G^r(p(z)) \right>}{\left< G^r(p(z)), G^r(p(z)) \right>} \cdot G^r(p(z)).$$

If we put (4) and (5) into (6), and simplify we get

$$G^{r+1}(p(z)) = \alpha(z) \cdot U \left[ \frac{\partial G^r}{\partial z}(z) - \frac{\left< \frac{\partial G^r}{\partial z}(z), G^r(z) \right>}{\left< G^r(z), G^r(z) \right>} \cdot G^r(z) \right] =$$

$$= \frac{q(z)}{[p'(z)]^{r+1}} \cdot U \cdot G^{r+1}(z).$$

**Theorem 3.2.** – If $p: S^2 \to S^2$ is a holomorphic isometry w.r.t. $ds^2_f$ and there exists a $U \in U(n)$, such that $\tilde{f} \circ p = U^* \circ \tilde{f}$, then $p$ is also an isometry w.r.t. $ds^2_\tilde{f}$ for all $\vec{\tilde{f}} \in h(\tilde{f})$. 


PROOF. – Since $p$ is holomorphic we get $|\det [dp]|^2 = |p'|^2$ and it follows from Lemma 3.1 that

$$\lambda^2_r(p(z)) = \frac{|G^{r+1}(p(z))|^2}{|G^r(p(z))|^2} = \frac{1}{|p'(z)|^2} \cdot \frac{|G^{r+1}(z)|^2}{|G^r(z)|^2} = \frac{1}{|\det[dp(z)]|^2} \cdot \lambda^2_r(z),$$

for $r \in \{0, 1, \ldots, n - 2\}$, and all except finite number of points $z \in S^2$. Formula (3) now shows that

$$\lambda^2_\infty(p(z)) = \frac{1}{|\det[dp(z)]|^2} \cdot \lambda^2_\infty(z),$$

so $p$ is an isometry w.r.t. $ds^2_\infty$ on $W$ and hence on the whole of $S^2$. ■

COROLLARY 3.3. – If $\bar{f}: S^2 \to CP^{n-1}$ is the Veronese-map given by

$$f(z) := \left[ \sqrt{\binom{n-1}{0}} \cdot z^0, \sqrt{\binom{n-1}{1}} \cdot z^1, \ldots, \sqrt{\binom{n-1}{n-1}} \cdot z^{n-1} \right]$$

and $\check{\varphi} \in h(\bar{f})$, then $\check{\varphi}(S^2)$ has constant curvature.

PROOF. – It is a known result from [2], that $(S^2, ds^2_\infty)$ has constant curvature. The elements of

$$R := \{ p_{z_0}: (S^2, ds^2_\infty) \to (S^2, ds^2_\infty) | p_{z_0} \text{ is the rotation about } z_0 \in S^2 \}$$

are therefore holomorphic isomorphisms of $(S^2, ds^2_\infty)$, and hence for $(S^2, ds^2_\infty)$. $R$ acts transitively on $S^2$, so $(S^2, ds^2_\infty)$ has constant curvature. ■

4. – Symmetric examples.

If $a_i \in \mathbb{C}^*$ and $a_i = a_{n-i+1}$ for $i \leq n/2$, then $\bar{f}: S^2 \to CP^{n-1}$, defined by

$$\bar{f}(z) = \begin{cases} [0, \ldots, 0, 1] & \text{for } z = \infty, \\ [a_0, a_1 \cdot z, \ldots, a_{n-1} \cdot z^{n-1}] & \text{otherwise}, \end{cases}$$
is a full holomorphic map. We now have

\[ f(e^{i\theta} \cdot z) = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & e^{i\theta} & 0 & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & e^{i(n-2)\theta} & 0 & \vdots \\
0 & 0 & \cdots & 0 & e^{i(n-1)\theta}
\end{pmatrix} \cdot f(z) \]

and

\[ f\left(\frac{1}{z}\right) = \frac{1}{z^{n-1}} \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 1 & 0 & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 1 & 0 & 0 & \vdots \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix} \cdot f(z). \]

This means, that \( r: S^2 \rightarrow S^2 \) defined by \( r(z) = e^{i\theta} \cdot z \) and \( i: S^2 \rightarrow S^2 \) with \( i: z \mapsto z^{-1} \) are isometries of \((S^2, ds_0^2)\). From the last theorem they are also isometries of \((S^2, ds_0^2)\) \( \forall \phi \in h(f) \). From the rotational symmetry follows, that the reflection about lines through the origin are isometries, so the reflection about the unit-circle \( S^1, R: S^2 \rightarrow S^2 \) given by \( R: z \mapsto \bar{z}^{-1} \) is one too.

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**REFERENCES**


Department of Pure Mathematics, University of Leeds
Leeds LS2 9JT, England

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