Harmonic Morphisms: Do They Exist?

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Cagliari - 17 February 2016
Outline

1. Classical Problems
   - Minimal Surfaces in $\mathbb{R}^3$
   - Harmonic Morphisms in $\mathbb{R}^3$
   - Holomorphic Functions in Several Variables
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   - Minimal Surfaces in $\mathbb{R}^3$
   - Harmonic Morphisms in $\mathbb{R}^3$
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2 Harmonic Morphisms
   - Basics
   - Geometric Motivation
   - Homogeneous Spaces
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   - Definition
   - A Useful Machine
   - The Classical Semisimple Lie Groups
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Definition 1.1 (minimal surface)

Let $\Sigma_t$ with $t \in (-\epsilon, \epsilon)$ be a family of surfaces with a common boundary curve i.e. $\partial \Sigma_0 = \partial \Sigma_t$ for all $t \in (-\epsilon, \epsilon)$. Then $\Sigma_0$ is said to be a **minimal surface** if it is a critical point for the area functional i.e.

$$\left. \frac{d}{dt} \left( \int_{\Sigma_t} dA \right) \right|_{t=0} = 0.$$
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$$\frac{d}{dt} \left( \int_{\Sigma_t} dA \right)|_{t=0} = 0.$$  

Theorem 1.2 (zero mean curvature)

*A surface $\Sigma$ in $\mathbb{R}^3$ is minimal if and only if its mean curvature vanishes i.e.*

$$H = \frac{k_1 + k_2}{2} \equiv 0.$$
There is an interesting connection between the theory of minimal surfaces, harmonic functions and hence complex analysis.

**Theorem 1.3 (the Weierstrass representation - 1866)**

*Every minimal surface $\Sigma$ in $\mathbb{R}^3$ can locally be parametrized by a conformal and harmonic map $\phi : W \subset \mathbb{C} \rightarrow \mathbb{R}^3$ of the form*

$$
\phi : z \mapsto \text{Re} \int_{z_0}^{z} f(w) \left[ 1 - g^2(w), i(1 + g^2(w)), 2g(w) \right] dw,
$$

*where $f, g : W \subset \mathbb{C} \rightarrow \mathbb{C}$ are two holomorphic functions.*

$$
\Delta \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,
$$

$$
\left( \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rangle = 0 \text{ and } |\frac{\partial \phi}{\partial x}|^2 = |\frac{\partial \phi}{\partial y}|^2 \right) \Leftrightarrow \left( \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial x} + i \frac{\partial \phi}{\partial y} \right) = 0.
$$
Definition 1.4 (harmonic morphism)

The map \( \phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C} \) is said to be a \textbf{harmonic morphism} if the composition \( f \circ \phi \) with any \textbf{holomorphic} function \( f : W \subset \mathbb{C} \to \mathbb{C} \) is \textbf{harmonic}. 

\[ \Delta (f \circ \phi) = \frac{\partial f}{\partial z} \cdot \Delta \phi + \frac{\partial^2 f}{\partial z^2} \cdot \langle \nabla \phi, \nabla \phi \rangle = 0. \]
**Definition 1.4 (harmonic morphism)**

The map $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is said to be a **harmonic morphism** if the composition $f \circ \phi$ with any **holomorphic** function $f : W \subset \mathbb{C} \to \mathbb{C}$ is harmonic.

**Theorem 1.5 (Jacobi 1848)**

The map $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is a harmonic morphism if and only if it is harmonic and horizontally (weakly) conformal i.e.

$$\Delta u = \Delta v = 0, \quad \left( \langle \nabla u, \nabla v \rangle = 0 \text{ and } |\nabla u|^2 = |\nabla v|^2 \right).$$
Definition 1.4 (harmonic morphism)

The map $\phi = u + iv : U \subset \mathbb{R}^3 \to \mathbb{C}$ is said to be a harmonic morphism if the composition $f \circ \phi$ with any holomorphic function $f : W \subset \mathbb{C} \to \mathbb{C}$ is harmonic.

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Proof.

$$\Delta (f \circ \phi) = \frac{\partial f}{\partial z} \cdot \Delta \phi + \frac{\partial^2 f}{\partial z^2} \cdot \langle \nabla \phi, \nabla \phi \rangle = 0.$$
Theorem 1.6 (the Jacobi representation - 1848)

Let \( f, g : W \subset \mathbb{C} \rightarrow \mathbb{C} \) be holomorphic functions, then every local solution \( z : U \subset \mathbb{R}^3 \rightarrow \mathbb{C} \) to the equation

\[
\langle f(z(x))\left[1 - g^2(z(x)), i(1 + g^2(z(x))), 2g(z(x))\right], x \rangle = 1
\]

is a harmonic morphism.
Theorem 1.6 (the Jacobi representation - 1848)

Let $f, g : W \subset \mathbb{C} \to \mathbb{C}$ be **holomorphic** functions, then every local solution $z : U \subset \mathbb{R}^3 \to \mathbb{C}$ to the equation

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$$

is a harmonic morphism.

Theorem 1.7 (Baird, Wood 1988)

Every local harmonic morphism $z : U \to \mathbb{C}$ in the Euclidean $\mathbb{R}^3$ is obtained this way.
Example 1.8 (the outer disc example)

Let \( r \in \mathbb{R}^+ \) and choose \( g(z) = z, f(z) = -1/2irz \) then we obtain

\[
(x_1 - ix_2)z^2 - 2(x_3 + ir)z - (x_1 + ix_2) = 0
\]

with the two solutions

\[
z_r^\pm = \frac{-(x_3 + ir) \pm \sqrt{x_1^2 + x_2^2 + x_3^2 - r^2 + 2irx_3}}{x_1 - ix_2}.
\]
Let $\phi = u + iv : U \to \mathbb{C}$ be a **holomorphic** function defined on an open subset $U$ of $\mathbb{C}^m$. Then it satisfies the Cauchy-Riemann equations i.e. if for each $k = 1, 2, \ldots, m$

$$\frac{\partial u}{\partial x_k} = \frac{\partial v}{\partial y_k} \quad \text{and} \quad \frac{\partial u}{\partial y_k} = -\frac{\partial v}{\partial x_k}.$$ 

As a direct consequence of these equations we get

$$\frac{\partial^2 u}{\partial x_k^2} = \frac{\partial^2 v}{\partial x_k \partial y_k} = -\frac{\partial^2 u}{\partial y_k^2} \quad \text{and} \quad \frac{\partial^2 v}{\partial x_k^2} = -\frac{\partial^2 u}{\partial x_k \partial y_k} = -\frac{\partial^2 v}{\partial y_k^2}.$$
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This implies that the functions $u, v : U \to \mathbb{R}$ are **harmonic**, hence

$$\Delta \phi = \Delta (u + iv) = \Delta u + i\Delta v = 0.$$  

For the two gradients $\nabla u, \nabla v$ we have the following

$$\langle \nabla \phi, \nabla \phi \rangle = \langle \nabla (u + iv), \nabla (u + iv) \rangle = 0.$$  

If $\psi : U \to \mathbb{C}$ is another **holomorphic** function then the polar identity

$$4\langle \nabla \phi, \nabla \psi \rangle = \langle \nabla (\phi + \psi), \nabla (\phi + \psi) \rangle - \langle \nabla (\phi - \psi), \nabla (\phi - \psi) \rangle$$  

gives

$$\langle \nabla \phi, \nabla \psi \rangle = 0.$$
Definition 2.1 (harmonic morphism)

A map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called a harmonic morphism if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.
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Theorem 2.2 (Fuglede 1978, Ishihara 1979)

A map \( \phi : (M, g) \to (N, h) \) between Riemannian manifolds is a harmonic morphism if and only if it is **harmonic and horizontally (weakly) conformal**.
For local coordinates $x$ on $M$ and $y$ on $N$, we have the \textbf{non-linear} system

$$\tau(\phi) = \sum_{i,j=1}^{m} g^{ij}_{\gamma} \left( \frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}^{k}_{ij} \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma^{\gamma}_{\alpha\beta} \circ \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) = 0,$$

where $\phi^\alpha = y^\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.
(harmonicity)

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$$

where $\phi^\alpha = y^\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.

(horizontal conformality)

There exists a continuous function $\lambda : M \to \mathbb{R}_0^+$ such that for all $\alpha, \beta = 1, 2, \ldots, n$

$$
\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial \phi^\alpha}{\partial x_i}(x) \frac{\partial \phi^\beta}{\partial x_j}(x) = \lambda^2(x) h^{\alpha \beta}(\phi(x)).
$$

This is a first order non-linear system of $\left(\binom{n+1}{2} - 1\right)$ equations.
Theorem 2.3 (Baird, Eells 1981)

Let $\phi : (M, g) \to (N^2, h)$ be a horizontally conformal map from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if its fibres are minimal at regular points $\phi$. 
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Let $\phi : (M, g) \to (N^2, h)$ be a horizontally conformal map from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if its fibres are minimal at regular points $\phi$.

The problem is invariant under isometries on $(M, g)$. If the codomain is a surface ($n = 2$) then it is also invariant under conformal changes $\sigma^2 h$ of the metric on $(N^2, h)$. This means, at least for local studies, that $(N^2, h)$ can be chosen to be the standard complex plane $\mathbb{C}$. 
Definition 2.4 (Riemannian homogeneous space)

A Riemannian manifold \((M, g)\) is said to be **homogeneous** if it possesses a transitive group \(G\) of isometries i.e. if for all \(p, q \in M\) there exists an isometry \(\phi_{qp} : M \to M\) such that \(\phi_{qp}(p) = q\).
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Example 2.5 (Riemannian Lie group)

Every **Lie group** \((G, g)\) equipped with a left-invariant Riemannian metric acts transitively on itself.
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**Example 2.5 (Riemannian Lie group)**

Every **Lie group** $(G, g)$ equipped with a left-invariant Riemannian metric acts transitively on itself.

**Example 2.6 (Riemannian symmetric space)**

Every Riemannian **symmetric space** $M = (G/K, g)$ is homogeneous.
Example 2.7 (the nilpotent Lie group \( \text{Nil}^3 \))

\[(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).\]

The left-invariant metric, with orthonormal basis

\[\{ X = \partial / \partial x, \ Y = \partial / \partial y, \ Z = \partial / \partial z \} \]

at the neutral element \( e = (0, 0, 0) \), is given by

\[ds^2 = dx^2 + dy^2 + (dz - xdy)^2.\]
Example 2.7 (the nilpotent Lie group Nil$^3$)

$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R})$.

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at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$  

(Baird, Wood 1990): Every solution is a restriction of the globally defined harmonic morphism $\phi : \text{Nil}^3 \rightarrow \mathbb{C}$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto x + iy.$$
Example 2.8 (the solvable Lie group $\text{Sol}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial / \partial x, \ Y = \partial / \partial y, \ Z = \partial / \partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = e^{2z} \, dx^2 + e^{-2z} \, dy^2 + dz^2.$$
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The left-invariant metric, with orthonormal basis

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\{ \ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \ \}
\]

at the neutral element $e = (0, 0, 0)$, is given by

\[ds^2 = e^{2z} \, dx^2 + e^{-2z} \, dy^2 + dz^2.\]

(Baird, Wood 1990): No solutions exist, not even locally.

\[e^{-2z} \frac{\partial^2 \phi}{\partial x^2} + e^{2z} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,\]

\[e^{-2z} \left( \frac{\partial \phi}{\partial x} \right)^2 + e^{2z} \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 = 0.\]
Definition 3.1 (Laplacian, conformality operator)

For functions $\phi, \psi : (M, g) \to \mathbb{C}$ the metric $g$ induces the complex-valued **Laplacian** $\tau(\phi)$ and the symmetric bilinear **conformality operator** $\kappa$ by

$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$
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$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$

The harmonicity and the horizontal conformality of $\phi : (M, g) \to \mathbb{C}$ are then given by the following relations

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.$$
Definition 3.1 (Laplacian, conformality operator)

For functions $\phi, \psi : (M, g) \to \mathbb{C}$ the metric $g$ induces the complex-valued \textbf{Laplacian} $\tau(\phi)$ and the symmetric bilinear \textbf{conformality operator} $\kappa$ by

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\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).
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The \textbf{harmonicity} and the \textbf{horizontal conformality} of $\phi : (M, g) \to \mathbb{C}$ are then given by the following relations

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\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.
$$

Definition 3.2 (eigenfamily)

A set $\mathcal{E} = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an \textbf{eigenfamily} on $(M, g)$ if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$
\tau(\phi) = \lambda \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \phi \psi.
$$
Theorem 3.3 (SG, Sakovich 2008)

Let \((M, g)\) be a Riemannian manifold and \(\mathcal{E} = \{\phi_1, \ldots, \phi_n\}\) be a \textbf{finite eigenfamily} of complex-valued functions on \(M\). If \(P, Q : \mathbb{C}^n \to \mathbb{C}\) are linearly independent homogeneous polynomials of the same positive degree then the quotient

\[
P(\phi_1, \ldots, \phi_n)/Q(\phi_1, \ldots, \phi_n)
\]

is a \textbf{non-constant harmonic morphism} on the open and dense subset

\[
\{p \in M | Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0\}.
\]

The authors apply this machine to construct solutions on the \textbf{classical semisimple} Lie groups \(\text{SO}(n), \text{SU}(n), \text{Sp}(n), \text{SL}_n(\mathbb{R}), \text{SU}^*(2n)\) and \(\text{Sp}(n, \mathbb{R})\) equipped with their standard Riemannian metrics.

They also develop a \textbf{duality principle} and use this to construct solutions from the \textbf{semisimple} Lie groups \(\text{SO}(n), \text{SU}(n), \text{Sp}(n), \text{SL}_n(\mathbb{R}), \text{SU}^*(2n), \text{Sp}(n, \mathbb{R}), \text{SO}^*(2n), \text{SO}(p, q), \text{SU}(p, q)\) and \(\text{Sp}(p, q)\) equipped with their standard dual semi-Riemannian metrics.
Equip the special orthogonal group

\[ \text{SO}(n) = \{ x \in \text{GL}_n(\mathbb{R}) \mid x^t \cdot x = I_n, \ \text{det} \ x = 1 \} \]

with the standard Riemannian metric \( g \) induced by the Euclidean scalar product \( g(X, Y) = \text{trace}(X^t \cdot Y) \) on the Lie algebra

\[ \text{so}(n) = \{ X \in \mathfrak{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}. \]
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\[ \text{so}(n) = \{ X \in \text{gl}_n(\mathbb{R}) \mid X^t + X = 0 \}. \]

**Lemma 3.4 (SG, Sakovich 2008)**

For \( 1 \leq i, j \leq n \), let \( x_{ij} : \text{SO}(n) \to \mathbb{R} \) be the real valued coordinate functions given by \( x_{ij} : x \mapsto \langle e_i, x \cdot e_j \rangle \) where \( \{ e_1, \ldots, e_n \} \) is the canonical basis for \( \mathbb{R}^n \). Then the following relations hold

\[ \tau(x_{ij}) = -\frac{(n-1)}{2} x_{ij}, \quad \kappa(x_{ij}, x_{kl}) = -\frac{1}{2} (x_{il} x_{kj} - \delta_{jl} \delta_{ik}). \]
Theorem 3.5 (SG, Sakovich 2008)

Let \( p \in \mathbb{C}^n \) be a non-zero isotropic element i.e. \( \langle p, p \rangle = 0 \). Then

\[
\mathcal{E}_p = \{ \phi_a : \text{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle, \ a \in \mathbb{C}^n \}. 
\]

is an eigenfamily on \( \text{SO}(n) \).
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Let $p \in \mathbb{C}^n$ be a non-zero isotropic element i.e. $\langle p, p \rangle = 0$. Then

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\mathcal{E}_p = \{ \phi_a : \text{SO}(n) \to \mathbb{C} \mid \phi_a(x) = \langle p, x \cdot a \rangle, \ a \in \mathbb{C}^n \}.
$$

is an eigenfamily on $\text{SO}(n)$

Example 3.6 (eigenfamilies on $\text{SO}(n)$)

For $z, w \in \mathbb{C}$, let $p$ be the isotropic element of $\mathbb{C}^4$ given by

$$
p(z, w) = (1 + zw, i(1 - zw), i(z + w), z - w).
$$

This gives us the complex 2-dimensional deformation of eigenfamilies $\mathcal{E}_p$ each consisting of functions $\phi_a : \text{SO}(4) \to \mathbb{C}$ with

$$
\phi_a(x) = (1 + zw)(x_{11}a_1 + x_{21}a_2 + x_{31}a_3 + x_{41}a_4) \\
+ i(1 - zw)(x_{12}a_1 + x_{22}a_2 + x_{32}a_3 + x_{42}a_4) \\
+ i(z + w)(x_{13}a_1 + x_{23}a_2 + x_{33}a_3 + x_{43}a_4) \\
+ (z - w)(x_{14}a_1 + x_{24}a_2 + x_{34}a_3 + x_{44}a_4)
$$
Definition 4.1 (orthogonal harmonic family)

A set $\Omega = \{\phi_\alpha : (M, g) \to \mathbb{C} \mid \alpha \in I\}$ of complex-valued functions is called an **orthogonal harmonic family** on $(M, g)$ if for all $\phi, \psi \in \Omega$

$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$
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$$\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \psi) = 0.$$ 

Example 4.2

Let $\Omega = \{\phi_\alpha : (M, g, J) \to \mathbb{C} \mid \alpha \in I\}$ be a collection of holomorphic functions on a Kähler manifold. Then $\Omega$ is an orthogonal harmonic family.
Theorem 4.3 (SG 1997)

Let \((M, g)\) be a Riemannian manifold and \(U\) be an open subset of \(\mathbb{C}^n\) containing the image of \(\Phi = (\phi_1, \ldots, \phi_n) : M \to \mathbb{C}^n\). Further let

\[ H = \{ F_\alpha : U \to \mathbb{C} \mid \alpha \in I \} \]

be a collection of \textbf{holomorphic} functions defined on \(U\). If the finite set

\[ \Omega = \{ \phi_k : (M, g) \to \mathbb{C} \mid k = 1, \ldots, n \} \]

is an \textbf{orthogonal harmonic family} on \((M, g)\) then

\[ \Omega_H = \{ \psi : M \to \mathbb{C} \mid \psi = F(\phi_1, \ldots, \phi_n), \ F \in H \} \]

is again an \textbf{orthogonal harmonic family}.
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a natural group epimorphism $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.
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**Fact 4.4 (semisimple - solvable - nilpotent)**

*If the group $G$ is semisimple then $d = 0$, if $G$ is solvable then $d \geq 1$ and if $G$ is nilpotent then $d \geq 2$.***
Let $G$ be a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$. Then the natural projection $\pi : \mathfrak{g} \to \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ to the abelian algebra $\mathfrak{a}$ is a Lie algebra homomorphism inducing a natural group epimorphism $\Phi : G \to \mathbb{R}^d$ with $d = \dim \mathfrak{a}$.

**Fact 4.4 (semisimple - solvable - nilpotent)**

*If the group $G$ is semisimple then $d = 0$, if $G$ is solvable then $d \geq 1$ and if $G$ is nilpotent then $d \geq 2$.*

Equip $\mathbb{R}^d$ with its standard Euclidean metric and the Lie group $G$ with a left-invariant Riemannian metric $g$ such that the natural group epimorphism $\Phi : G \to \mathbb{R}^d$ is a *Riemannian submersion*. Then the kernel $[\mathfrak{g}, \mathfrak{g}]$ of the linear map $\pi : \mathfrak{g} \to \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ generates a left-invariant Riemannian foliation $\mathcal{V}$ on $(G, g)$ with orthogonal distribution $\mathcal{H} = [\mathfrak{g}, \mathfrak{g}]^\perp$. 


Theorem 4.5 (SG, Svensson 2009)

Let $\mathcal{B} = \{X_1, \ldots, X_d\}$ be an ONB for the horizontal subspace $[\mathfrak{g}, \mathfrak{g}]^\perp$ of $\mathfrak{g}$ and $\xi \in \mathbb{C}^d$ be given by $\xi = (\text{trace } \text{ad} X_1, \ldots, \text{trace } \text{ad} X_d)$. For a maximal isotropic subspace $V$ of $\mathbb{C}^d$ put

$$V_{\xi} = \{v \in V \mid \langle \xi, v \rangle = 0\}.$$  

If the real dimension of the isotropic subspace $V_{\xi}$ is at least 2 then

$$\Omega = \{\phi_v(x) = \langle \Phi(x), v \rangle \mid v \in V_{\xi}\}$$  

is an orthogonal harmonic family on $(G, g)$. 

Proof. The tension field of $\Phi$ satisfies

$$\tau(\Phi)(p) = d\sum_{k=1}^{d} (\text{trace } \text{ad} X_k) d\Phi e(X_k).$$
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Proof.

The tension field of $\Phi$ satisfies

$$\tau(\Phi)(p) = \sum_{k=1}^d (\text{trace } \text{ad} X_k) d\Phi_e(X_k).$$
Example 4.6

For the nilpotent Riemannian Lie group

\[
N_n = \left\{ \begin{pmatrix}
1 & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\
0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & 1 & x_{n-1,n} \\
0 & \cdots & \cdots & 0 & 1
\end{pmatrix} \in \text{SL}_n(\mathbb{R}) \mid x_{ij} \in \mathbb{R} \right\}.
\]

the natural group epimorphism \( \Phi : N_n \to \mathbb{R}^{n-1} \) is given by

\[
\Phi(x) = (x_{12}, \ldots, x_{n-1,n}).
\]
Example 4.7

For the solvable Riemannian Lie group

\[
S_n = \left\{ \begin{pmatrix}
e^{t_1} & x_{12} & \cdots & x_{1,n-1} & x_{1n} \\
0 & e^{t_2} & \cdots & x_{2,n-1} & x_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \cdots & 0 & e^{t_{n-1}} & x_{n-1,n} \\
0 & \cdots & 0 & 0 & e^{t_n}
\end{pmatrix} \in \text{GL}_n(\mathbb{R}) \mid x_{ij}, t_i \in \mathbb{R}\right\}
\]

the natural group epimorphism \( \Phi : S_n \to \mathbb{R}^n \) is given by

\[
\Phi(x) = (t_1, t_2, \ldots, t_n)
\]

and the vector \( \xi \in \mathbb{C}^n \) satisfies

\[
\xi = ((n + 1) - 2, (n + 1) - 4, \ldots, (n + 1) - 2n).
\]
Let \((M, g)\) be an irreducible Riemannian \textbf{symmetric space} of \textbf{non-compact type} and write \(M = G/K\) with \(G\) a semisimple, connected and simply connected Lie group and \(K\) a maximal compact subgroup of \(G\).
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**Fact 4.8 (solvable Lie group - rank)**

The symmetric space $(M, g)$ can be identified with the solvable subgroup $S = NA$ of $G$ and its rank $r$ is the dimension of abelian subgroup $A$. 
Let \((M, g)\) be an irreducible Riemannian symmetric space of non-compact type and write \(M = G/K\) with \(G\) a semisimple, connected and simply connected Lie group and \(K\) a maximal compact subgroup of \(G\).

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**Fact 4.8 (solvable Lie group - rank)**

*The symmetric space \((M, g)\) can be identified with the solvable subgroup \(S = NA\) of \(G\) and its rank \(r\) is the dimension of abelian subgroup \(A\).*

Let \(s, n, a\) be the Lie algebras of \(S, N, A\), respectively. For this situation we have \(s = a + n = a + [s, s]\), hence

\[a = s/[s, s].\]
With a series of different methods we have obtained the following result:

**Theorem 4.9 (SG, Svensson 2009)**

Let \((M, g)\) be an irreducible Riemannian symmetric space other than \(G^*_2/\text{SO}(4)\) or its compact dual \(G_2/\text{SO}(4)\). Then for each point \(p \in M\) there exists a non-constant complex-valued harmonic morphism \(\phi : U \rightarrow \mathbb{C}\) defined on an open neighbourhood \(U\) of \(p\). If the space \((M, g)\) is of non-compact type then the domain \(U\) can be chosen to be the whole of \(M\).
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An essential tool is the following **Duality Principle**:

**Theorem 4.10 (SG, Svensson 2006)**

Let \(\mathcal{F}\) be a family of local maps \(\phi : W \subset G/K \rightarrow \mathbb{C}\) and \(\mathcal{F}^*\) be the dual family consisting of the local maps \(\phi^* : W^* \subset U/K \rightarrow \mathbb{C}\). Then \(\mathcal{F}^*\) is a local orthogonal harmonic family on \(U/K\) if and only if \(\mathcal{F}\) is a local orthogonal harmonic family on \(G/K\).
The **Duality Principle** explains the following.

**Example 4.11 (Baird, Eells 1981)**

The map \( \phi : U \subset S^3 \subset \mathbb{R}^4 = \mathbb{C}^2 \rightarrow \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 + ix_4}
\]

is a **locally defined** harmonic morphism.
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**Example 4.12 (SG 1996)**

The map \( \phi : H^3 \subset \mathbb{R}^4 \rightarrow \mathbb{C} \) given by

\[
\phi : (x_1, x_2, x_3, x_4) \mapsto \frac{x_1 + ix_2}{x_3 - x_4}
\]

is a **globally defined** harmonic morphism.
Our existence result for symmetric spaces has the following **interesting** consequence:

**Theorem 4.13 (SG, Svensson 2013)**

Let \((M, g)\) be a Riemannian **homogeneous space of positive curvature** other than the Berger space \(\text{Sp}(2)/\text{SU}(2)\). Then \(M\) admits local complex-valued harmonic morphisms.
Fact 5.1

Every Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either symmetric or a Lie group with a left-invariant metric.
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(SG, Svensson 2011): Give a classification for 3-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing \(\text{Sol}^3\), not carrying any left-invariant metric admitting complex-valued harmonic morphisms.
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*(SG, Svensson 2013):* Give a classification for 4-dimensional Riemannian Lie groups admitting left-invariant solutions. Most of the solutions constructed are NOT holomorphic with respect to any (integrable) Hermitian structure.
Fact 5.1

*Every* Riemannian homogeneous space \((M, g)\) of dimension 3 or 4 is either *symmetric* or a *Lie group* with a left-invariant metric.

*(SG, Svensson 2011):* Give a classification for 3-dimensional Riemannian Lie groups admitting solutions. Find a continuous family of groups, containing \(\text{Sol}^3\), not carrying any left-invariant metric admitting complex-valued harmonic morphisms.

*(SG, Svensson 2013):* Give a classification for 4-dimensional Riemannian Lie groups admitting *left-invariant* solutions. Most of the solutions constructed are *NOT holomorphic* with respect to any (integrable) Hermitian structure.

*(SG 2013):* Gives a partial classification for 5-dimensional Riemannian Lie groups admitting *left-invariant* solutions.


