Lie Foliations Producing Harmonic Morphisms

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Definition 1.1 (Harmonic Morphisms (Fuglede 1978, Ishihara 1979))

A map $\phi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is called a \textbf{harmonic morphism} if, for any harmonic function $f : U \to \mathbb{R}$ defined on an open subset $U$ of $N$ with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is a harmonic function.
Definition 1.1 (Harmonic Morphisms (Fuglede 1978, Ishihara 1979))

A map \( \phi : (M^m, g) \rightarrow (N^n, h) \) between Riemannian manifolds is called a **harmonic morphism** if, for any harmonic function \( f : U \rightarrow \mathbb{R} \) defined on an open subset \( U \) of \( N \) with \( \phi^{-1}(U) \) non-empty, \( f \circ \phi : \phi^{-1}(U) \rightarrow \mathbb{R} \) is a harmonic function.

Theorem 1.2 (Fuglede 1978, Ishihara 1979)

A map \( \phi : (M, g) \rightarrow (N, h) \) between Riemannian manifolds is a harmonic morphism if and only if it is **harmonic** and **horizontally** (weakly) conformal.
(Harmonicity)

For local coordinates $x$ on $(M, g)$ and $y$ on $(N, h)$, we have the **non-linear** system

$$
\tau(\phi) = \sum_{i,j=1}^{m} g^{ij} \left( \frac{\partial^2 \phi^\gamma}{\partial x_i \partial x_j} - \sum_{k=1}^{m} \hat{\Gamma}^k_{ij} \frac{\partial \phi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n} \Gamma^\gamma_{\alpha\beta} \circ \phi \frac{\partial \phi^\alpha}{\partial x_i} \frac{\partial \phi^\beta}{\partial x_j} \right) = 0,
$$

where $\phi^\alpha = y^\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.
Harmonic Morphisms

3-dimensional Lie groups
4-dimensional Lie groups
5-dimensional Lie groups

Riemannian Geometry - Fuglede 1978, Ishihara 1979
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(Harmonicity)

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$$

where $\phi^\alpha = y^\alpha \circ \phi$ and $\hat{\Gamma}, \Gamma$ are the Christoffel symbols on $M, N$, resp.

(Horizontal (weak) Conformality)

There exists a continuous function $\lambda : M \to \mathbb{R}_0^+$ such that for all $\alpha, \beta = 1, 2, \ldots, n$

$$
\sum_{i,j=1}^{m} g^{ij}(x) \frac{\partial \phi^\alpha}{\partial x_i}(x) \frac{\partial \phi^\beta}{\partial x_j}(x) = \lambda^2(x) h^{\alpha \beta}(\phi(x)).
$$

This is a first order non-linear system of $\left[\binom{n+1}{2} - 1\right]$ equations.
Theorem 1.3 (Baird, Eells 1981)

Let \( \phi : (M, g) \to (N^2, h) \) be a horizontally conformal map from a Riemannian manifold to a surface. Then \( \phi \) is harmonic if and only if its fibres are minimal at regular points \( \phi \).
Theorem 1.3 (Baird, Eells 1981)

Let $\phi : (M, g) \to (N^2, h)$ be a horizontally conformal map from a Riemannian manifold to a surface. Then $\phi$ is harmonic if and only if its fibres are minimal at regular points $\phi$.

The problem is invariant under isometries on $(M, g)$. If the codomain $(N, h)$ is a surface ($n = 2$) then it is also invariant under conformal changes $\sigma^2 h$ of the metric on $N^2$. This means, at least for local studies, that $(N^2, h)$ can be chosen to be the standard complex plane $\mathbb{C}$. 
Let \( \phi : (M, g) \to (N^2, h) \) be a submersive harmonic morphism from a Riemannian manifold to a surface. Then this induces a conformal foliation \( \mathcal{F} \) on \((M, g)\) with minimal leaves of codimension 2.
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Let \( \mathcal{V} \) be the integrable subbundle of \( TM \) tangent to the fibres of \( \mathcal{F} \) and \( \mathcal{H} \) be its orthogonal complement. Then the second fundamental form for \( \mathcal{H} \) is given by

\[
B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).
\]
Let $\phi : (M, g) \to (N^2, h)$ be a submersive harmonic morphism from a Riemannian manifold to a surface. Then this induces a \textbf{conformal} foliation $\mathcal{F}$ on $(M, g)$ with \textbf{minimal} leaves of codimension 2.

Let $\mathcal{F}$ be a \textbf{conformal} foliation on $(M, g)$ with \textbf{minimal} leaves of codimension 2. Then $\mathcal{F}$ produces locally \textbf{submersive harmonic morphisms} from $(M, g)$ to surfaces.

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$$B^\mathcal{H}(X, Y) = \frac{1}{2} \mathcal{V}(\nabla_X Y + \nabla_Y X) \quad (X, Y \in \mathcal{H}).$$

$\mathcal{F}$ is said to be \textbf{conformal} if there is a vector field $V \in \mathcal{V}$ such that

$$B^\mathcal{H} = g \otimes V.$$

$\mathcal{F}$ is said to be \textbf{Riemannian} if $V = 0$. 
Example 1.4 (The Nilpotent Lie Group $\text{Nil}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = dx^2 + dy^2 + (dz - xdy)^2.$$
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(Baird, Wood 1990): Every local solution is a restriction of the globally defined harmonic morphism $\phi : \text{Nil}^3 \to \mathbb{C}$ with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto x + iy.$$
Example 1.5 (The Solvable Lie Group Sol³)

\[(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).\]

The left-invariant metric, with orthonormal basis

\[\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \} \]

at the neutral element \( e = (0, 0, 0) \), is given by

\[ ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2. \]
Example 1.5 (The Solvable Lie Group $\text{Sol}^3$)

$$(x, y, z) \in \mathbb{R}^3 \mapsto \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \text{SL}_3(\mathbb{R}).$$

The left-invariant metric, with orthonormal basis

$$\{ X = \partial/\partial x, \ Y = \partial/\partial y, \ Z = \partial/\partial z \}$$

at the neutral element $e = (0, 0, 0)$, is given by

$$ds^2 = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$

(Baird, Wood 1990): No solutions exist, not even locally.

$$e^{-2z} \frac{\partial^2 \phi}{\partial x^2} + e^{2z} \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

$$e^{-2z} \left( \frac{\partial \phi}{\partial x} \right)^2 + e^{2z} \left( \frac{\partial \phi}{\partial y} \right)^2 + \left( \frac{\partial \phi}{\partial z} \right)^2 = 0.$$
At the end of the 19th century, L. Bianchi classified the 3-dimensional real Lie algebras. They fall into nine disjoint types I-IX. Each contains a single isomorphy class except types VI and VII which are continuous families of different classes.
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**Example 2.1 (Type I)**

The Abelian Lie algebra $\mathbb{R}^3$; the corresponding simply connected Lie group is of course the Abelian group $\mathbb{R}^3$ which we equip with the standard flat metric.
At the end of the 19th century, L. Bianchi classified the 3-dimensional real Lie algebras. They fall into nine disjoint types I-IX. Each contains a single isomorphy class except types VI and VII which are continuous families of different classes.

Example 2.1 (Type I)

The Abelian Lie algebra \( \mathbb{R}^3 \); the corresponding simply connected Lie group is of course the Abelian group \( \mathbb{R}^3 \) which we equip with the standard flat metric.

Example 2.2 (Type II)

The Lie algebra \( \mathfrak{n}_3 \) with a basis \( X, Y, Z \) satisfying

\[
[X, Y] = Z.
\]

The corresponding simply connected Lie group is the nilpotent Heisenberg group \( \text{Nil}^3 \).
Example 2.3 (Type III)

The Lie algebra \( \mathfrak{h}^2 \oplus \mathbb{R} = \text{span}\{X, Y, Z\} \), where \( \mathfrak{h}^2 \) is the two-dimensional Lie algebra with basis \( X, Y \) satisfying

\[ [Y, X] = X. \]

The corresponding simply connected Lie group is denoted by \( H^2 \times \mathbb{R} \). Here \( H^2 \) is the standard hyperbolic plane.
### Example 2.3 (Type III)

The Lie algebra $\mathfrak{h}^2 \oplus \mathbb{R} = \text{span}\{X, Y, Z\}$, where $\mathfrak{h}^2$ is the two-dimensional Lie algebra with basis $X, Y$ satisfying

$$[Y, X] = X.$$ 

The corresponding simply connected Lie group is denoted by $H^2 \times \mathbb{R}$. Here $H^2$ is the standard hyperbolic plane.

### Example 2.4 (Type IV)

The Lie algebra $\mathfrak{g}_4$ with a basis $X, Y, Z$ satisfying

$$[Z, X] = X, \quad [Z, Y] = X + Y.$$ 

The corresponding simply connected Lie group is denoted by $G_4$. 
Example 2.5 (Type V)

The Lie algebra $\mathfrak{h}^3$ with a basis $X, Y, Z$ satisfying

$$[Z, X] = X, \quad [Z, Y] = Y.$$ 

The corresponding simply connected Lie group $H^3$ is the standard hyperbolic 3-space of constant sectional curvature $-1$.
Example 2.5 (Type V)

The Lie algebra $\mathfrak{h}^3$ with a basis $X, Y, Z$ satisfying

\[ [Z, X] = X, \quad [Z, Y] = Y. \]

The corresponding simply connected Lie group $H^3$ is the standard hyperbolic 3-space of constant sectional curvature $-1$.

Example 2.6 (Type VI)

The Lie algebra $\mathfrak{sol}_\alpha^3$, where $\alpha > 0$, is the Lie algebra with basis $X, Y, Z$ satisfying

\[ [Z, X] = \alpha X, \quad [Z, Y] = -Y. \]

The corresponding simply connected Lie group is denoted by $\text{Sol}_\alpha^3$. The group Sol mentioned in the introduction is actually $\text{Sol}_1^3$. 
Example 2.7 (Type VII)

The Lie algebra $\mathfrak{g}_7(\alpha)$, where $\alpha \in \mathbb{R}$, is the Lie algebra with basis $X, Y, Z$ satisfying

\[
[Z, X] = \alpha X - Y, \quad [Z, Y] = X + \alpha Y.
\]

The corresponding simply connected Lie group is denoted by $G_7(\alpha)$. 
Example 2.7 (Type VII)

The Lie algebra $\mathfrak{g}_7(\alpha)$, where $\alpha \in \mathbb{R}$, is the Lie algebra with basis $X, Y, Z$ satisfying

$$\begin{align*}
[Z, X] &= \alpha X - Y, \\
[Z, Y] &= X + \alpha Y.
\end{align*}$$

The corresponding simply connected Lie group is denoted by $G_7(\alpha)$.

Example 2.8 (Type VIII)

The Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ with a basis $X, Y, Z$ satisfying

$$\begin{align*}
[X, Y] &= -2Z, \\
[Z, X] &= 2Y, \\
[Y, Z] &= 2X.
\end{align*}$$

The corresponding simply connected Lie group is denoted by $\tilde{\mathbf{SL}}_2(\mathbb{R})$ as it is the universal cover of the special linear group $\mathbf{SL}_2(\mathbb{R})$. 
Example 2.9 (Type IX)

The Lie algebra $\mathfrak{su}(2)$ with a basis $X, Y, Z$ satisfying

\[
\begin{align*}
[X, Y] &= 2Z, \\
[Z, X] &= 2Y, \\
[Y, Z] &= 2X.
\end{align*}
\]

The corresponding simply connected Lie group is of course $\text{SU}(2)$. This is isometric to the standard 3-sphere of constant curvature $+1$. 
Example 2.9 (Type IX)

The Lie algebra $\mathfrak{su}(2)$ with a basis $X, Y, Z$ satisfying

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X.$$  

The corresponding simply connected Lie group is of course $\text{SU}(2)$. This is isometric to the standard 3-sphere of constant curvature +1.

Definition 2.10 (Lie Foliations)

Let $G$ be a Lie group equipped with a left-invariant Riemannian metric and $K$ be a subgroup of $G$. Then the natural projection $\pi : G \to G/K$ induces a foliation $\mathcal{F}$ on $G$. The leaves of $\mathcal{F}$ are the fibres $\pi^{-1}(gK)$ of $\pi$. 
Theorem 2.11 (G, Svensson (2009))

Let $G$ be a connected 3-dimensional Lie group with a left-invariant metric of non-constant sectional curvature. Then any local conformal foliation by geodesics of a connected open subset of $G$ can be extended to a global conformal foliation $\mathcal{F}$ by geodesics of $G$. This is given by the left-translation of a 1-parameter subgroup of $G$ i.e. $\mathcal{F}$ is a Lie foliation.
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Let $G$ be a 3-dimensional Riemannian Lie group with Lie algebra $\mathfrak{g}$ generated by the elements of the orthonormal basis $\{X, Y, Z\}$. Let $\mathcal{F}$ be a 1-dimensional conformal and minimal foliation of $G$ generated by the left-invariant vector field $Z$. Then the Lie brackets of $\mathfrak{g}$ must satisfy

\[
\begin{align*}
[Z, X] &= aX + bY, \\
[Z, Y] &= -bX + aY, \\
[X, Y] &= xX + yY + zZ,
\end{align*}
\]

where $a, b, x, y, z \in \mathbb{R}$. The Jacobi identity shows that here we have a Lie algebra if and only if the following system of 3 quadratic equations in 5 variables are satisfied

\[
a z = 0, \quad a x + b y = 0, \quad b x - a y = 0.
\]
Theorem 2.12 (G, Svensson (2011))

The following THREE families of 3-dimensional Lie algebras give a complete classification.

Example 2.13
When $x = y = a = 0$ we obtain a 2-dimensional family with the bracket relations
\[
[Z, X] = bY, \\
[Z, Y] = -bX, \\
[X, Y] = zZ,
\]
When $bz < 0$ the Lie algebra is of type VIII and of type IX if $bz > 0$. The case when $z = 0$ and $b \neq 0$ is of type VII ($\alpha = 0$), and the case when $b = 0$ and $z \neq 0$ is of type II. The case when $b = z = 0$ is of type I.
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Example 2.14

With $a = b = 0$ we yield a 3-dimensional family of Lie groups with bracket relation

$$[X, Y] = xX + yY + zZ.$$ 

If $x = y = z = 0$ then the type is I. If $x$ or $y$ non-zero then we have type III.
If $z \neq 0$ and $x = y = 0$, then the type is II.
Example 2.14

With $a = b = 0$ we yield a 3-dimensional family of Lie groups with bracket relation

$$[X, Y] = xX + yY + zZ.$$  

If $x = y = z = 0$ then the type is I. If $x$ or $y$ non-zero then we have type III. If $z \neq 0$ and $x = y = 0$, then the type is II.

Example 2.15

In the case of $x = y = z = 0$ we get semi-direct products $\mathbb{R}^2 \rtimes \mathbb{R}$ with bracket relations

$$[Z, X] = aX + bY,$$

$$[Z, Y] = -bX + aY.$$  

If $b \neq 0$ then the Lie algebra is of type VII. If $b = 0$ then the Lie algebra is of type V or of type I if also $a = 0$. 
Theorem 2.16

Let $G$ be a 3-dimensional Lie group with Lie algebra $\mathfrak{g}$. Then there exists a left-invariant Riemannian metric $g$ on $G$ and a left-invariant horizontally conformal foliation on $(G, g)$ by geodesics if and only if the Lie algebra $\mathfrak{g}$ is neither of type IV nor of type VI.
Theorem 2.16

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Type IV: The single case of $G_4$. 

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Theorem 2.16

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Type IV: The single case of $G_4$.

Type VI: The continuous family of $\text{Sol}_\alpha^3$, where $\alpha > 0$.

Note that in the cases of type I, II, III, V, VII, VIII and IX the possible left-invariant Riemannian metrics are completely determined via isomorphisms to the standard examples.
Let $G$ be a 4-dimensional Riemannian Lie group with Lie algebra $\mathfrak{g}$ generated by the elements of the orthonormal basis $\{X, Y, Z, W\}$. Let $\mathcal{F}$ be a 2-dimensional conformal and minimal Lie foliation of $G$ generated by the left-invariant vector fields $Z$ and $W$. Then the Lie bracket of $\mathfrak{g}$ must satisfy the following SIX relations

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + z_1 Z + w_1 W, \quad [Z, Y] = -\beta X + \alpha Y + z_2 Z + w_2 W, \\
[W, X] &= aX + bY + z_3 Z - z_1 W, \quad [W, Y] = -bX + aY + z_4 Z - z_2 W, \\
[Y, X] &= rX + \theta_1 Z + \theta_2 W
\end{align*}
\]
Let $G$ be a 4-dimensional Riemannian Lie group with Lie algebra $\mathfrak{g}$ generated by the elements of the orthonormal basis $\{X, Y, Z, W\}$. Let $\mathcal{F}$ be a 2-dimensional conformal and minimal Lie foliation of $G$ generated by the left-invariant vector fields $Z$ and $W$. Then the Lie bracket of $\mathfrak{g}$ must satisfy the following SIX relations

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[W, X] &= aX + bY + z_3 Z - z_1 W, \\
[Y, X] &= rX + \theta_1 Z + \theta_2 W
\end{align*}
\]

**Proposition 1**

Let $G$ be a 4-dimensional Lie group and $\{X, Y, Z, W\}$ be an orthonormal basis for its Lie algebra as above. Then

(i) $\mathcal{F}$ is totally geodesic if and only if $z_1 = z_2 = z_3 + w_1 = z_4 + w_2 = 0$,

(ii) $\mathcal{F}$ is Riemannian if and only if $\alpha = a = 0$, and

(iii) $\mathcal{H}$ is integrable if and only if $\theta_1 = \theta_2 = 0$. 
0 = \lambda a,
0 = \lambda b,
0 = -w_2 z_3 + w_1 z_4 - 2\alpha \theta_1 + rz_1,
0 = -2z_4 z_1 + 2z_3 z_2 - 2a \theta_1 + rz_3,
0 = -\lambda z_3 - z_2 b + z_4 \beta - z_1 a + z_3 \alpha,
0 = -\lambda z_4 - z_2 a + z_4 \alpha + z_1 b - z_3 \beta,
0 = \lambda \theta_1 - w_1 z_4 + w_2 z_3 - 2a \theta_2 - rz_1,
0 = -\lambda \theta_2 + 2z_1 w_2 - 2z_2 w_1 - 2\alpha \theta_2 + rw_1,
0 = -w_2 a - w_1 b - z_2 \alpha - z_1 \beta - \alpha r,
0 = -w_2 b + w_1 a - z_2 \beta + z_1 \alpha + r \beta,
0 = \lambda z_1 - w_2 b - z_2 \beta - w_1 a - z_1 \alpha,
0 = z_2 a + z_1 b - z_4 \alpha - z_3 \beta - ar,
0 = z_2 b - z_1 a - z_4 \beta + z_3 \alpha + rb,
0 = \lambda z_2 - w_2 a - z_2 \alpha + w_1 b + z_1 \beta.
Harmonic Morphisms
3-dimensional Lie groups
4-dimensional Lie groups
5-dimensional Lie groups

Lie Foliations Producing Harmonic Morphisms

Case (A) - \((\lambda \neq 0 \text{ and } (\lambda - \alpha)^2 + \beta^2 \neq 0)\)

Case (B) - \((\lambda \neq 0 \text{ and } (\lambda - \alpha)^2 + \beta^2 = 0)\)

Case (C-F) - \((\lambda = 0)\)

0 = \lambda a,
0 = \lambda b,
0 = -w_2 z_3 + w_1 z_4 - 2\alpha \theta_1 + rz_1,
0 = -2z_4 z_1 + 2z_3 z_2 - 2a \theta_1 + rz_3,
0 = -\lambda z_3 - z_2 b + z_4 \beta - z_1 a + z_3 \alpha,
0 = -\lambda z_4 - z_2 a + z_4 \alpha + z_1 b - z_3 \beta,
0 = \lambda \theta_1 - w_1 z_4 + w_2 z_3 - 2a \theta_2 - rz_1,
0 = -\lambda \theta_2 + 2z_1 w_2 - 2z_2 w_1 - 2a \theta_2 + rw_1,
0 = -w_2 a - w_1 b - z_2 \alpha - z_1 \beta - \alpha r,
0 = -w_2 b + w_1 a - z_2 \beta + z_1 \alpha + r \beta,
0 = \lambda z_1 - w_2 b - z_2 \beta - w_1 a - z_1 \alpha,
0 = z_2 a + z_1 b - z_4 \alpha - z_3 \beta - ar,
0 = z_2 b - z_1 a - z_4 \beta + z_3 \alpha + rb,
0 = \lambda z_2 - w_2 a - z_2 \alpha + w_1 b + z_1 \beta.

Maple 17: Time 0.89 seconds - Memory: 38.18 Mb
Harmonic Morphisms
3-dimensional Lie groups
4-dimensional Lie groups
5-dimensional Lie groups

Lie Foliations Producing Harmonic Morphisms
Case (A) - $(\lambda \neq 0$ and $(\lambda - \alpha)^2 + \beta^2 \neq 0)$
Case (B) - $(\lambda \neq 0$ and $(\lambda - \alpha)^2 + \beta^2 = 0)$
Case (C-F) - $(\lambda = 0)$

---

Theorem 3.1 (G, Svensson (2014))

Let $G$ be a 4-dimensional Riemannian Lie group. Let $\mathcal{F}$ be a conformal Lie foliation on $G$ with minimal leaves of codimension 2. Then the Lie algebra $\mathfrak{g}$ of $G$ belongs to one of 20 multi-dimensional families $\mathfrak{g}_1, \ldots, \mathfrak{g}_{20}$ of Lie algebras and $\mathcal{F}$ is the corresponding foliation generated by $\mathfrak{g}$. 
An elementary calculation shows that the two Jacobi equations

\[
[[W, Z], X] + [[X, W], Z] + [[Z, X], W] = 0,
\]

\[
[[W, Z], Y] + [[Y, W], Z] + [[Z, Y], W] = 0
\]

are equivalent to the following relations for the real structure constants

\[
\begin{pmatrix}
\beta & \lambda - \alpha \\
\lambda - \alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
= \begin{pmatrix}
z_4 \\
-z_3
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\]
An elementary calculation shows that the two Jacobi equations

\[
[[W, Z], X] + [[X, W], Z] + [[Z, X], W] = 0, \tag{1}
\]
\[
[[W, Z], Y] + [[Y, W], Z] + [[Z, Y], W] = 0 \tag{2}
\]

are equivalent to the following relations for the real structure constants

\[
\begin{pmatrix}
\beta & \lambda - \alpha \\
\lambda - \alpha & -\beta
\end{pmatrix}
\begin{pmatrix}
z_1 \\
z_2
\end{pmatrix}
= 
\begin{pmatrix}
z_4 \\
z_3
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0
\end{pmatrix}. \tag{3}
\]

Applying equation (3) we see that \( z = 0 \) so the Lie brackets satisfy

\[
[W, Z] = \lambda W,
\]
\[
[Z, X] = \alpha X + \beta Y + w_1 W,
\]
\[
[Z, Y] = -\beta X + \alpha Y + w_2 W,
\]
\[
[Y, X] = r X + \theta_1 Z + \theta_2 W.
\]
In this situation it is easily seen that the two remaining Jacobi equations are equivalent to \( \theta_1 = r\alpha = r\beta = 0 \) and \( \theta_2(\lambda + 2\alpha) = rw_1 \).
In this situation it is easily seen that the two remaining Jacobi equations are equivalent to $\theta_1 = r\alpha = r\beta = 0$ and $\theta_2(\lambda + 2\alpha) = rw_1$.

**Example 3.2 ($g_1(\lambda, r, w_1, w_2)$)**

If $r \neq 0$ then clearly $\alpha = \beta = 0$ and $rw_1 = \lambda \theta_2$. This gives a 4-dimensional family of solutions satisfying the following Lie bracket relations

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= w_1 W, \\
[Z, Y] &= w_2 W, \\
\lambda[Y, X] &= \lambda r X + rw_1 W.
\end{align*}
\]
On the other hand, if $r = 0$ then clearly $\theta_1 = \theta_2(\lambda + 2\alpha) = 0$ providing us with the following two examples.

**Example 3.3** ($g_2(\lambda, \alpha, \beta, w_1, w_2)$)

For $r = \theta_1 = \theta_2 = 0$ we have the family $g = g_2(\lambda, \alpha, \beta, w_1, w_2)$ given by

\[
[W, Z] = \lambda W, \\
[Z, X] = \alpha X + \beta Y + w_1 W, \\
[Z, Y] = -\beta X + \alpha Y + w_2 W.
\]
On the other hand, if \( r = 0 \) then clearly \( \theta_1 = \theta_2(\lambda + 2\alpha) = 0 \) providing us with the following two examples.

**Example 3.3 \( (g_2(\lambda, \alpha, \beta, w_1, w_2)) \)**

For \( r = \theta_1 = \theta_2 = 0 \) we have the family \( g = g_2(\lambda, \alpha, \beta, w_1, w_2) \) given by

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \alpha X + \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + w_2 W.
\end{align*}
\]

**Example 3.4 \( (g_3(\alpha, \beta, w_1, w_2, \theta_2)) \)**

If \( r = \theta_1 = 0 \) and \( \theta_2 \neq 0 \) then \( \lambda = -2\alpha \) provides us with the family \( g = g_3(\alpha, \beta, w_1, w_2, \theta_2) \) of solutions satisfying

\[
\begin{align*}
[W, Z] &= -2\alpha W, \\
[Z, X] &= \alpha X + \beta Y + w_1 W, \\
[Z, Y] &= -\beta X + \alpha Y + w_2 W, \\
[Y, X] &= \theta_2 W.
\end{align*}
\]
Under the assumptions that $\alpha = \lambda$ and $\beta = 0$ we have

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \lambda X + z_1 Z + w_1 W, \\
[W, X] &= z_3 Z - z_1 W, \\
[Y, X] &= r X + \theta_1 Z + \theta_2 W.
\end{align*}
\]

The last two Jacobi equations are easily seen to be equivalent to

\[z_1 = z_3 = z_4 = \theta_1 = 0, \quad z_2 = -r \quad \text{and} \quad \lambda \theta_2 = -z_2 w_1.\]

**Example 3.5 ($g_4(\lambda, z_2, w_1, w_2)$)**

In this case we have the family of solutions given by

\[
\begin{align*}
[W, Z] &= \lambda W, \\
[Z, X] &= \lambda X + w_1 W, \\
[Z, Y] &= \lambda Y + z_2 Z + w_2 W, \\
[W, Y] &= -z_2 W, \\
\lambda[Y, X] &= -z_2 \lambda X - z_2 w_1 W.
\end{align*}
\]
That story is far too long for this talk :-)

We get the following collection of 4-dimensional Lie foliations producing harmonic morphisms.

\[ g_5(\alpha, a, \beta, b, r), \quad g_6(z_1, z_2, z_3, r, \theta_1, \theta_2), \]
\[ g_7(z_2, w_1, w_2, \theta_1, \theta_2), \quad g_8(z_2, z_4, w_2, r, \theta_1, \theta_2), \]
\[ g_9(z_2, z_3, z_4, \theta_1, \theta_2), \quad g_{10}(\alpha, a, \beta, b), \]
\[ g_{11}(z_1, z_2, z_3, w_1, \theta_1, \theta_2), \quad g_{12}(z_3, w_1, w_2, \theta_1, \theta_2), \]
\[ g_{13}(z_3, z_4, \theta_1, \theta_2), \quad g_{14}(z_2, z_4, w_2, \theta_1, \theta_2), \]
\[ g_{15}(\alpha, w_1, w_2), \quad g_{16}(\beta, w_1, w_2, \theta_1, \theta_2), \]
\[ g_{17}(\alpha, a, w_1, w_2), \quad g_{18}(\beta, b, z_3, z_4, \theta_1, \theta_2), \]
\[ g_{19}(\alpha, \beta, w_1, w_2), \quad g_{20}(\alpha, a, \beta, w_1, w_2). \]
Theorem 4.1 ($\mathfrak{f} = \mathfrak{su}(2)$)

Let $G$ be a 5-dimensional Riemannian Lie group carrying a conformal and minimal Lie foliation $\mathcal{F}$, generated by the subalgebra $\mathfrak{su}(2)$ of $\mathfrak{g}$ as above. Then the Lie bracket relations are of the form

$$
[X, Y] = 2\lambda Z, \quad [Z, X] = 2\lambda Y, \quad [Y, Z] = 2X/\lambda.
$$

$$
[X, A] = -\lambda^2 x_3 Y - \lambda^2 x_5 Z, \quad [X, B] = -\lambda^2 x_4 Y - \lambda^2 x_6 Z,
$$

$$
[Y, A] = x_3 X + z_3 Z, \quad [Y, B] = x_4 X + z_4,
$$

$$
[Z, A] = x_5 X - z_3 Y, \quad [Z, B] = x_6 X - z_4 Y,
$$

$$
[A, B] = rA + \theta_1 X + \theta_2 Y + \theta_3 Z,
$$

where $\theta_1, \theta_2, \theta_3$ are given by

$$
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
rx_3/\lambda + \lambda(x_3x_6 - x_4x_5) \\
\lambda(rx_5 - x_3z_4 + x_4z_3) \\
-\lambda(rx_3 + z_4x_5 - z_3x_6)
\end{pmatrix}.
$$

The foliation $\mathcal{F}$ is Riemannian. It is totally geodesic if and only if $\lambda = 1$ i.e. the leaves are 3-dimensional round spheres.
Theorem 4.2 ($\mathfrak{f} = \mathfrak{sl}_2(\mathbb{R})$)

Let $G$ be a 5-dimensional Riemannian Lie group carrying a left-invariant, minimal and conformal distribution $\mathcal{V}$, generated by the Riemannian subalgebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathfrak{g}$ as above. Then the Lie bracket relations are of the form

\[
[X, Y] = 2\lambda Z, \quad [Z, X] = 2\lambda Y, \quad [Y, Z] = -2X/\lambda.
\]

\[
[X, A] = \lambda^2 x_3 Y + \lambda^2 x_5 Z, \quad [X, B] = \lambda^2 x_4 Y + \lambda^2 x_6 Z,
\]

\[
[Y, A] = x_3 X + z_3 Z, \quad [Y, B] = x_4 X + z_4 Z,
\]

\[
[Z, A] = x_5 X - z_3 Y, \quad [Z, B] = x_6 X - z_4 Y,
\]

\[
[A, B] = rA + \theta_1 X + \theta_2 Y + \theta_3 Z,
\]

where $\theta_1, \theta_2, \theta_3$ are given by

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix}
= \frac{1}{2} \begin{pmatrix}
rz_3/\lambda - \lambda(x_3x_6 - x_4x_5) \\
-\lambda(rx_5 - x_3z_4 + x_4z_3) \\
\lambda(rx_3 + z_4x_5 - z_3x_6)
\end{pmatrix}.
\]

The corresponding foliation $\mathcal{F}$ is Riemannian but not totally geodesic for any values of $\lambda \in \mathbb{R}^+$. 
