KÄHLER STRUCTURES ON GENERALIZED FLAG MANIFOLDS

DAG NILSSON

Master's thesis
2013:E17

LUND UNIVERSITY

Faculty of Science
Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

Abstract

In this Master's thesis we study complex structures and Kähler forms on homogeneous spaces $M = G/K$, where G is a Lie group and K a closed subgroup of G. We show some general results on these manifolds regarding G-invariant complex structures and G-invariant metrics.

In particular we study the case when G is a compact connected semisimple Lie group and K is the centralizer of a torus in G. In this case M is called a generalized flag manifold. We investigate these manifolds by looking at the corresponding root space decompositions, with respect to some Cartan subalgebra.

Our main goal is to show that for each G-invariant complex structure on a generalized flag manifold M there exists a family of G-invariant Kähler metrics on M. For this result we follow [4] by Bordemann, Forger and Römer.

For a more detailed discussion regarding complex structures on homogeneous spaces, we recommend [5] by Borel and Hirzebruch and [13] by Wang.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known

Sammanfattning

Ett homogent rum är en mångfald M tillsammans med en Lie grupp G med egenskapen att givet två punkter $x, y \in M$, så existerar det $g \in G$ så att $g \cdot x = y$. I Proposition 2.6 så visar vi att ett homogent rum också kan skrivas som G/G_x, där $G_x = \{g \in G \mid g \cdot x = y\}$. Vidare så noterar vi att varje mångfald på formen G/K, där K är en sluten delgrupp av G, är ett homogent rum.

En generaliserad flagmångfald är ett homogent rum på formen G/K, där G är en kompakt Lie grupp och

$$K = C(T) = \{ \ g \in G \mid ghg^{-1} = h \text{ för alla } h \in T \ \}$$

där T är en torus i G.

Låt M vara en reell mångfald. En nästan komplex struktur J på M är ett tensorfält sådant att $J_x^2 = -I$ för alla $x \in M$. Med andra ord är J_x en komplex struktur på T_xM för alla $x \in M$. Om den nästan komplexa strukturen kommer från en holomorf struktur på mångfalden så kallas den för en komplex struktur.

Låt M vara en mångfald med nästan komplex struktur J. En Kähler metrik g är en metrik sådan att $g(JX, JY) = g(X, Y)$ och den bilineära formen $\omega(X, Y) = g(JX, Y)$ är sluten.

Låt $\mathfrak{g}$ vara en komplex semienkel Lie algebra och låt $\mathfrak{h}$ vara en Cartan subalgebra. För ett element $\alpha \in \mathfrak{h}^*$ så definierar vi

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}(H)(X) - \alpha(X)I)^n = 0 \text{ för alla } H \in \mathfrak{h}\}$$

och där n är något heltal. Låt R vara mängden av alla element $\alpha \in \mathfrak{h}^*$ sådana att $\mathfrak{g}_\alpha \neq \{0\}$. Mängden R kallas för rotsystemet hörande till $\mathfrak{g}$ med avseende på $\mathfrak{h}$. Vi har då att

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha.$$

För en generaliserad flagmångfald $M = G/K$ så har vi att

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

där $\mathfrak{h}^{\mathbb{C}}$ är en Cartan subalgebra av $\mathfrak{g}^{\mathbb{C}}$ sådan att $\mathfrak{h}^{\mathbb{C}} \subseteq \mathfrak{k}^{\mathbb{C}}$. På grund av detta så existerar det $R_K \subseteq R$ sådant att

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha.$$

Man kan sedan visa att det existerar en bijektion mellan G-invarianta komplexa strukturer på M och speciella delrum av $\mathfrak{g}^{\mathbb{C}}$ som kallas Weylkammare.

Slutligen så kan man visa att varje generaliserad flagmångfald med G-invariant komplex struktur J har en Kähler metrik. För att bevisa detta så definierar vi helt enkelt en metrik på M och visar att denna är Kähler. Konstruktionen av denna metrik kommer att bero på den Weylkammare som motsvarar J.

Acknowledgments

I would like to thank my supervisor Sigmundur Gudmundsson for his advice when writing this thesis and also for suggesting this interesting topic.

Dag Nilsson

Contents

1 Basic Lie Theory **1**
1.1 Root systems . . . 1
1.2 Derivations of Lie Algebras . . . 3

2 Homogeneous Spaces **7**
2.1 Homogeneous Spaces . . . 7
2.2 Reductive Homogeneous Spaces . . . 10

3 Kähler Manifolds **13**
3.1 Complex Vector Spaces . . . 13
3.2 Complex Manifolds . . . 15
3.3 Complex Homogeneous Spaces . . . 16
3.4 Kähler Manifolds . . . 20

4 Generalized Flag Manifolds **23**
4.1 Complex Structures on Generalized Flag Manifolds . . . 23
4.2 Weyl Chambers . . . 31
4.3 The Construction of the Kähler Metrics . . . 34

Appendix:

A Differentiable Manifolds **47**

Bibliography **49**

Chapter 1

Basic Lie Theory

1.1 Root systems

We will in this chapter state some results regarding Lie algebras which will be needed when constructing complex structures on generalized flag manifolds. The presentation given here follows [10]. We begin to state some preliminary definitions.

Definition 1.1. A real (complex) vector space V together with an operation $[,] : V \times V \to V$ is said to be a real (complex) **Lie algebra** if

a) $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$

b) $[X, Y] = -[Y, X]$

c) $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

for all $X, Y, Z \in V$ and $\lambda, \mu \in \mathbb{R}$ $(\mathbb{C})$. The equality c) is called the **Jacobi identity**.

We will denote Lie algebras by Gothic letters $\mathfrak{g}, \mathfrak{h}, \mathfrak{m} \ldots$ and will only consider finite-dimensional Lie algebras.

Let $\mathfrak{g}$ be a Lie algebra. For $j \geq 0$, we define

$$\mathfrak{g}_0 = \mathfrak{g}, \mathfrak{g}_1 = [\mathfrak{g}_0, \mathfrak{g}_0], \ldots, \mathfrak{g}_{j+1} = [\mathfrak{g}, \mathfrak{g}_j].$$

A Lie algebra $\mathfrak{g}$ is said to be **nilpotent** if $\mathfrak{g}_j = 0$ for some j. In a similar way, for $j \geq 0$, define

$$\mathfrak{g}^0 = \mathfrak{g}, \mathfrak{g}^1 = [\mathfrak{g}^0, \mathfrak{g}^0], \ldots, \mathfrak{g}^{j+1} = [\mathfrak{g}^j, \mathfrak{g}^j].$$

A Lie algebra $\mathfrak{g}$ is said to be **solvable** if $\mathfrak{g}^j = 0$ for some j.

A **subalgebra** $\mathfrak{h}$ of $\mathfrak{g}$ is a subspace of $\mathfrak{g}$, such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$. An **ideal** $\mathfrak{h}$ in $\mathfrak{g}$ is a subspace of $\mathfrak{g}$ satisfying $[\mathfrak{h}, \mathfrak{g}] \subseteq \mathfrak{h}$.

A Lie algebra is said to be **semisimple** if it has no non-zero solvable ideals. Next we define $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}_{\mathbb{C}}(\mathfrak{g})$ by

$$\mathrm{ad}(X)(Y) = [X, Y].$$

This is a linear map since the bracket is linear in both its arguments. We then define the symmetric bilinear form B:

$$\mathrm{B}(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

B is called the **Killing form** of $\mathfrak{g}$. There is a way of defining semisimplicity in terms of the Killing form, which is equivalent to the definition given above. We state this as a theorem.

Theorem 1.2. [10] *A Lie algebra $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate.*

Now let $\mathfrak{g}$ be a complex Lie algebra with a nilpotent subalgebra $\mathfrak{h}$. Let $\alpha \in \mathfrak{h}^*$ be an element of the dual of $\mathfrak{h}$. We then define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}(H) - \alpha(H)I)^n X = 0 \quad \text{for all } H \in \mathfrak{h}\},$$

where I denotes the identity mapping of $\mathfrak{g}$ and n is some positive integer depending on both H and X. If $\mathfrak{g}_\alpha \neq 0$, then α is called a **root** of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and $\mathfrak{g}_\alpha$ is called the corresponding **root space**. The elements of $\mathfrak{g}_\alpha$ are called **root vectors**. The following theorem is a key result.

Theorem 1.3. [10] *Let $\mathfrak{g}$ be a complex Lie algebra with a nilpotent subalgebra $\mathfrak{h}$. Then the following holds:*

a) $\mathfrak{g} = \oplus_\alpha \mathfrak{g}_\alpha$

b) $\mathfrak{h} \subseteq \mathfrak{g}_0 = \{X \in \mathfrak{g} \mid ad(H)^n X = 0 \quad \textit{for all } H \in \mathfrak{h}\}$

c) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$*, where* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$ *if* $\alpha + \beta$ *is not a root.*

From c) we see that $\mathfrak{g}_0$ is a subalgebra, since $[\mathfrak{g}_0, \mathfrak{g}_0] \subseteq \mathfrak{g}_{0+0} = \mathfrak{g}_0$.

Definition 1.4. A nilpotent Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a **Cartan subalgebra** if $\mathfrak{h} = \mathfrak{g}_0$.

It can be shown that any complex Lie algebra has a Cartan subalgebra and that Cartan subalgbras are conjugate to each other by an automorphism of $\mathfrak{g}$. In particular this means that every Cartan subalgebra of $\mathfrak{g}$ has the same dimension.

We will be particularly interested in the case when $\mathfrak{g}$ is semisimple. If $\mathfrak{h}$ is a Cartan subalgebra of a complex semisimple Lie algebra, then one can show that $\mathfrak{h}$ is Abelian.

Let $\mathfrak{g}$ be a complex Lie algebra with Cartan subalgebra $\mathfrak{h}$. Let R be the set of nonzero roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set R is called the **root system** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. From Theorem 1.3 we then have the following

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

We call this the **root space decomposition** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. We will usually write this decomposition as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

where the sum is understood to be direct. We now state some additional results regarding roots.

Proposition 1.5. [10] *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and let R be the corresponding root system. Then*

a) If $\alpha, \beta \in R \cup \{0\}$ *and* $\alpha + \beta \neq 0$*, then* $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.

b) If $\alpha \in R \cup \{0\}$*, then* $B(X, \mathfrak{g}_{-\alpha}) \neq 0$ *for all nonzero* X *in* $\mathfrak{g}_\alpha$.

c) If $\alpha \in R$*, then* $-\alpha \in R$.

d) B is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$

e) For each $\alpha \in R$ *there exist an element* $E_\alpha \in \mathfrak{g}_\alpha$ *such that* $ad(H)(E_\alpha) = \alpha(H)E_\alpha$ *for* $H \in \mathfrak{h}$.

f) If $\alpha \in R$*, then* $\dim \mathfrak{g}_\alpha = 1$.

g) If $\alpha, \beta \in R \cup \{0\}$*, such that* $\alpha + \beta \neq 0$*, then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

h) Let $\alpha \in R$*. The only roots proportional to* α *are* $-\alpha, 0$ *and* α.

From e) and f) we see that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid (\mathrm{ad}(H) - \alpha(H)I)X = 0 \quad \text{for all } H \in \mathfrak{h}\},$$

so we need no longer worry about the integer n. From d) we get that for all $\psi \in \mathfrak{h}^*$, there exists a unique $H_\psi \in \mathfrak{h}$ such that $\psi(H) = B(H, H_\psi)$. We let

$$\mathfrak{h}_0 = \mathrm{span}_{\mathbb{R}}\{H_\alpha\}_{\alpha \in R}.$$

One can show that $(\mathfrak{h}_0, \mathrm{B})$ is a real Euclidean vector space and that

$$\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0.$$

Since $\mathrm{B}(X, Y) \in \mathbb{R}$ for all $X, Y \in \mathfrak{h}_0$, we have in particular that $\alpha(H) = \mathrm{B}(H_\alpha, H) \in \mathbb{R}$, for all $H \in \mathfrak{h}_0$. So the roots are real-valued on $\mathfrak{h}_0$.

Since the root spaces are 1-dimensional and $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ for $\alpha + \beta \neq 0$ we define the number $N_{\alpha,\beta}$ such that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$$

and

$$N_{\alpha+\beta} = 0 \text{ if } \alpha + \beta \notin R.$$

The following proposition will be very useful later on.

Proposition 1.6. [10] *For each $\alpha \in R$ the basis vectors $E_\alpha \in \mathfrak{g}_\alpha$ can be chosen such that*

$$B(E_\alpha, E_{-\alpha}) = 1$$
$$[E_\alpha, E_{-\alpha}] = H_\alpha.$$

Next we introduce the notion of positivity on R. The idea is to choose a subset R^+ of R, and saying that α is positive if $\alpha \in R^+$. The set R^+ must satisfy the following criterions:

i) given $\alpha \in R$, exactly one of α, $-\alpha$ is positive.

ii) Any sum of positive roots, which belongs to R, is positive.

A root α is said to be **simple**, if $\alpha > 0$ and there does not exist $\beta, \gamma > 0$ such that $\alpha = \beta + \gamma$. Let Π be the set of simple roots of R. We then have the following result:

Proposition 1.7. [10] *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with the set of nonzero roots R, corresponding to some Cartan subalgebra of $\mathfrak{g}$. The set of simple roots $\Pi = \{\alpha_1, \ldots, \alpha_l\} \subset R$, is a basis of R, such that every $\alpha \in R$ can be written uniquely as $\alpha = \sum_{i=1}^{l} n_i \alpha_i$, where the n_i are integers which are either all non-negative or all non-positive. Moreover, the set of simple roots are linearly independent over $\mathbb{R}$.*

Definition 1.8. A Lie algebra is compact if it is the Lie algebra of a compact Lie group.

We have the following result regarding compact semisimple Lie algebras over $\mathbb{R}$.

Proposition 1.9. [9] *Let $\mathfrak{g}$ be a compact semisimple Lie algebra over $\mathbb{R}$. Then $\mathfrak{g}$ is compact if and only if the Killing form of $\mathfrak{g}$ is strictly negative definite.*

1.2 Derivations of Lie Algebras

A derivation of a Lie algebra is a special kind of endomorphism. The goal of this section is to show that every derivation of a semisimple Lie algebra can be written as $\mathrm{ad}(X)$, where X is an element in the Lie algebra.

Throughout this section $\mathfrak{g}$ will be a (real or complex) finite dimensional Lie algebra.

Definition 1.10. An endomorphism D of $\mathfrak{g}$, which satisfies

$$D([X, Y]) = [D(X), Y] + [X, D(Y)] \quad \text{for all } X, Y \in \mathfrak{g}$$

is called a **derivation** of $\mathfrak{g}$.

Let $\delta(\mathfrak{g})$ be the set of all derivations of $\mathfrak{g}$. Recall the Jacobi identity $[X,[Y,Z]]+[Z,[X,Y]]+[Y,[Z,X]]=0$ for $X,Y,Z\in\mathfrak{g}$. We may rewrite this as

$$[Z,[X,Y]]=[X,[Z,Y]]+[[Z,X],Y]$$

which is equivalent to

$$\mathrm{ad}(Z)[X,Y]=[X,\mathrm{ad}(Z)(Y)]+[\mathrm{ad}(Z)(X),Y]$$

which means that $\mathrm{ad}(Z)$ is a derivation for all $Z\in\mathfrak{g}$. We therefore have that $\mathrm{ad}(\mathfrak{g})\subseteq\delta(\mathfrak{g})$. We would like to show that $\mathrm{ad}(\mathfrak{g})=\delta(\mathfrak{g})$ when $\mathfrak{g}$ is semisimple. We will begin by proving the following lemmas.

Lemma 1.11. [10] *$\delta(\mathfrak{g})$ is a Lie algebra and ad: $\mathfrak{g}\to\delta(\mathfrak{g})$ is a Lie algebra homomorphism.*

Proof. [10] It is easy to see that $\delta(\mathfrak{g})$ is a vector space. So what we need to show is that $[D,E]\in\delta(\mathfrak{g})$ for all $D,E\in\delta(\mathfrak{g})$. Let $D,E\in\delta(\mathfrak{g})$ and $X,Y\in\mathfrak{g}$. Then

$$\begin{aligned}
[D,E]([X,Y]) &= (DE-ED)([X,Y])\\
&= DE([X,Y])-ED([X,Y])\\
&= D([EX,Y]+[X,EY])-E([DX,Y]+[X,DY])\\
&= [DEX,Y]+[EX,DY]+[DX,EY]+[X,DEY]\\
&\quad -[EDX,Y]-[DX,EY]-[EX,DY]-[X,EDY]\\
&= [DEX,Y]-[EDX,Y]+[X,DEY]-[X,EDY]\\
&= [(DE-ED)X,Y]+[X,(DE-ED)Y]\\
&= [[D,E]X,Y]+[X,[D,E]Y]
\end{aligned}$$

which implies that $[D,E]\in\delta(\mathfrak{g})$.

Let $X,Y,Z\in\mathfrak{g}$. Then, using the Jacobi identity, we see that

$$\begin{aligned}
\mathrm{ad}([X,Y])(Z) &= [[X,Y],Z]\\
&= -[[Y,Z],X]-[[Z,X],Y]\\
&= [X,[Y,Z]]-[Y,[X,Z]]\\
&= (\mathrm{ad}(X)\circ\mathrm{ad}(Y))(Z)-(\mathrm{ad}(Y)\circ\mathrm{ad}(X))(Z)\\
&= [\mathrm{ad}(X),\mathrm{ad}(Y)](Z)
\end{aligned}$$

which shows that ad is a Lie algebra homomorphism. □

Lemma 1.12. [9] *Let B be the Killing form of $\mathfrak{g}$ and let σ be an automorphism of $\mathfrak{g}$, then*

$$\begin{aligned}
B(\sigma(X),\sigma(Y)) &= B(X,Y)\\
B([X,Z],Y) &= B(X,[Z,Y])
\end{aligned}$$

for all $X,Y,Z\in\mathfrak{g}$.

Proof. [2],[9] Since σ is an automorphism of $\mathfrak{g}$, we have that $\sigma([X,Y])=[\sigma(X),\sigma(Y)]$ for all $X,Y\in\mathfrak{g}$. Using this we see that

$$\begin{aligned}
\mathrm{ad}(\sigma(X))(Y) &= [\sigma(X),Y]\\
&= \sigma([X,\sigma^{-1}(Y)])\\
&= (\sigma\circ\mathrm{ad}(X)\circ\sigma^{-1})(Y).
\end{aligned}$$

This holds for all $Y\in\mathfrak{g}$, hence $\mathrm{ad}(\sigma(X))=\sigma\circ\mathrm{ad}(X)\circ\sigma^{-1}$. Using this and the definition of the Killing form, we get

$$\mathrm{B}(\sigma(X),\sigma(Y))=\mathrm{Tr}(\mathrm{ad}(\sigma(X))\circ\mathrm{ad}(\sigma(Y)))$$

$$
\begin{aligned}
&= \mathrm{Tr}(\sigma \circ \mathrm{ad}(X) \circ \sigma^{-1} \circ \sigma \circ \mathrm{ad}(Y) \circ \sigma^{-1}) \\
&= \mathrm{Tr}(\sigma \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \sigma^{-1}) \\
&= \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) \\
&= \mathrm{B}(X, Y).
\end{aligned}
$$

To show the second equality we will use the Jacobi identity.

$$
\begin{aligned}
[Z, [X, [Y, W]]] &= [X, [Z, [Y, W]]] + [[Z, X], [Y, W]] \\
&= [X, [Y, [Z, W]] + [[Z, Y], W]] + [[Z, X], [Y, W]] \\
&= [X, [Y, [Z, W]]] + [X, [[Z, Y], W]] + [[Z, X], [Y, W]]
\end{aligned}
$$

from which it follows that

$$
\begin{aligned}
&\mathrm{ad}(Z) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) = \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) \\
&\quad + \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)(Y)) + \mathrm{ad}(\mathrm{ad}(Z)(X)) \circ \mathrm{ad}(Y) \\
&\Leftrightarrow \mathrm{ad}(Z) \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) = \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)(Y)) \\
&\quad + \mathrm{ad}(\mathrm{ad}(Z)(X)) \circ \mathrm{ad}(Y) \\
&\Leftrightarrow [\mathrm{ad}(Z), \mathrm{ad}(X) \circ \mathrm{ad}(Y)] = \mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)(Y)) + \mathrm{ad}(\mathrm{ad}(Z)(X)) \circ \mathrm{ad}(Y).
\end{aligned}
$$

This implies that

$$
\begin{aligned}
&\mathrm{B}(\mathrm{ad}(Z)(X), Y) + \mathrm{B}(X, \mathrm{ad}(Z)(Y)) \\
&= \mathrm{Tr}(\mathrm{ad}(\mathrm{ad}(Z)(X)) \circ \mathrm{ad}(Y)) + \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(\mathrm{ad}(Z)(Y))) \\
&= \mathrm{Tr}([\mathrm{ad}(Z), \mathrm{ad}(X) \circ \mathrm{ad}(Y)]) \\
&= 0.
\end{aligned}
$$

So we get that

$$-\mathrm{B}([Z, X], Y) = \mathrm{B}(X, [Z, Y])$$

or equivalently

$$\mathrm{B}([X, Z], Y) = \mathrm{B}(X, [Z, Y]),$$

which is what we wanted to show. □

Lemma 1.13. [10] *Let V be a finite dimensional vector space and C a nondegenerate bilinear form on $V \times V$. Then the map $\varphi : V \to V^*$ given by $\varphi(u)(v) = C(u, v)$ is bijective.*

Proof. [10] By assumption C is nondegenerate. This implies that

$$
\begin{aligned}
\ker(\varphi) &= \{v \in V \mid \varphi(v)(u) = 0 \text{ for all } u \in V\} \\
&= \{v \in V \mid C(v, u) = 0 \text{ for all } u \in V\} \\
&= \{0\}.
\end{aligned}
$$

So φ is injective and hence bijective, since $\dim(V)$=$\dim(V^*)$. □

We are now ready to prove the main result of this section.

Proposition 1.14. [10] *If $\mathfrak{g}$ is semisimple, then $ad(\mathfrak{g}) = \delta(\mathfrak{g})$.*

Proof. [10] Let $D \in \delta(\mathfrak{g})$. Then

$$
\begin{aligned}
\mathrm{ad}(DX)(Y) &= [DX, Y] \\
&= D[X, Y] - [X, DY] \\
&= (D \circ \mathrm{ad}(X))(Y) - (\mathrm{ad}(X) \circ D)(Y) \\
&= [D, \mathrm{ad}(X)](Y)
\end{aligned}
$$

which implies that $\mathrm{ad}(DX) = [D, \mathrm{ad}(X)]$.

From Lemma 1.13 we know that the map $\varphi : \mathfrak{g} \to \mathfrak{g}^*$ given by $\varphi(X)(Y) = \mathrm{B}(X, Y)$ is bijective, since B is nondegenerate. Given $D \in \delta(\mathfrak{g})$, we let $l(Y) = \mathrm{Tr}(D \circ \mathrm{ad}(Y))$ for $Y \in \mathfrak{g}$. It is clear that $l \in \mathfrak{g}^*$. We know that there exists a unique $X \in \mathfrak{g}$ such that $l = \varphi(X)$. We therefore have that

$$\mathrm{Tr}(D \circ \mathrm{ad}(Y)) = l(Y) = \varphi(X)(Y) = \mathrm{B}(X, Y).$$

Using this we get that for $Y, Z \in \mathfrak{g}$,

$$\begin{aligned}
\mathrm{B}(DY, Z) &= \mathrm{Tr}(\mathrm{ad}(DY) \circ \mathrm{ad}(Y)) \\
&= \mathrm{Tr}([D, \mathrm{ad}(Y)] \circ \mathrm{ad}(Y)) \\
&= \mathrm{Tr}(D \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z) - \mathrm{ad}(Y) \circ D \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}(D \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z)) - \mathrm{Tr}(\mathrm{ad}(Y) \circ D \circ \mathrm{ad}(Z)) \\
&= \mathrm{Tr}(D \circ \mathrm{ad}(Y) \circ \mathrm{ad}(Z)) - \mathrm{Tr}(D \circ \mathrm{ad}(Z) \circ \mathrm{ad}(Y)) \\
&= \mathrm{Tr}(D(\mathrm{ad}(Y) \circ \mathrm{ad}(Z) - \mathrm{ad}(Z) \circ \mathrm{ad}(Y))) \\
&= \mathrm{Tr}(D([\mathrm{ad}(Y), \mathrm{ad}(Z)])) \\
&= \mathrm{Tr}(D \circ \mathrm{ad}([X, Y])) \\
&= \mathrm{B}(X, [Y, Z]) \\
&= \mathrm{B}([X, Y], Z).
\end{aligned}$$

We have shown that $\mathrm{B}(DY, Z) = \mathrm{B}(\mathrm{ad}(X)(Y), Z)$ for all $Y, Z \in \mathfrak{g}$. Since B is nondegenerate this implies that $DY = \mathrm{ad}(X)(Y)$ for all $Y \in \mathfrak{g}$ or equivalently, $D = \mathrm{ad}(X) \in \mathrm{ad}(\mathfrak{g})$. It is then clear that $\delta(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$, since we already knew that $\mathrm{ad}(\mathfrak{g}) \subseteq \delta(\mathfrak{g})$. $\square$

Chapter 2

Homogeneous Spaces

In this chapter, we introduce a special class of manifolds called homogeneous spaces. We then state and prove several important results regarding these manifolds. In general we use the same notation as in [8] throughout this text. We assume that the manifolds are Hausdorff and second countable.

2.1 Homogeneous Spaces

We begin by introducing the coset manifolds and the action of a Lie group on a manifold. We shall see that the two are essentially the same. Let G be a Lie group and K be a closed subgroup of G.

Definition 2.1. The set

$$G/K = \{gK : g \in G\}$$

is called a **coset space**. The map

$$\pi : G \to G/K, \quad \pi(g) = gK.$$

is called the **natural projection**.

We will use the following topology on the coset spaces.

Definition 2.2. The topology on G/K which is uniquely determined by the condition that π should be open and continuous, is called the **natural topology**.

It is possible to equip G/K with a manifold structure, in fact we have the following result:

Theorem 2.3. [14] *Let G be a Lie group and K be a closed subgroup of G. Then G/K has a unique manifold structure such that the projection π is a smooth submersion.*

For the moment we call G/K, together with the above differentiable structure, a **coset manifold**.

Definition 2.4. Let M be a differentiable manifold and G be a Lie group. A smooth map $\alpha : G \times M \to M$ is a **group action** of G on M if

$$\alpha(e, x) = x, \text{ where } e \text{ is the identity element in G}$$
$$\alpha(g_1 g_2, x) = \alpha(g_1, \alpha(g_2, x))$$

for all $g_1, g_2 \in G$ and $x \in M$.

Definition 2.5. A group action α of G on M is said to be **transitive**, if for any $x, y \in M$ there exist a $g \in G$ such that $\alpha(g, x) = y$.

The set $G_x = \{g \in G : \alpha(g, x) = x\}$ is called the **isotropy subgroup** at $x \in M$. The **orbit** of a point $x \in M$ is the set $\alpha(G, x) = \{\alpha(g, x) : g \in G\}$.

On a coset manifold G/K we can define an action α of G in the following natural way:

$$\alpha : G \times G/K \to G/K \quad \text{with} \quad \alpha(g, g_1K) = gg_1K$$

This is a transitive action, since given $g_1K, g_2K \in G/K$ we may take $g = g_2g_1^{-1}$, and so

$$\alpha(g_2g_1^{-1}, g_1K) = g_2K.$$

Before we write the definition of a homogeneous space, we state the following result.

Theorem 2.6. [14] *Let α be a transitive action of a Lie group G on a differentiable manifold M and let $x \in M$. Then the isotropy group G_x is a closed subgroup of G and the map $\overline{\beta}$ defined by*

$$\overline{\beta} : G/G_x \to M \quad \textit{with} \quad \overline{\beta}(gG_x) = \alpha(g, x)$$

is a diffeomorphism.

Proof. [14] Since M is Hausdorff, the singleton sets are closed in M. Let $\alpha_x(g) = \alpha(g, x)$. It is easy to see that

$$\alpha_x^{-1}(\{x\}) = G_x.$$

It then follows from the inverse function theorem that G_x is a submanifold of G. In particular α_x is a continuous map from G to M, so $\alpha_x^{-1}(\{x\}) = G_x$ is a closed subset of G.

Let $g, h \in G_x$. Then

$$\alpha(gh, x) = \alpha(g, \alpha(h, x)) = \alpha(g, x) = x$$

so $gh \in G_x$. Clearly $e \in G_x$ and for any $g \in G_x$

$$\begin{aligned} x &= \alpha(e, x) \\ &= \alpha(g^{-1}g, x) \\ &= \alpha(g^{-1}, \alpha(g, x)) \\ &= \alpha(g^{-1}, x) \end{aligned}$$

which implies that $g^{-1} \in G_x$. Hence G_x is a closed subgroup of G.

We need to show that the map $\overline{\beta}$ is well defined. Let $gG_x = hG_x$. Then $h = gg_1$ for some $g_1 \in G_x$. So

$$\begin{aligned} \overline{\beta}(hG_x) &= \alpha(h, x) \\ &= \alpha(gg_1, x) \\ &= \alpha(g, \alpha(g_1, x)) \\ &= \alpha(g, x) \\ &= \overline{\beta}(gG_x) \end{aligned}$$

We would now like to show that $\overline{\beta}$ is a diffeomorphism. By Lemma A.1 it is enough to show that $\overline{\beta}$ is smooth, bijective and everywhere non-singular. Let us begin by showing that $\overline{\beta}$ is bijective.

Given any $y \in M$ there exists $g \in G$ such that $\alpha(g, x) = y$ since the action α is transitive. Then $\overline{\beta}(gG_x) = \alpha(g, x) = y$ which implies that $\overline{\beta}$ is surjective.

Suppose $\overline{\beta}(gG_x) = \overline{\beta}(hG_x)$ for some $g, h \in G$. This is equivalent to $\alpha(g, x) = \alpha(h, x)$. We therefore get that

$$\begin{aligned} \overline{\beta}(h^{-1}gG_x) &= \alpha(h^{-1}g, x) \\ &= \alpha(h^{-1}, \alpha(g, x)) \\ &= \alpha(h^{-1}, \alpha(h, x)) \end{aligned}$$

$$
\begin{aligned}
&= \alpha(e,x)\\
&= x
\end{aligned}
$$

which means that $h^{-1}g \in G_x$ or equivalently $gG_x = hG_x$. Hence $\overline{\beta}$ is injective.

Since $\overline{\beta}$ is smooth if and only if $\overline{\beta}\circ\pi$ is smooth we let $\beta = \overline{\beta}\circ\pi$ and consider β instead of $\overline{\beta}$. For any $g \in G$ we have

$$\beta(g) = \overline{\beta}(\pi(g)) = \alpha(g,x) = (\alpha \circ i_x)(g),$$

where i_x is the smooth map from G into $G \times M$ defined by $i_x(g) = (g,x)$. Since α and i_x are both smooth it follows that their composition is smooth, hence β is smooth.

What is left to show is that $\overline{\beta}$ is nonsingular. To do this we show that $d\overline{\beta}_{gG_x}$ has kernel $\{0\}$ for all $gG_x \in G/G_x$. We again consider β. Suppose $\ker(d\beta_g) = \ker(d\pi_g)$. Then $\ker(d\overline{\beta}_{gG_x}) = \{0\}$. If this was not the case there would exist a nonzero element $X_{gG_x} \in T_{gG_x}G$ such that $d\overline{\beta}_{gG_x}(X_{gG_x}) = 0$. Since π is a smooth submersion, there exists $X_g \in T_gG$ such that $d\pi_g(X_g) = X_{gG_x}$. Clearly $X_g \notin \ker(d\pi_g)$ since $X_{gG_x} \neq 0$. But

$$(d\beta)_g(X_g) = (d\overline{\beta})_{gG_x}(X_{gG_x}) = 0$$

and this contradicts the fact that $X_g \notin \ker(d\pi_g)$, since we assumed $\ker(d\beta_g) = \ker(d\pi_g)$. We must therefore have that $\ker(d\overline{\beta}_{gG_x}) = \{0\}$. From this we see that it is sufficient to show that $\ker(d\beta_g) = \ker(d\pi_g)$ for all $g \in G$. Later on in this section we will see that the kernel of $d\pi_e$ is T_eG_x and from this it is easy to see that $\ker(d\pi_g) = T_g gG_x$. So to show that $\overline{\beta}$ is nonsingular we will show $\ker(d\beta_g) = T_g gG_x$ for all $g \in G$.

Let $\alpha_g(y) = \alpha(g,y)$. We then see that

$$
\begin{aligned}
(\alpha_g \circ \beta \circ L_{g^{-1}})(g) &= \alpha_g(\beta(e))\\
&= \alpha_g(\overline{\beta}(G_x))\\
&= \alpha(g, \alpha(e,x))\\
&= \alpha(g,x)\\
&= \beta(g),
\end{aligned}
$$

where L_g denotes the left translation. This holds for all $g \in G$, hence $\beta = \alpha_g \circ \beta \circ L_{g^{-1}}$. From this we get that

$$d\beta_g = (d\alpha_g)_x \circ d\beta_e \circ (dL_{g^{-1}})_g. \tag{2.1}$$

Suppose we know that $\ker(d\beta_e) = T_eG_x$. Then it follows easily from (2.1) that the kernel of $d\beta_g$ is $T_g gG_x$. It is therefore sufficient to show that $\ker(d\beta_e) = T_eG_x$.

It is clear that $T_eG_x \subset \ker(d\beta_g)$ since $d\beta = d\overline{\beta}\circ d\pi$. Let $X_e \in \ker(d\beta_e) \subseteq T_eG$. Let $X \in \mathfrak{g}$ be the left-invariant vector field which corresponds to X_e. Our problem then translates into showing that $X \in \mathfrak{g}_x$, where $\mathfrak{g}_x$ is the Lie algebra of G_x. This is equivalent to showing that $\exp(tX) \in G_x$ for all $t \in \mathbb{R}$, where $\exp(tX)$ is the one parameter subgroup generated by X. If we can show that the curve

$$t :\longrightarrow \beta(\exp(tX))$$

is constant then we would be done, because $\beta(\exp(0)) = \beta(e) = x$ so we would have $\beta(\exp(tX)) = x$ for all $t \in \mathbb{R}$. From this we then get that $\exp(tX) \in G_x$ for all $t \in \mathbb{R}$. To show that the curve is constant we consider the tangent vector at t. Using (2.1), the left-invariance of X and the fact that $X_e \in \ker(d\beta_e)$ we obtain

$$
\begin{aligned}
d\beta_{\exp(tX)}(X_{\exp(tX)}) &= ((d\alpha_{\exp(tX)})_x \circ d\beta_e \circ dL_{\exp(-tX)})(X_{\exp(tX)})\\
&= ((d\alpha_{\exp(tX)})_x \circ d\beta_e(X_e))\\
&= 0.
\end{aligned}
$$

Since this holds for all $t \in \mathbb{R}$ it follows that the curve $\beta(\exp(tX))$ is constant. By what we have shown this implies that $\overline{\beta}$ is non-singular. $\square$

Definition 2.7. A **homogeneous space** is a differentiable manifold M together with a transitive action of a Lie group G.

From the discussion before Proposition 2.6 we saw that every coset manifold can be equipped with a transitive action. This means that every coset manifold can be seen as a homogeneous space. Moreover, from Proposition 2.6, we see that a homogeneous space is diffeomorphic to a coset manifold. So equivalently we can define a homogeneous space to be a coset manifold i.e. a manifold of the form G/K, where G is a Lie group and K a closed subgroup of G.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. We now consider the tangent space at $o = \pi(e)$, where e is the identity element of G. For $X \in \mathfrak{g}$, we have that

$$d\pi_e(X) = \frac{d}{dt}(\pi(\exp(tX)))|_{t=0} = \frac{d}{dt}(\exp(tX)K)|_{t=0},$$

where $\exp(tX)$ is the one-parameter subgroup generated by X. If $X \in \mathfrak{k}$ then $\exp(tX) \in K$, for all $t \in \mathbb{R}$. From the above calculation it follows that $\mathfrak{k} \subseteq \ker(d\pi_e)$. In fact $\mathfrak{k} = \ker(d\pi_e)$, since $\exp(tX) \in K$ implies $X \in \mathfrak{k}$.

From Theorem 2.3 we know that π is a submersion, which means that $d\pi_g$ is onto for all $g \in G$. So $d\pi_e(\mathfrak{g}) = T_o(G/K)$. This gives us the following isomorphism,

$$\mathfrak{g}/\mathfrak{k} \cong T_o(G/K).$$

We may choose $\mathfrak{m} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

Here $\mathfrak{m}$ is a vector subspace of $\mathfrak{g}$ and not necessarily a subalgebra. Since $d\pi_e : \mathfrak{g} \to T_oM$ has kernel $\mathfrak{k}$, it follows that $d\pi_e : \mathfrak{m} \to T_oM$ is an isomorphism.

For a fixed $g \in G$ we define I_g by,

$$I_g : G \to G \quad I_g(x) = gxg^{-1}.$$

It is easily seen that I_g is a homomorphism, moreover if we let L_g, R_g denote left and right translation, we see that $I_g = R_{-g} \circ L_g$. This implies that I_g is a diffeomorphism, since left and right translation are diffeomorphisms.

Definition 2.8. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the map

$$\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g}) \text{ with } \mathrm{Ad}(g) = (dI_g)_e$$

is called the **adjoint representation** of G.

Since $I_{gh} = I_g \circ I_h$, we have that $\mathrm{Ad}(gh) = (dI_{gh})_e = (dI_g)_e \circ (dI_h)_e = \mathrm{Ad}(g) \circ Ad(h)$. So Ad really is a representation of G.

In Chapter 1 we defined $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$, by $\mathrm{ad}(X)(Y) = [X, Y]$. One can in fact show that $\mathrm{ad}(X) = (d\mathrm{Ad})_e(X)$.

2.2 Reductive Homogeneous Spaces

In this section we show that there is a one to one correspondence between certain metrics on a reductive homogeneous space G/K and certain scalar products on $\mathfrak{m}$.

Let $M = G/K$ be a homogeneous space. On M we define $\tau_g : G/K \to G/K$

$$\tau_g(hK) = ghK, \quad hK \in G/K, g \in G.$$

Recall the left translation $L_g : G \to G$, $L_g(h) = gh$. We note that

$$(\pi \circ L_g)(h) = \pi(gh) = ghK = \tau_g(hK) = (\tau_g \circ \pi)(h),$$

which implies that for all $g, h \in G$, and all $X \in T_hG$,

$$((d\pi)_{gh} \circ (dL_g)_h)(X) = ((d\tau_g)_{hK} \circ (d\pi)_h)(X).$$

We will have use of the following lemma.

Lemma 2.9. *Let $M = G/K$ and let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Then*

$$d\pi_e(Ad(k)(X)) = (d\tau_k)_o(d\pi_e(X))$$

for all $k \in K$ and $X \in \mathfrak{m}$.

Proof. We know that $I_g = L_g \circ R_{g^{-1}}$. For $k \in K$ and $g \in G$, we have

$$(\pi \circ R_{k^{-1}})(g) = \pi(gk^{-1}) = gk^{-1}K = gK = \pi(g)$$

which implies that for $X \in \mathfrak{g}$,

$$(d\pi_{k^{-1}} \circ (dR_{k^{-1}})_e)(X) = d(\pi \circ R_{k^{-1}})_e(X) = d\pi_e(X).$$

So for $X \in \mathfrak{m}$,

$$\begin{aligned} d\pi_e(\mathrm{Ad}(k)(X)) &= d\pi_e((dI_k)_e(X)) \\ &= d\pi_e(((dL_k)_{k^{-1}} \circ (dR_{k^{-1}})_e)(X)) \\ &= (d\tau_k)_o((d\pi_{k^{-1}} \circ (dR_{k^{-1}})_e)(X)) \\ &= (d\tau_k)_o(d\pi_e(X)). \end{aligned}$$

□

Definition 2.10. A homogeneous space G/K is said to be **reductive**, if there exists a subspace $\mathfrak{m}$ of $\mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \tag{2.2}$$

and $\mathrm{Ad}(k)(\mathfrak{m}) \subset \mathfrak{m}$ for all $k \in K$.

Definition 2.11. Let $M = G/K$ be a homogeneous space. A metric g on M is said to be G**-invariant** if

$$g_{\tau_g(hK)}((d\tau_g)_{hK}X_{hK}, (d\tau_g)_{hK}Y_{hK}) = g_{hK}(X_{hK}, Y_{hK})$$

for all $hK \in M$, $g \in G$ and $X_{hK}, Y_{hK} \in T_{hK}M$.

Proposition 2.12. [15] *Let $M = G/K$ be a reductive homogeneous space with a reductive decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

Then there is a bijection between G-invariant metrics on M and $Ad(K)$-invariant inner products on $\mathfrak{m}$.

Proof. [15] Let g be a G-invariant metric on M. We define an inner product $\langle , \rangle$ on $\mathfrak{m}$ by

$$\langle X, Y \rangle = g_o(d\pi_e(X), d\pi_e(Y))$$

for $X, Y \in \mathfrak{m}$. We show that $\langle , \rangle$ is $\mathrm{Ad}(K)$-invariant, i.e.

$$\langle \mathrm{Ad}(k)(X), \mathrm{Ad}(k)(Y) \rangle = \langle X, Y \rangle \quad \text{for all } k \in K \text{ and } X, Y \in \mathfrak{m}.$$

We have that

$$\begin{aligned} \langle \mathrm{Ad}(k)(X), \mathrm{Ad}(k)(Y) \rangle &= g_o(d\pi_e(\mathrm{Ad}(k)(X)), d\pi_e(\mathrm{Ad}(k)(Y))) \\ &= g_o((d\tau_k)_o(d\pi_e(X)), (d\tau_k)_o(d\pi_e(Y))) \\ &= g_o(d\pi_e(X), d\pi_e(Y)) \\ &= \langle X, Y \rangle \end{aligned}$$

where we have used Lemma 2.9 and the G-invariance of g. This shows that $\langle , \rangle$ is $\mathrm{Ad}(K)$-invariant.

Suppose now instead that we are given an $\mathrm{Ad}(K)$-invariant inner product $\langle , \rangle$ on $\mathfrak{m}$. Since $\tau_g : M \to M$ is a diffeomorphism, it follows that $(d\tau_g)_o : T_oM \to T_{gK}M$ is an isomorphism. So given $X_{gK}, Y_{gK} \in T_{gK}M$ there exist unique $X_0, Y_0 \in T_oM$ such that

$$(d\tau_g)_o(X_0) = X_{gK},\ (d\tau_g)_o(Y_0) = Y_{gK}.$$

Moreover, we know that $d\pi_e : \mathfrak{m} \to T_oM$ is an isomorphism, so there exist unique $X, Y \in \mathfrak{m}$ such that

$$d\pi_e(X) = X_0,\ d\pi_e(Y) = Y_0.$$

We define

$$g_{gK}(X_{gK}, Y_{gK}) = \langle X, Y \rangle.$$

We need to show that g is well-defined. We may choose another representative h of gK such that $hK = gK$. This implies that $h = gk$ for some $k \in K$. For $g_1K \in M$,

$$\begin{aligned}
\tau_h(g_1K) &= \tau_{gk}(g_1K) \\
&= gkg_1K \\
&= \tau_g(kg_1K) \\
&= (\tau_g \circ \tau_k)(g_1K),
\end{aligned}$$

from which it follows that $\tau_h = \tau_g \circ \tau_k$. As before we know that there exist unique $X_0', Y_0' \in T_oM$ such that

$$\begin{aligned}
(d\tau_h)_o(X_0') = X_{hK} = X_{gK} = (d\tau_g)_o(X_0) \\
(d\tau_h)_o(Y_0') = Y_{hK} = Y_{gK} = (d\tau_g)_o(Y_0).
\end{aligned}$$

Using this and the fact that $\tau_h = \tau_g \circ \tau_k$, we get that $(d\tau_g)_o(X_0) = (d\tau_h)_o(X_0') = (d\tau_g)_o((d\tau_k)_o(X_0'))$ and since $(d\tau_g)_o$ is an isomorphism, we must have that

$$(d\tau_k)_o(X_0') = X_0$$

and in the same way

$$(d\tau_k)_o(Y_0') = Y_0.$$

Also in the same way as before, there exist unique $X', Y' \in \mathfrak{m}$ such that

$$d\pi_e(X') = X_0',\ d\pi_e(Y') = Y_0'.$$

Using the above equalities, we see that

$$\begin{aligned}
X' &= (d\pi_e|_{\mathfrak{m}})^{-1}(X_0') \\
&= (d\pi_e|_{\mathfrak{m}})^{-1}((d\tau_{k^{-1}})_o(X_0)) \\
&= (d\pi_e|_{\mathfrak{m}})^{-1}((d\tau_{k^{-1}})_o(d\pi_e(X))) \\
&= \mathrm{Ad}(k^{-1})(X)
\end{aligned}$$

where we used Lemma 2.9 in the last equality. In the same way we have that $Y' = \mathrm{Ad}(k^{-1})(Y)$. So

$$\begin{aligned}
g_{hK}(X_{hK}, Y_{hK}) &= \langle X', Y' \rangle \\
&= \langle \mathrm{Ad}(k^{-1})X, \mathrm{Ad}(k^{-1})Y \rangle \\
&= \langle X, Y \rangle \\
&= g_{gK}(X_{gK}, Y_{gK}).
\end{aligned}$$

which shows that g is well-defined. By construction g is G-invariant. $\square$

Chapter 3

Kähler Manifolds

In this chapter we introduce the notion of complex structures, first on vector spaces and then on manifolds. In particular we consider the case when the manifold is a homogeneous space.

3.1 Complex Vector Spaces

We will here go through some of the theory for complex structures on vector spaces.

Let V be a finite-dimensional real vector space. A linear map $J : V \to V$ such that

$$J^2 = -I,$$

where I is the identity mapping of V, is called a complex structure on V. The space V together with J can be turned into a complex vector space, by defining a multiplication on V, by complex numbers, in the following way:

$$(a + bi)X = aX + bJX \quad \text{for all} \quad a, b \in \mathbb{R} \text{ and } X \in V.$$

Let $\{e_1, e_2, \ldots, e_n\}$ be a complex basis of V, with the multiplication defined as above. Then

$$(a_1 + ib_1)e_1 + (a_2 + ib_2)e_2 + \ldots + (a_n + ib_n)e_n = 0$$

if and only if $a_i = b_i = 0$ for all $i = 1, 2, \ldots, n$. But by definition

$$\sum_{i=1}^{n}(a_i + ib_i)e_i = \sum_{i=1}^{n} a_i e_i + b_i J(e_i)$$

which implies that the vectors

$$\{e_1, e_2, \ldots, e_n, J(e_1), J(e_2), \ldots, J(e_n)\}$$

are linearly independent over $\mathbb{R}$. It is clear that these vectors also span V, since we assumed that $\{e_1, e_2, \ldots, e_n\}$ was a complex basis of V. This means that

$$\{e_1, e_2, \ldots, e_n, J(e_1), J(e_2), \ldots, J(e_n)\}$$

is a real basis of V. So if it is possible to define a complex structure on a real vector space V, then V must necessarily be of even dimension.

Suppose now instead that V is a complex vector space of dimension n. Then we define $JX = iX$ for $X \in V$. The space V can be considered as a real vector space of dimension $2n$. If we consider V in this way it is clear that J is a complex structure on V.

For a real vector space V we let the **complexification** $V^{\mathbb{C}}$ of V be given by

$$V^{\mathbb{C}} = \{X + iY \mid X, Y \in V\}.$$

If J is a complex structure on V we may extend J to $V^{\mathbb{C}}$ by

$$J(X+iY) = JX + iJY.$$

It is clear that $J^2 = -I$ on $V^{\mathbb{C}}$ as well.

Definition 3.1. Let V be a real vector space with a complex structure J. We then define

$$V^{1,0} = \{Z \in V^{\mathbb{C}} \mid JZ = iZ\}$$
$$V^{0,1} = \{Z \in V^{\mathbb{C}} \mid JZ = -iZ\}.$$

Proposition 3.2. [16] *Let V be a real vector space with complex structure J. Then*

i) $V^{1,0} = \{X - iJX \mid X \in V\}$ *and* $V^{0,1} = \{X + iJX \mid X \in V\}$.

ii) $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$

iii) $\overline{V^{1,0}} = V^{0,1}$

Proof. [16] Let $Z = X + iY \in V^{\mathbb{C}}$. Consider the eigenvalue equation,

$$\begin{aligned} & JZ = iZ \\ \Leftrightarrow\ & JX + iJY = iX - Y \\ \Leftrightarrow\ & JX + Y + i(JY - X) = 0 \\ \Leftrightarrow\ & Y = -JX. \end{aligned}$$

From the above calculation we see that $JZ = iZ$ if and only if $Z = X - iJX$. In a similar way way see that $JZ = -iZ$ if and only if $Z = X + iJX$. It then follows that

$$V^{1,0} = \{X - iJX \mid X \in V\}$$
$$V^{0,1} = \{X + iJX \mid X \in V\}.$$

It is easily seen that $\overline{V^{1,0}} = V^{0,1}$.

Now let $Z = X + iY$ be an arbitrary element in $V^{\mathbb{C}}$. Let

$$Z_1 = \frac{1}{2}(X - iJX + i(Y - iJY))$$
$$Z_2 = \frac{1}{2}(i(Y + iJY) + X + iJX).$$

We see that $Z_1 \in V^{1,0}$ and $Z_2 \in V^{0,1}$, and

$$Z_1 + Z_2 = \frac{1}{2}(2X + 2iY + JY - JY - iJX + iJX) = X + iY = Z.$$

So every element in $V^{\mathbb{C}}$ can be written as a sum of two elements, one from $V^{1,0}$ and one from $V^{0,1}$. Suppose that $Z \in V^{1,0} \cap V^{0,1}$, so $Z = X - iJX$, and $Z = Y + iJY$ for some $X, Y \in V$. This implies

$$\begin{aligned} & X - iJX = Y - iJY \\ \Leftrightarrow\ & Y - X + iJ(Y + X) = 0 \\ \Leftrightarrow\ & Y = X,\ Y = -X \\ \Leftrightarrow\ & Y = X = 0. \end{aligned}$$

Hence $V^{1,0} \cap V^{0,1} = \{0\}$. □

We have the natural correspondence between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$,

$$(x_1 + iy_1, \ldots, x_n + iy_n) \to (x_1, \ldots, x_n, y_1, \ldots, y_n).$$

Multiplication by i in $\mathbb{C}^n$, $i(x_1 + iy_1, \ldots, x_n + iy_n) = (ix_1 - y_1, \ldots, ix_n - y_n)$, corresponds to the linear map $J_n : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$,

$$J_n = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$$

where I_n is the identity mapping on $\mathbb{R}^n$. The map J_n is called the canonical complex structure on $\mathbb{R}^{2n}$. Multiplication by i considered as an operator on $\mathbb{C}^n$, will also be denoted by J_n.

3.2 Complex Manifolds

We would now like to transfer the notion of complex structures to the manifold setting. We do this in the following way.

Definition 3.3. Let M be a real differentiable manifold. An **almost complex structure** J on M is a tensor field

$$J : C^\infty(TM) \to C^\infty(TM)$$

such that $J_x^2 = -I$, for all $x \in M$.

As before we see that a manifold M together with an almost complex structure must be of even dimension. The pair (M, J) is called an **almost complex manifold**. We also make the following definition.

Definition 3.4. Let M be a topological manifold with atlas $\{(U_\alpha, z^\alpha) \mid \alpha \in I\}$, where I is some index set, and $z^\alpha(U_\alpha) = V_\alpha$ is an open set in $\mathbb{C}^n$. M is called a **complex manifold** if

$$z^\alpha \circ (z^\beta)^{-1} : z^\beta(U_\alpha \cap U_\beta) \to \mathbb{C}^n,$$

are holomorphic functions, for all $\alpha, \beta \in I$. The collection of all charts (z_α, U_α), is called a holomorphic structure.

Suppose that we are given a complex manifold M of dimension n. Let $z^\alpha = (z_1^\alpha, \dots, z_n^\alpha)$ be a chart. Then, for each $1 \leq k \leq n$, we have $z_k^\alpha = x_k^\alpha + iy_k^\alpha$, where $x_k^\alpha, y_k^\alpha : U_\alpha \to \mathbb{R}$. The pair $(\phi^\alpha = (x_1^\alpha, \dots, x_n^\alpha, y_1^\alpha, \dots y_n^\alpha), U_\alpha)$ is a real chart. This means that any complex manifold with dimension n can be seen as a real manifold with dimension $2n$.

We now show that a complex manifold carries a natural almost complex structure. Let $x \in M$, and choose a neighborhood U_α such that $x \in U_\alpha$. For $X \in T_xM$, we define

$$J_x^\alpha(X) = d((z^\alpha)^{-1} \circ J_n \circ z^\alpha)_x(X).$$

One can show that this definition is independent of the choice of neighborhood U_α. It is clear that J defined in this way is an almost complex structure.

Now let (M, J) be an almost complex manifold. We say that J is a **complex structure** if J comes from a holomorphic structure as described above. For an almost complex manifold (M, J), we define the **Nijenhuis tensor** N of J by,

$$N(X, Y) = 2([JX, JY] - [X, Y] - J[X, JY] - J[JX, Y]), \quad X, Y \in C^\infty(TM).$$

The following theorem, which we state without proof, will be useful when determining whether or not J is a complex structure on M.

Theorem 3.5. *An almost complex structure J on a manifold M, is a complex structure on M, if and only if $N \equiv 0$.*

A proof of this theorem, for the special case when M is real analytic, can be found in [11].

We continue to let (M, J) be an almost complex manifold. Let $x \in M$. Since T_xM is a vector space and J_x is a complex structure on T_xM, we have

$$T_x^{\mathbb{C}}M = T_x^{1,0}M \oplus T_x^{0,1}M,$$

where

$$T_x^{1,0}M = \{Z \in T_x^{\mathbb{C}}M \mid J_xZ = iZ\}$$
$$T_x^{0,1}M = \{Z \in T_x^{\mathbb{C}}M \mid J_xZ = -iZ\}.$$

A complex vector field on M is a vector field of the form $Z = X + iY$, where $X, Y \in C^\infty(TM)$. A complex vector field Z is said to be of type $(1, 0)$ if $Z_x \in T_x^{1,0}M$, for all $x \in M$. From Section 3.1 we see that Z is of type $(1, 0)$ if and only if $Z = X - iJX$ for some vector field $X \in C^\infty(TM)$. Similarly, we say that a vector field Z is of type $(0, 1)$, if $Z_x \in T_x^{0,1}M$, for all $x \in M$. In the same way as before we see that Z is of type $(0, 1)$ if and only if $Z = X + iJX$, for some $X \in C^\infty(TM)$. It is clear that conjugation takes vector fields of type $(1, 0)$ into vector fields of type $(0, 1)$ and vice versa.

Proposition 3.6. [16] *Let (M, J) be an almost complex manifold. Then the following are equivalent:*

a) If Z, W is of type $(1,0)$, then so is $[Z, W]$.

b) If Z, W is of type $(0,1)$, then so is $[Z, W]$.

c) J is a complex structure on M.

Proof. [16] We first show that *a*) is equivalent to *b*). Let $Z = X_1 + iY_1$ and $W = X_2 + iY_2$, where $X_1, X_2, Y_1, Y_2 \in C^\infty(TM)$. It is then clear that

$$[Z, W] = ([X_1, X_2] - [Y_1, Y_2]) + i([X_1, Y_2] + [Y_1, X_2]),$$

so

$$\begin{aligned}\overline{[Z, W]} &= ([X_1, X_2] - [Y_1, Y_2]) - i([X_1, Y_2] + [Y_1, X_2]) \\ &= [X_1 - iY_1, X_2 - iY_2] \\ &= [\overline{Z}, \overline{W}].\end{aligned}$$

Now assume that *a*) holds, and that Z, W is of type $(1,0)$. By assumption $[Z, W]$ is of type $(1,0)$, which implies that $\overline{[Z, W]}$ is of type $(0,1)$. We just saw that $\overline{[Z, W]} = [\overline{Z}, \overline{W}]$, which then implies that $[\overline{Z}, \overline{W}]$ is of type $(0,1)$ for all Z, W of type $(1,0)$.

Given any two complex vector fields Z^*, W^* of type $(0,1)$, we know that $Z = \overline{Z^*}, W = \overline{W^*}$ is of type $(1,0)$. By our previous argument $[\overline{Z}, \overline{W}]$ is of type $(0,1)$. But $\overline{Z} = Z^*, \overline{W} = W^*$, which means that $[Z^*, W^*]$ is of type $(0,1)$ for all vector fields Z^*, W^* of type $(0,1)$, which is exactly *b*). By symmetry we get that *a*) is equivalent to *b*).

Recall that every vector field of type $(1,0)$ is of the form $X - iJX$, for some $X \in C^\infty(TM)$. Let

$$Z = [X - iJX, Y - iJY].$$

We see that *a*) holds if and only if Z is of type $(1,0)$, for all $X, Y \in C^\infty(TM)$. That Z is of type $(1,0)$ means in particular that $JZ = iZ$, which is equivalent to $Z + iJZ = 0$, and

$$\begin{aligned}Z + iJZ &= [X - iJX, Y - iJY] + iJ([X - iJX, Y - iJY]) \\ &= [X, Y] - i[X, JY] - i[JX, Y] - [JX, JY] \\ &\quad + iJ[X, Y] + J[X, JY] + J[JX, Y] - iJ[JX, JY] \\ &= -([JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]) \\ &\quad - iJ([JX, JY] - J[X, JY] - J[JX, Y] - [X, Y]) \\ &= -\frac{N(X, Y)}{2} - \frac{iJN(X, Y)}{2}.\end{aligned}$$

From this calculation we see that Z is of type $(1,0)$ if and only if $N(X, Y) = 0$. From our previous discussion this means that *a*) holds if and only if $N \equiv 0$. From Theorem 3.5 we know that J is a complex structure on M if and only if $N \equiv 0$, which then implies the equivalence of *a*) and *c*). □

3.3 Complex Homogeneous Spaces

In this section we will specialize to the case when M is a homogeneous space. Let $M = G/K$, where G is a Lie group, and K is a closed subgroup of G. We choose $\mathfrak{m} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}.$$

Let J be an almost complex structure on M. We say that J is G-**invariant** if

$$(J_{\tau_g(hK)} \circ (d\tau_g)_{hK})(X_{hK}) = ((d\tau_g)_{hK} \circ J_{hK})(X_{hK})$$

for all $g, h \in G$ and all $X_{hK} \in T_{hK}M$. We have the following result:

Proposition 3.7. [11] *There is a bijection between the set of G-invariant almost complex structures on M and the set of linear endomorphisms I_0 on $\mathfrak{m}$ satisfying*

(1) $I_0^2 = -I$

(2) $I_0((Ad(k)(X))_{\mathfrak{m}}) = (Ad(k)(I_0X))_{\mathfrak{m}}$ *for all* $k \in K$ *and* $X \in \mathfrak{m}$.

The subscript $\mathfrak{m}$ *denotes projection on the subspace* $\mathfrak{m}$.

Proof. [11], [15] Recall that $d\pi_e : \mathfrak{m} \to T_oM$ is a vector space isomorphism. We let $\psi = d\pi_e|_{\mathfrak{m}}$. Suppose first that we are given a G-invariant almost complex structure J on M. We then know that $J_o : T_oM \to T_oM$ is a linear endomorphism satisfying $J_o^2 = -I$. We define, for $X \in \mathfrak{m}$,

$$I_0(X) = (\psi^{-1} \circ J_o \circ d\pi_e)(X).$$

We see directly that

$$I_0^2(X) = (\psi^{-1} \circ J_o^2 \circ d\pi_e)(X) = -X,$$

hence I_0 satisfies (1). Using Lemma 2.9 and the G-invariance of J, we see that

$$\begin{aligned} I_0((\mathrm{Ad}(k)X)_{\mathfrak{m}}) &= (\psi^{-1} \circ J_o \circ d\pi_e)((\mathrm{Ad}(k)(X))_{\mathfrak{m}}) \\ &= (\psi^{-1} \circ J_o)(d\pi_e(\mathrm{Ad}(k)(X))) \\ &= (\psi^{-1} \circ J_o \circ (d\tau_k)_o \circ d\pi_e)(X) \\ &= ((\psi^{-1} \circ (d\tau_k)_o \circ J_o \circ d\pi_e)(X) \\ &= (\psi^{-1} \circ (d\tau_k)_o \circ d\pi_e \circ I_0)(X)) \\ &= (\psi^{-1} \circ d\pi_e \circ \mathrm{Ad}(k) \circ I_0)(X) \\ &= (\mathrm{Ad}(k)(I_0X))_{\mathfrak{m}}. \end{aligned}$$

So the existence of J on M implies the existence of I_0 on $\mathfrak{m}$ satisfying (1) and (2).

To prove the converse, let I_0 be as in the statement of the proposition. For $gK \in M$ we define

$$\begin{aligned} &J_{gK} : T_{gK}M \to T_{gK}M \\ &J_{gK}(X_{gK}) = ((d\tau_g)_o \circ d\pi_e \circ I_0 \circ \psi^{-1})(X_0), \end{aligned}$$

where $X_0 \in T_oM$ satisfies $X_{gK} = (d\tau_g)_o(X_0)$. Since τ_g is a diffeomorphism, $(d\tau_g)_o$ is a vector space isomorphism and X_0 is therefore uniquely determined.

We need to show that J_{gK} is independent of the choice of representative of the coset gK. We therefore suppose $gK = hK$, for some $h \in G$. This implies that $h = gk$ for some $k \in K$. Let Y_0 be the tangent vector such that $(d\tau_h)_o(Y_0) = X_{hK}$. Since $\tau_h = \tau_g \circ \tau_k$, we get that $(d\tau_h)_o = (d\tau_g)_o \circ (d\tau_k)_o$, so

$$X_{gK} = ((d\tau_g)_o \circ (d\tau_k)_o)(Y_0).$$

Since $X_{gK} = (d\tau_g)_o(X_0)$ and $(d\tau_g)_o$ is an isomorphism, we must have that $X_0 = (d\tau_k)_o(Y_0)$. Using Lemma 2.9, (2) and the above calculation, we see that

$$\begin{aligned} J_{hK}(X_{hK}) &= ((d\tau_h)_o \circ d\pi_e \circ I_0 \circ \psi^{-1})(Y_0) \\ &= ((d\tau_g)_o \circ (d\tau_k)_o \circ d\pi_e \circ I_0 \circ \psi^{-1})(Y_0) \\ &= ((d\tau_g)_o \circ d\pi_e \circ \mathrm{Ad}(k) \circ I_0 \circ \psi^{-1})(Y_0) \\ &= ((d\tau_g)_o \circ d\pi_e)((\mathrm{Ad}(k)(I_0(\psi^{-1}(Y_0)))_{\mathfrak{m}}) \\ &= ((d\tau_g)_o \circ d\pi_e \circ I_0)(\mathrm{Ad}(k)(\psi^{-1}(Y_0))_{\mathfrak{m}}) \\ &= ((d\tau_g)_o \circ d\pi_e \circ I_0 \circ \psi^{-1})((d\tau_k)_o(Y_0)) \\ &= ((d\tau_g)_o \circ d\pi_e \circ I_0 \circ \psi^{-1})(X_0) \\ &= J_{gK}(X_{gK}). \end{aligned}$$

It then follows that J is well defined. We see directly that

$$J_{gK}^2(X_{gK}) = ((d\tau_g)_o \circ J_o)(J_o(X_0)) = -(d\tau_g)_o(X_0) = -X_{gK}.$$

So J is an almost complex structure on M, and it is clear from the construction that J is G-invariant. □

When $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition we immediately get the following result from Proposition 3.7.

Proposition 3.8. [11] *If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ is a reductive decomposition there is a one-to-one correspondence between G-invariant almost complex structures on M and linear endomorphisms I_0 on $\mathfrak{m}$ satisfying*

(1) $I_0^2 = -I$

(2) $I_0(Ad(k)(X)) = (Ad(k)(I_0 X))$ *for all* $k \in K$ *and* $X \in \mathfrak{m}$.

Proof. [11] We may remove the projection on the subspace $\mathfrak{m}$, since $\mathrm{Ad}(k)(X) \in \mathfrak{m}$ for all $k \in K$ and $X \in \mathfrak{m}$ in a reductive space. □

Let $I_0 : \mathfrak{m} \to \mathfrak{m}$ satisfy (1) and (2). We may extend I_0 to $\mathfrak{g}$ by defining

$$\begin{aligned} &\widetilde{I}_0 : \mathfrak{g} \to \mathfrak{g} \\ &\widetilde{I}_0(X+Y) = I_0(Y) \quad \text{for all } Z = X+Y \in \mathfrak{g},\ X \in \mathfrak{k},\ Y \in \mathfrak{m} \end{aligned}$$

For $Z = X + Y \in \mathfrak{g}$ we have

$$\widetilde{I}_0^2(Z) = I_0^2(Y) = -Y = -Z + X = -Z \bmod \mathfrak{k}.$$

For $k \in K$ and $X \in \mathfrak{k}$, $\mathrm{Ad}(k)(X) \in \mathfrak{k}$, so

$$\begin{aligned} \widetilde{I}_0(\mathrm{Ad}(k)(Z)) &= \widetilde{I}_0(\mathrm{Ad}(k)(X)) + \widetilde{I}_0(\mathrm{Ad}(k)(Y)) \\ &= \widetilde{I}_0(\mathrm{Ad}(k)(Y)) \\ &= \widetilde{I}_0((\mathrm{Ad}(k)(Y))_{\mathfrak{k}} + (\mathrm{Ad}(k)(Y))_{\mathfrak{m}}) \\ &= I_0((\mathrm{Ad}(k)(Y))_{\mathfrak{m}}) \\ &= (\mathrm{Ad}(k)(I_0(Y)))_{\mathfrak{m}} \\ &= (\mathrm{Ad}(k)(\widetilde{I}_0(Z)))_{\mathfrak{m}} \\ &= (\mathrm{Ad}(k)(\widetilde{I}_0(Z)))_{\mathfrak{k}} + (\mathrm{Ad}(k)(\widetilde{I}_0(Z)))_{\mathfrak{m}} - (\mathrm{Ad}(k)(\widetilde{I}_0(Z)))_{\mathfrak{k}} \\ &= (\mathrm{Ad}(k)(\widetilde{I}_0(Z))) \bmod \mathfrak{k}. \end{aligned}$$

Hence $\widetilde{I}_0$ is a linear endomorphism of $\mathfrak{g}$ which satisfies

(a) $\widetilde{I}_0(X) = 0$,

(b) $\widetilde{I}_0^2(Z) = -Z \bmod \mathfrak{k}$,

(c) $\widetilde{I}_0(\mathrm{Ad}(k)(Z)) = \mathrm{Ad}(k)(\widetilde{I}_0(Z)) \bmod \mathfrak{k}$

for all $X \in \mathfrak{k}$, $k \in K$ and all $Z \in \mathfrak{g}$. If $\widetilde{I}_0$ and $\widetilde{I}_0^*$ are two linear endomorphisms of $\mathfrak{g}$ satisfying (a),(b) and (c), then we say that $\widetilde{I}_0 = \widetilde{I}_0^*$ if $\widetilde{I}_0 = \widetilde{I}_0^* \bmod \mathfrak{k}$. We may now state and prove the following result:

Proposition 3.9. [11] *There is a one to one correspondence between the G-invariant complex structures on M, and the linear endomorphisms of $\mathfrak{g}$ satisfying (a),(b) and (c).*

Proof. [11] Given an almost complex structure J on M, we know from Proposition 3.7 that there exists a unique linear endomorphism I_0 of $\mathfrak{m}$ satisfying (1) and (2). In the discussion above we constructed a linear endomorphism $\widetilde{I}_0$ of $\mathfrak{g}$, which satisfied (a),(b) and (c). It is clear that this correspondence is injective.

Conversely, suppose that we are given a linear endomorphism $\widetilde{I}_0$ of $\mathfrak{g}$ satisfying (a),(b) and (c). We then define, for $X \in \mathfrak{m}$, $I_0(X) = (\widetilde{I}_0(X))_{\mathfrak{m}}$. It is easy to see that I_0 satisfies (1) and (2). By Proposition 3.7 I_0 gives rise to a unique G-invariant complex structure on M. □

Let J be a G-invariant almost complex structure on M. We would like to determine whether or not J is a complex structure by considering the corresponding linear endomorphism $\widetilde{I}_0$. The following theorem will allow us to do this.

Theorem 3.10. [11] *A G-invariant almost complex structure J on M is a complex structure if and only if*

$$[\widetilde{I}_0 X, \widetilde{I}_0 Y] - [X,Y] - \widetilde{I}_0[X, \widetilde{I}_0 Y] - \widetilde{I}_0[\widetilde{I}_0 X, Y] \in \mathfrak{k},$$

for all $X, Y \in \mathfrak{g}$.

Proof. [11] Let $U \in C^\infty(TG)$. We define the tensor field $\widehat{J}$ in the following way:

$$\widehat{J}_g(U_g) = ((dL_g)_e \circ \widetilde{I}_0 \circ (dL_g)_e^{-1})(U_g).$$

We see that for $h \in G$

$$\begin{aligned}
\widehat{J}_{hg}((dL_h)_g(U_g)) &= ((dL_{hg})_e \circ \widetilde{I}_0 \circ (dL_{hg})_e^{-1})((dL_h)_g(U_g)) \\
&= ((dL_{hg})_e \circ \widetilde{I}_0 \circ (dL_{g^{-1}h^{-1}})_{hg} \circ (dL_h)_g)(U_g) \\
&= ((dL_{hg})_e \circ \widetilde{I}_0 \circ (d(L_{g^{-1}h^{-1}} \circ L_h))_e)(U_g) \\
&= ((dL_{hg})_e \circ \widetilde{I}_0 \circ (dL_{g^{-1}})_g)(U_g) \\
&= (dL_h)_g((dL_g)_e \circ \widetilde{I}_0 \circ (dL_g)_e^{-1})(U_g) \\
&= (dL_h)(\widehat{J}_g(U_g)).
\end{aligned}$$

This means that $\widehat{J}$ is left invariant i.e.

$$(dL_h) \circ \widehat{J} = \widehat{J} \circ (dL_h)$$

for all $h \in G$.

We say that a vector field $U \in C^\infty(TG)$ is **projectable** if there exists $U' \in C^\infty(TM)$ such that

$$d\pi_g(U_g) = U'_{\pi(g)} \quad \text{for all } g \in G,$$

we write this as $d\pi(U) = U'$. Let $P(G)$ be the set of all projectable vector fields. We shall show that $P(G)$ is a subalgebra of $C^\infty(TG)$. Let $U, V \in P(G)$. Then

$$\begin{aligned}
d\pi(U+V) &= d\pi(U) + d\pi(V) \\
d\pi(cU) &= c d\pi(U) \quad c \in \mathbb{R},
\end{aligned}$$

so $U + V, cU \in P(G)$ for all $U, V \in P(G)$ and $c \in \mathbb{R}$. This means that $P(G)$ is a vector subspace of $C^\infty(TG)$. Moreover,

$$d\pi([U,V]) = [d\pi(U), d\pi(V)]$$

so $[U,V] \in P(G)$ for all $U, V \in P(G)$, which shows that $P(G)$ is a subalgebra of $C^\infty(TG)$.

We now want to show that if $U \in P(G)$ then $\widehat{J}(U) \in P(G)$. Let $U_g \in T_gG$ and let $W_e = (dL_g)_e^{-1}(U_g)$. Then

$$\begin{aligned}
d\pi_g(\widehat{J}_g(U_g)) &= d\pi_g(\widehat{J}_g((dL_g)_e(W_e))) \\
&= (d\pi_g \circ (dL_g)_e \circ \widehat{J}_e)(W_e) \\
&= ((d\tau_g)_o \circ d\pi_e \circ \widetilde{I}_0)(W_e) \\
&= ((d\tau_g)_o \circ J_o \circ d\pi_e)(W_e) \\
&= (J_{gK} \circ (d\tau_g)_o \circ d\pi_e)(W_e) \\
&= (J_{gK} \circ d\pi_g \circ (dL_g)_e)(W_e) \\
&= J_{gK}(d\pi_g(U_g)),
\end{aligned}$$

where we used the definition of $\widetilde{I}_0$ and the fact that $\widehat{J}_e = \widetilde{I}_0$. So if $U \in P(G)$, then $d\pi(\widehat{J}(U)) = J(d\pi(U))$, which implies that $\widehat{J}(U) \in P(G)$.

We now define the tensor field

$$\widehat{N}(U,V) = [\widehat{J}(U), \widehat{J}(V)] + \widehat{J}^2([U,V]) - \widehat{J}([U, \widehat{J}(V)]) - \widehat{J}([\widehat{J}(U), V])$$

for $U, V \in C^\infty(TG)$. What we have shown so far implies that $\widehat{N}(U,V) \in P(G)$ if $U, V \in P(G)$, and also that $\widehat{N}$ is a left invariant tensor field. Applying $d\pi$ to $\widehat{N}(U,V)$, where $U, V \in P(G)$ gives us the following:

$$\begin{aligned} d\pi(\widehat{N}(U,V)) &= d\pi([\widehat{J}(U), \widehat{J}(V)] + \widehat{J}^2([U,V]) - \widehat{J}([U, \widehat{J}(V)]) - \widehat{J}([\widehat{J}(U), V])) \\ &= [J(d\pi(U)), J(d\pi(V))] + J^2([d\pi(U), d\pi(V)]) \\ &- J([d\pi(U), J(d\pi(V))]) - J([J(d\pi(U)), d\pi(V)]) \\ &= \frac{1}{2} N(d\pi(U), d\pi(V)) \end{aligned}$$

Note that the map $d\pi : P(G) \to C^\infty(TM)$ is surjective, so

$$\begin{aligned} J \text{ complex structure on } M \ &\Leftrightarrow \\ N(U', V') = 0 \quad \text{for all } U', V' \in C^\infty(TM) &\Leftrightarrow \\ N(d\pi(U), d\pi(V)) = 0 \quad \text{for all } U, V \in P(G) &\Leftrightarrow \\ d\pi(\widehat{N}(U,V)) = 0 \quad \text{for all } U, V \in P(G). \end{aligned}$$

Since $\widehat{N}$ is left-invariant the last statement is equivalent to

$$d\pi_e(\widehat{N}(X,Y)) = 0$$

for all $X, Y \in \mathfrak{g}$ which is equivalent to $\widehat{N}(X,Y) \in \mathfrak{k}$, for all $X, Y \in \mathfrak{g}$. For left invariant vector fields X, Y, we have that

$$\widehat{N}(X,Y) = [\widetilde{I}_0 X, \widetilde{I}_0 Y] - [X,Y] - \widetilde{I}_0[X, \widetilde{I}_0 Y] - \widetilde{I}_0[\widetilde{I}_0 X, Y],$$

which concludes the proof of this theorem. □

3.4 Kähler Manifolds

So far we have not considered Riemannian metrics on complex manifolds. We will in this section introduce Riemannian metrics with some special properties with respect to the complex structure. In particular we will consider the case when the manifold is a homogeneous space.

Definition 3.11. Let M be an almost complex manifold with an almost complex structure J. A Riemannian metric g on M is said to be **Hermitian** if

$$g(JX, JY) = g(X,Y)$$

for all $X, Y \in C^\infty(TM)$. An almost complex manifold with a Hermitian metric is called an almost Hermitian manifold, and a complex manifold with a Hermitian metric is called a **Hermitian manifold**.

Definition 3.12. Let M be an almost Hermitian manifold with almost complex structure J and Hermitian metric g. The alternating 2-form ω defined by

$$\omega(X,Y) = g(JX, Y)$$

is called the **Kähler form** of g. If ω is closed, that is if $d\omega = 0$, where d denotes the exterior derivative, then g is called a **Kähler metric** and a complex manifold equipped with a Kähler metric is called a **Kähler manifold**.

Let $M = G/K$ be a reductive homogeneous space with a G-invariant almost complex structure J and let g be a G-invariant metric on M. Let I_0 be the $\mathrm{Ad}(K)$-invariant complex structure on $\mathfrak{m}$ which corresponds to J, and let $\langle , \rangle$ be the $\mathrm{Ad}(K)$-invariant metric in $\mathfrak{m}$ which corresponds to g. We then have the following proposition.

Proposition 3.13. *The metric g is Hermitian with respect to J if and only if $\langle I_0 X, I_0 Y\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{m}$.*

Proof. Suppose first that g is Hermitian with respect to J. So $g(JU, JV) = g(U, V)$ for all $U, V \in C^\infty(TM)$. From the definition of $\langle , \rangle$ we have that

$$\langle I_0 X, I_0 Y\rangle = g_o(d\pi_e(I_0 X), d\pi_e(I_0 Y))$$

and I_0 is defined by

$$I_0 = \psi^{-1} \circ J_o \circ d\pi_e,$$

where $\psi = d\pi_e|_{\mathfrak{m}}$. Using this we get that

$$\begin{aligned}\langle I_0 X, I_0 Y\rangle &= g_o(d\pi_e(I_0 X), d\pi_e(I_0 Y)) \\ &= g_o(J_o(d\pi_e(X)), J_o(d\pi_e(Y))) \\ &= g_o(d\pi_e(X), d\pi_e(Y)) \\ &= \langle X, Y\rangle.\end{aligned}$$

Suppose now instead that $\langle I_0 X, I_0 Y\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{m}$. Recall from the proof of Proposition 2.12 that $g_{gK}(X_{gK}, Y_{gK}) = \langle X, Y\rangle$, where $X, Y \in \mathfrak{m}$ such that

$$X_{gK} = (d\tau_g)_o(d\pi_e(X)),\ Y_{gK} = (d\tau_g)_o(d\pi_e(Y)).$$

Using this and the G-invariance of J, we see that

$$\begin{aligned}J_{gK}(X_{gK}) &= (J_{gK} \circ (d\tau_g)_o \circ d\pi_e)(X) \\ &= ((d\tau_g)_o \circ J_o)(d\pi_e(X)) \\ &= (d\tau_g)_o(d\pi_e(I_0(X)))\end{aligned}$$

and in the same way, we have that

$$J_{gK}(Y_{gK}) = (d\tau_g)_o(d\pi_e(I_0(Y))).$$

So

$$g_{gK}(J_{gK}(X_{gK}), J_{gK}(Y_{gK})) = \langle I_0(X), I_0(Y)\rangle = \langle X, Y\rangle = g_{gK}(X_{gK}, Y_{gK}),$$

which shows that g is Hermitian with respect to J.

□

Definition 3.14. On a reductive homogeneous space with G-invariant almost complex structure J and Hermitian G-invariant metric g we define the bilinear form ω_0 on $\mathfrak{m}$ by

$$\omega_0(X, Y) = \langle I_0(X), Y\rangle.$$

The following Lemma will be of importance later on.

Lemma 3.15. *An $Ad(K)$ invariant scalar product on $\mathfrak{m}$ satisfies*

$$\langle [Z, X], Y\rangle = \langle X, [Y, Z]\rangle$$

for all $X, Y \in \mathfrak{m}$ and $Z \in \mathfrak{k}$.

Proof.

$$\begin{aligned}\langle [Z, X], Y\rangle &= \langle \mathrm{ad}(Z)(X), Y\rangle \\ &= \langle \frac{d}{dt}(\mathrm{Ad}(\exp(tZ))(X))|_{t=0}, Y\rangle \\ &= \frac{d}{dt}(\langle \mathrm{Ad}(\exp(tZ))(X), Y\rangle)|_{t=0} \\ &= \frac{d}{dt}(\langle X, \mathrm{Ad}(-\exp(tZ))(Y)\rangle)|_{t=0}\end{aligned}$$

$$
\begin{aligned}
&= \langle X, \frac{d}{dt}(\mathrm{Ad}(-\exp(tZ))(Y))|_{t=0}\rangle \\
&= \langle X, -\mathrm{ad}(Z)(Y)\rangle \\
&= \langle X, [Y, Z]\rangle
\end{aligned}
$$

□

Corollary 3.16. *The bilinear form ω_0 satisfies*

$$\omega_0(X, [Y, Z]) = -\omega_0(Y, [Z, X])$$

for all $X, Y \in \mathfrak{m}$ and all $Z \in \mathfrak{k}$.

Proof. It is easy to see from the $\mathrm{Ad}(K)$-invariance of I_0, that $I_0(\mathrm{ad}(Z)(X)) = \mathrm{ad}(Z)(I_0(X))$ for all $Z \in \mathfrak{k}$ and $X \in \mathfrak{m}$. We therefore have that

$$
\begin{aligned}
\omega_0(X, [Y, Z]) &= \langle I_0(X), [Y, Z]\rangle \\
&= \langle [Z, I_0(X)], Y\rangle \\
&= \langle I_0([Z, X]), Y\rangle \\
&= -\langle I_0(Y), [Z, X]\rangle \\
&= -\omega_0(Y, [Z, X]).
\end{aligned}
$$

□

Finally we have the following result, which we will state without proof.

Proposition 3.17. [4] *The 2-form ω is closed if and only if*

$$\omega_0([X, Y]_{\mathfrak{m}}, Z) + \omega_0([Y, Z]_{\mathfrak{m}}, X) + \omega_0([Z, X]_{\mathfrak{m}}, Y) = 0$$

for all $X, Y, Z \in \mathfrak{m}$.

Chapter 4

Generalized Flag Manifolds

We now introduce a special kind of homogeneous spaces, namely the generalized flag manifolds. The additional structure of these manifolds allows us to use the theory gathered in Chapter 1 about semisimple Lie algebras. The root space decomposition in particular will be of great importance throughout this chapter.

4.1 Complex Structures on Generalized Flag Manifolds

In this section we show that there is a one-to-one correspondence between G-invariant complex structures on a generalized flag manifold and certain subsets of the corresponding root system R.

A Lie group G is said to be semisimple if its Lie algebra is semisimple. Recall that a **torus** in a Lie group G is a subgroup T such that $T \cong S^1 \times S^1 \times \ldots \times S^1$. A torus $\widehat{T}$ is said to be maximal in G if for any torus T in G such that $\widehat{T} \subset T \subset G$ we have $\widehat{T} = T$.

Definition 4.1. A **flag manifold** is a homogeneous space of the form $G/\widehat{T}$, where G is a compact Lie group and $\widehat{T}$ is a maximal torus in G.

Definition 4.2. A **generalized flag manifold** is a homogeneous space of the form G/K, where G is a compact Lie group, and K is the centralizer of a torus T in G i.e.

$$K = C(T) = \{g \in G \mid ghg^{-1} = h \quad \text{for all } h \in T\}.$$

We state the following result.

Proposition 4.3. [6] *Let G be compact Lie group and T a torus in G. Then $C(T)$ is a closed subgroup of G. If G is connected then $C(T)$ is also connected.*

By Proposition 4.3, $C(T)$ is a closed subgroup of G, so $G/C(T)$ really is a homogeneous space. Note that $C(T) = T$ when T is a maximal torus in G, so a flag manifold is a special case of a generalized flag manifold.

Throughout this text we will make two more assumptions on G, namely that G should be connected and semisimple.

Let G/K be a generalized flag manifold and $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G and K, respectively. We will now use the theory gathered in Chapter 1 to obtain a root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ and $\mathfrak{k}^{\mathbb{C}}$.

Let $\widehat{T}$ be a maximal torus in G which contains T. Since $\widehat{T}$ is an abelian subgroup of G, $T \subset C(T)$. Let $\mathfrak{h}$ be the Lie algebra of $\widehat{T}$. Then $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. We therefore obtain a root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha,$$

where R is the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. Since $\widehat{T} \subset K$ we have that $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$. Because of this, we obtain a root space decomposition of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$:

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha,$$

where R_K is the root system of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$.

We start by considering a more general case. We follow the outline given in [12]. Let $M = G/K$ be a homogeneous space, so G is a Lie group and K a closed subgroup of G. As usual we let $\mathfrak{g}, \mathfrak{k}$ denote the Lie algebras of G,K respectively. We also let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ which satisfies $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Furthermore we suppose that there exists a G-invariant complex structure J on M. From Chapter 3 we know that J gives rise to the decomposition

$$T_o^{\mathbb{C}}M = T_o^{1,0}M \oplus T_o^{0,1}M,$$

where

$$T_o^{1,0}M = \{X \in T_o^{\mathbb{C}}M \mid JX = iX\}, \; T_o^{0,1}M = \{X \in T_o^{\mathbb{C}}M \mid JX = -iX\}.$$

We extend the map $d\pi_e$ to $\mathfrak{g}^{\mathbb{C}}$ in the following way: For $Z = X + iY \in \mathfrak{g}^{\mathbb{C}}$,

$$d\pi_e(Z) = d\pi_e(X) + id\pi_e(Y).$$

Clearly $d\pi_e(\mathfrak{g}^{\mathbb{C}}) = T_o^{\mathbb{C}}M$ and it is easy to see that the kernel of this extension is $\mathfrak{k}^{\mathbb{C}}$. We define

$$\mathfrak{a}^+ = (d\pi_e)^{-1}(T_o^{1,0}M), \; \mathfrak{a}^- = (d\pi_e)^{-1}(T_o^{0,1}M).$$

Let $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ be the conjugation with respect to $\mathfrak{g}$. Since $T_o^{\mathbb{C}}M = T_o^{1,0}M \oplus T_o^{0,1}M$, $\overline{T_o^{1,0}M} = T_o^{0,1}M$ and $d\pi_e(\mathfrak{g}^{\mathbb{C}}) = T_o^{\mathbb{C}}M$, it follows that $\theta(\mathfrak{a}^+) = \mathfrak{a}^-$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^+ + \theta(\mathfrak{a}^+)$. Moreover, since $\ker(d\pi_e) = \mathfrak{k}^{\mathbb{C}}$, it also follows that

$$\mathfrak{a}^+ \cap \theta(\mathfrak{a}^+) = \{Z \in \mathfrak{g}^{\mathbb{C}} \mid d\pi_e(Z) \in T_o^{1,0}M \cap T_o^{0,1}M\} = \ker(d\pi_e) = \mathfrak{k}^{\mathbb{C}}.$$

Next we define subspaces $\mathfrak{m}^+$ and $\mathfrak{m}^-$ of $\mathfrak{g}^{\mathbb{C}}$ in the following way:

$$\mathfrak{m}^+ = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{a}^+, \; \mathfrak{m}^- = \mathfrak{m}^{\mathbb{C}} \cap \mathfrak{a}^-.$$

It is then easy to see that

$$\mathfrak{a}^+ = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+, \; \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-, \; \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-.$$

We know that $d\pi_e : \mathfrak{m}^{\mathbb{C}} \to T_o^{\mathbb{C}}M$ is an isomorphism, since the kernel of $d\pi_e$ is $\mathfrak{k}^{\mathbb{C}}$. From this and the definition of $\mathfrak{a}^{\pm}$, it follows that $\mathfrak{m}^+ \simeq T_o^{1,0}M$, $\mathfrak{m}^- \simeq T_o^{0,1}M$ in the same way.

Recall that J gives rise to a unique linear map $I_0 : \mathfrak{m} \to \mathfrak{m}$ such that

$$\begin{aligned} &I_0^2(X) = -X \quad \text{for all } X \in \mathfrak{m}, \\ &I_0((\mathrm{Ad}(k)(X))_{\mathfrak{m}}) = (\mathrm{Ad}(k)(I_0(X)))_{\mathfrak{m}} \quad \text{for all } X \in \mathfrak{m} \text{ and all } \; k \in K. \end{aligned}$$

Lemma 4.4. *The subspaces $\mathfrak{m}^{\pm}$ satisfies*

$$\mathfrak{m}^{\pm} = \{Z \in \mathfrak{m}^{\mathbb{C}} \mid I_0(Z) = \pm iZ \; \}.$$

Proof. We know that

$$\mathfrak{a}^+ = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+.$$

By definition

$$I_0(Z) = ((d\pi_e|_{\mathfrak{m}})^{-1} \circ J_o \circ d\pi_e)(Z)$$

so for $Z \in \mathfrak{m}^+$ we have that

$$I_0(Z) = ((d\pi_e|_{\mathfrak{m}})^{-1} \circ J_o \circ d\pi_e)(Z) = (d\pi_e|_{\mathfrak{m}})^{-1}(id\pi_e(Z)) = iZ,$$

where we used that $d\pi_e(\mathfrak{m}^+) = T_o^{1,0}M$. So

$$\mathfrak{m}^+ \subseteq \{Z \in \mathfrak{m}^{\mathbb{C}} \mid I_0(Z) = iZ \; \}.$$

Conversely, suppose $Z \in \mathfrak{m}^{\mathbb{C}}$ satisfies $I_0(Z) = iZ$. Then

$$iZ = I_0(Z) = ((d\pi_e|_{\mathfrak{m}})^{-1} \circ J_o \circ d\pi_e)(Z)$$

which implies

$$id\pi_e(Z) = J_o(d\pi_e(Z))$$

and this is equivalent to $d\pi_e(Z) \in T_o^{1,0}M$. Clearly this means that $Z \in \mathfrak{a}^+$. By assumption $Z \in \mathfrak{m}^{\mathbb{C}}$, so

$$Z \in \mathfrak{a}^+ \cap \mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+.$$

Hence

$$\{Z \in \mathfrak{m}^{\mathbb{C}} \mid I_0(Z) = iZ \ \} \subseteq \mathfrak{m}^+$$

which together with our previous result implies

$$\mathfrak{m}^+ = \{Z \in \mathfrak{m}^{\mathbb{C}} \ |I_0(Z) = iZ\}.$$

Performing similar calculations for $\mathfrak{a}^-, \mathfrak{m}^-$ gives us

$$\mathfrak{m}^- = \{Z \in \mathfrak{m}^{\mathbb{C}} \ |I_0(Z) = -iZ\}$$

□

We may now present the following lemma:

Lemma 4.5. [12] *The subspace $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Moreover, the correspondence between the set of G-invariant complex structures on M and the set of subspaces of $\mathfrak{g}^{\mathbb{C}}$, given by $\mathfrak{a}^+ = (d\pi_e)^{-1}(T_o^{1,0}M)$ is injective.*

Proof. Suppose J' is another G-invariant complex structure on M with corresponding decomposition

$$T_o^{\mathbb{C}}M = T_o^{1,0}M' \oplus T_o^{0,1}M',$$

such that $\mathfrak{a}^+ = (d\pi_e)^{-1}(T_o^{1,0}M')$. Then

$$(d\pi_e)^{-1}(T_o^{1,0}M) = (d\pi_e)^{-1}(T_o^{1,0}M').$$

Let $X_o \in T_o^{1,0}M'$. Since $d\pi_e$ is onto there exists $X \in \mathfrak{g}^{\mathbb{C}}$ such that $d\pi_e(X) = X_o$. By definition $X \in (d\pi_e)^{-1}(T_o^{1,0}M')$. But we knew that $(d\pi_e)^{-1}(T_o^{1,0}M) = (d\pi_e)^{-1}(T_o^{1,0}M')$, so $X \in (d\pi_e)^{-1}(T_o^{1,0}M)$ which implies that $X_o = d\pi_e(X) \in T_o^{1,0}M$. This means that $T_o^{1,0}M' \subseteq T_o^{1,0}M$. The opposite inclusion is shown in the same way, hence $T_o^{1,0}M' = T_o^{1,0}M$. It follows from this that $J_o = J'_o$ and since both J and J' are G-invariant, we must in fact have that $J = J'$.

Let $X, Y \in \mathfrak{a}^+$. We define vector fields $X', Y' \in C^\infty(TM)$ in the following way:

$$X'_{gK} = ((d\tau_g)_o \circ d\pi_e)(X), \ Y'_{gK} = ((d\tau_g)_o \circ d\pi_e)(Y).$$

Since $X, Y \in \mathfrak{a}^+$ it follows that $d\pi_e(X), d\pi_e(Y) \in T_o^{1,0}M$. Using this and the G-invariance of J, we see that

$$\begin{aligned} J_{gK}(X'_{gK}) &= (J_{gK} \circ (d\tau_g)_o)(d\pi_e(X)) \\ &= ((d\tau_g)_o \circ J_o)(d\pi_e(X)) \\ &= (d\tau_g)_o(id\pi_e(X)) \\ &= iX'_{gK}. \end{aligned}$$

Similarly we see that

$$J_{gK}(Y'_{gK}) = iY'_{gK}.$$

This holds for all $g \in G$, which means that X' and Y' are of type (1,0). We also see that

$$d\pi_g(X_g) = (d\pi_g \circ (dL_g)_e)(X_e) = ((d\tau_g)_o \circ d\pi_e)(X) = X'_{gK} = X'_{\pi(g)}$$

or equivalently that $d\pi(X) = X'$. In the same way we see that $d\pi(Y) = Y'$. This implies that

$$d\pi([X, Y]) = [d\pi(X), d\pi(Y)] = [X', Y'].$$

In particular we have that

$$d\pi_e([X, Y]) = [X', Y']_o \in T_o^{1,0}M,$$

where we used that J is a complex structure together with Proposition 3.6. This shows that $[X, Y] \in \mathfrak{a}^+$ and this holds for all $X, Y \in \mathfrak{a}^+$, which implies that $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. □

We have now shown that the existence of a complex G-invariant structure on M implies the existence of a unique subalgebra $\mathfrak{a}^+$ satisfying

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^+ + \theta(\mathfrak{a}^+), \quad \mathfrak{a}^+ \cap \theta(\mathfrak{a}^+) = \mathfrak{k}^{\mathbb{C}}. \tag{4.1}$$

Lemma 4.6. [12] *Let $\mathfrak{a}^+ \subset \mathfrak{g}^{\mathbb{C}}$ be a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ which satisfy (4.1). Then there exists a G-invariant complex structure J on M such that $\mathfrak{a}^+ = (d\pi_e)^{-1}(T_o^{1,0}M)$.*

Proof. Let

$$\begin{aligned} T_o^{1,0}M &= d\pi_e(\mathfrak{a}^+) \\ T_o^{0,1}M &= d\pi_e(\mathfrak{a}^-) = d\pi_e(\theta(\mathfrak{a}^+)) = \overline{d\pi_e(\mathfrak{a}^+)} = \overline{T_o^{1,0}M}. \end{aligned}$$

We see that

$$\begin{aligned} T_o^{\mathbb{C}}M &= d\pi_e(\mathfrak{g}^{\mathbb{C}}) \\ &= d\pi_e(\mathfrak{a}^+ + \mathfrak{a}^-) \\ &= d\pi_e(\mathfrak{a}^+) + d\pi_e(\mathfrak{a}^-) \\ &= T_o^{1,0}M + T_o^{0,1}M. \end{aligned}$$

Suppose that $Z \in T_oM^+ \cap T_oM^-$. Then

$$Z = d\pi_e(X) = d\pi_e(Y),$$

where $X \in \mathfrak{a}^+$ and $Y \in \mathfrak{a}^-$. This implies that $d\pi_e(X - Y) = 0$ or equivalently, that $X - Y \in \ker(d\pi_e) = \mathfrak{k}^{\mathbb{C}}$. So there exists some $Y' \in \mathfrak{k}^{\mathbb{C}}$ such that $X = Y + Y'$. Since $Y \in \mathfrak{a}^-$ and $Y' \in \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{a}^-$ it follows that $Y + Y' \in \mathfrak{a}^-$. This means that $X \in \mathfrak{a}^+ \cap \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}}$. Hence, $Z = d\pi_e(X) = 0$. We then get that

$$T_o^{\mathbb{C}}M = T_o^{1,0}M \oplus T_o^{0,1}M.$$

This allows us to define $J_o : T_o^{\mathbb{C}}M \to T_o^{\mathbb{C}}M$, by

$$J_o(Z) = iZ \quad \text{for } Z \in T_o^{1,0}M, \; J_o(Z) = -iZ \quad \text{for } Z \in T_o^{0,1}M.$$

It is clear that J_o is a complex structure on the vector space $T_o^{\mathbb{C}}M$. We extend J_o to the whole of $C^\infty(TM)$ by defining

$$J_{gK}(X_{gK}) = ((d\tau_g)_o \circ J_o \circ (d\tau_g)_o^{-1})(X_{gK})$$

for $g \in G$ and $X_{gK} \in T_{gK}^{\mathbb{C}}M$. We need to show that J_{gK} does not depend on the choice of representative of the coset gK. To do this we first note that for $X \in \mathfrak{a}^+$ and $k \in K$, $\mathrm{Ad}(k)(X) \in \mathfrak{a}^+$. This is true since $\mathfrak{k}^{\mathbb{C}} \subseteq \mathfrak{a}^+$ and $\mathfrak{a}^+$ is a Lie algebra. If we combine this with Lemma 2.9, we get that

$$(d\tau_k)_o(d\pi_e(X)) = d\pi_e(\mathrm{Ad}(k)(X)) \in T_o^{1,0}M$$

for all $X \in \mathfrak{a}^+$ and all $k \in K$. Hence it follows that $(d\tau_k)_o(T_o^{1,0}M) = T_o^{1,0}M$ for all $k \in K$. In the same way we have that $(d\tau_k)_o(T_o^{0,1}M) = T_o^{0,1}M$ for all $k \in K$. It is then easy to see that $J_o \circ (d\tau_k)_o = (d\tau_k)_o \circ J_o$ for all $k \in K$. Suppose now that $hK = gK$. Then $h = gk$ for some $k \in K$, which implies that $(d\tau_h)_o = (d\tau_g)_o \circ (d\tau_k)_o$. We then see that

$$\begin{aligned} J_{hK}(X_{hK}) &= ((d\tau_h)_o \circ J_o \circ (d\tau_h)_o^{-1})(X_{hK}) \\ &= ((d\tau_g)_o \circ (d\tau_k)_o \circ J_o \circ (d\tau_{k^{-1}g^{-1}})_{gK})(X_{gK}) \\ &= ((d\tau_g)_o \circ (d\tau_k)_o \circ J_o \circ (d\tau_{k^{-1}})_o \circ (d\tau_{g^{-1}})_{gK})(X_{gK}) \\ &= ((d\tau_g)_o \circ J_o \circ (d\tau_g)_o^{-1})(X_{gK}) \\ &= J_{gK}(X_{gK}). \end{aligned}$$

So J is well-defined. It is clear that $J_{gK}^2 = -I$ for all $g \in G$ and that J is G-invariant follows directly from the definition. Moreover, it is easy to see from the definition of $T_o^{1,0}M$ that $(d\pi_e)^{-1}(T_o^{1,0}M) = \mathfrak{a}^+$.

We would now like to show that J is a complex structure. To do this we will use Theorem 3.10. We let $\mathfrak{m}^{\pm}$ be defined as before. We define, for $Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$,

$$N(Z_1, Z_2) = [\widetilde{I}_0(Z_1), \widetilde{I}_0(Z_2)] - [Z_1, Z_2] - \widetilde{I}_0([Z_1, \widetilde{I}_0(Z_2)]) - \widetilde{I}_0([\widetilde{I}_0(Z_1), Z_2]).$$

We have that $Z_i = X_i + Y_i$, $X_i \in \mathfrak{k}^{\mathbb{C}}$,$Y_i \in \mathfrak{m}^{\mathbb{C}}$, $i = 1, 2$. Since $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ we may write $Y_i = Y_i^+ + Y_i^-$, where $Y_i^+ \in \mathfrak{m}^+$ and $Y_i^- \in \mathfrak{m}^-$. We want to show that $N(Z_1, Z_2) \in \mathfrak{k}^{\mathbb{C}}$, for all $Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$. We begin by looking at each term in $N(Z_1, Z_2)$ individually. Using Lemma 4.4 and the properties of $\widetilde{I}_0$, we have that

$$\widetilde{I}_0(X_i) = 0,\ \widetilde{I}_0(Y_i^+) = iY_i^+,\ \widetilde{I}_0(Y_i^-) = -iY_i^-$$

for $i = 1, 2$. We use this to obtain the following identities:

$$\begin{aligned}
[\widetilde{I}_0(Z_1), \widetilde{I}_0(Z_2)] &= [iY_1^+ - iY_1^-, iY_2^+ - iY_2^-] \\
&= -[Y_1^+, Y_2^+] + [Y_1^+, Y_2^-] + [Y_1^-, Y_2^+] - [Y_1^-, Y_2^-],
\end{aligned}$$

$$\begin{aligned}
[Z_1, Z_2] &= [X_1 + Y_1, X_2 + Y_2] \\
&= [X_1, X_2] + [X_1, Y_2^+] + [X_1, Y_2^-] + [Y_1^+, X_2] + [Y_1^-, X_2] \\
&\quad + [Y_1^+, Y_2^+] + [Y_1^+, Y_2^-] + [Y_1^-, Y_2^+] + [Y_1^-, Y_2^+],
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_0([Z_1, \widetilde{I}_0(Z_2)]) &= \widetilde{I}_0([X_1 + Y_1^+ + Y_1^-, iY_2^+ - iY_2^-]) \\
&= i\widetilde{I}_0([X_1, Y_2^+]) - i\widetilde{I}_0([X_1, Y_2^-]) + i\widetilde{I}_0([Y_1^+, Y_2^+]) - i\widetilde{I}_0([Y_1^+, Y_2^-]) \\
&\quad + i\widetilde{I}_0([Y_1^-, Y_2^+]) - i\widetilde{I}_0([Y_1^-, Y_2^-]),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I}_0([\widetilde{I}_0(Z_1), Z_2]) &= \widetilde{I}_0([iY_1^+ - iY_1^-, X_2 + Y_2^+ + Y_2^-] \\
&= i\widetilde{I}_0([Y_1^+, X_2]) - i\widetilde{I}_0([Y_1^-, X_2]) + i\widetilde{I}_0[Y_1^+, Y_2^+]) - i\widetilde{I}_0([Y_1^-, Y_2^+]) \\
&\quad + i\widetilde{I}_0([Y_1^+, Y_2^-]) - i\widetilde{I}_0([Y_1^-, Y_2^-]).
\end{aligned}$$

From these identities we get that:

$$\begin{aligned}
N(Z_1, Z_2) = -2\Big([Y_1^+, Y_2^+] + i\widetilde{I}_0([Y_1^+, Y_2^+])\Big) - 2\Big([Y_1^-, Y_2^-] - i\widetilde{I}_0([Y_1^-, Y_2^-])\Big) \\
- \Big([X_1, Y_2^+] + i\widetilde{I}_0([X_1, Y_2^+])\Big) - \Big([X_1, Y_2^-] - i\widetilde{I}_0([X_1, Y_2^-])\Big) \\
- \Big([Y_1^+, X_2] + i\widetilde{I}_0([Y_1^+, X_2])\Big) - \Big([Y_1^-, X_2] - i\widetilde{I}_0([Y_1^-, X_2])\Big) - [X_1, X_2].
\end{aligned}$$

Let us start by considering the first parenthesis in the above expression. Both Y_1^+ and Y_2^+ belong to $\mathfrak{m}^+$ which is a subspace of $\mathfrak{a}^+$. By assumption $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. This means that $[Y_1^+, Y_2^+] \in \mathfrak{a}^+$. So we may write

$$[Y_1^+, Y_2^+] = [Y_1^+, Y_2^+]_{\mathfrak{k}} + [Y_1^+, Y_2^+]_{\mathfrak{m}^+}$$

where $[Y_1^+, Y_2^+]_{\mathfrak{k}} \in \mathfrak{k}^{\mathbb{C}}$ and $[Y_1^+, Y_2^+]_{\mathfrak{m}^+} \in \mathfrak{m}^+$. This implies that

$$\begin{aligned}
[Y_1^+, Y_2^+] + i\widetilde{I}_0([Y_1^+, Y_2^+]) &= [Y_1^+, Y_2^+]_{\mathfrak{k}} + [Y_1^+, Y_2^+]_{\mathfrak{m}^+} + i(i[Y_1^+, Y_2^+]_{\mathfrak{m}^+}) \\
&= [Y_1^+, Y_2^+]_{\mathfrak{k}} \in \mathfrak{k}^{\mathbb{C}}
\end{aligned}$$

By using the same method it is easy to see that the other parenthesis belong to $\mathfrak{k}^{\mathbb{C}}$ as well. Also, since $\mathfrak{k}^{\mathbb{C}}$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, it is clear that $[X_1, X_2] \in \mathfrak{k}^{\mathbb{C}}$. It now follows that $N(Z_1, Z_2) \in \mathfrak{k}^{\mathbb{C}}$ for all $Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$ and by Theorem 3.10 this is equivalent to J being a complex structure. $\square$

The following theorem is now obvious.

Theorem 4.7. [12] *The map*

F : *the set of G-invariant complex structures on $M \to$ the set of subalgebras of $\mathfrak{g}^{\mathbb{C}}$ satisfying* (4.1),

given by $F(J) = (d\pi_e)^{-1}(T_o^{1,0}M)$, is a bijection.

Proof. By Lemma 4.5, F is injective and by Lemma 4.6, F is also surjective. □

Let us now consider the case when $M = G/K$ is a generalized flag manifold. As described in the beginning of this section we have that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$
$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha.$$

Before moving on, we will need some properties regarding the values of the roots on $\mathfrak{h}$ and how the conjugation with respect to $\mathfrak{g}^{\mathbb{C}}$ acts on the root spaces. These results will also be of importance later on.

We have already mentioned the conjugation $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{g}$ defined by

$$\theta(X + iY) = X - iY \text{ for all } X, Y \in \mathfrak{g}.$$

The following properties are easy to verify.

$$\begin{aligned} &\theta(\theta(Z)) = Z, \quad \theta(Z_1 + Z_2) = \theta(Z_1) + \theta(Z_2), \\ &\theta(aZ) = \bar{a}\theta(Z), \quad \theta([Z_1, Z_2]) = [\theta(Z_1), \theta(Z_2)] \end{aligned}$$

for all $Z, Z_1, Z_2 \in \mathfrak{g}^{\mathbb{C}}$ and all $a \in \mathbb{C}$. The set of elements in $\mathfrak{g}^{\mathbb{C}}$ that are fixed by θ are exactly the elements in $\mathfrak{g}$, i.e.

$$\mathfrak{g} = \{X \in \mathfrak{g}^{\mathbb{C}} \mid \theta(X) = X\}$$

Proposition 4.8. [4] *$\alpha(H)$ is purely imaginary for all $\alpha \in R$ and all $H \in \mathfrak{h}$.*

Proof. The quotient $M = G/K$ is a generalized flag manifold, so $\mathfrak{g}$ is compact and semisimple. That $\mathfrak{g}$ is compact and semisimple implies that B is negative definite on $\mathfrak{g}$, by Proposition 1.9. It is then clear that $-\mathrm{B}$ is an inner product on $\mathfrak{g}$. Using this, we define an inner product $(\,,)$ on $\mathfrak{g}^{\mathbb{C}}$ by

$$(X, Y) = -\mathrm{B}(X, \theta(Y)).$$

By Lemma 1.12, $\mathrm{B}(\mathrm{ad}(X)(Y), Z) = \mathrm{B}(Y, -\mathrm{ad}(X)(Z))$ for all $X, Y, Z \in \mathfrak{g}^{\mathbb{C}}$. Note also that for $X \in \mathfrak{g}$ and $Y \in \mathfrak{g}^{\mathbb{C}}$,

$$\mathrm{ad}(X)(\theta(Y)) = [X, \theta(Y)] = \theta([\theta(X), Y]) = \theta([X, Y]) = \theta(\mathrm{ad}(X)(Y)).$$

Let $\alpha \in R$ and $E_\alpha \in \mathfrak{g}_\alpha$ such that $E_\alpha \neq 0$. For any $H \in \mathfrak{h}$ we have that $\mathrm{ad}(H)(E_\alpha) = \alpha(H)E_\alpha$. This implies that

$$(\mathrm{ad}(H)(E_\alpha), E_\alpha) = \alpha(H)(E_\alpha, E_\alpha).$$

But we also have that

$$\begin{aligned} (\mathrm{ad}(H)(E_\alpha), E_\alpha) &= -\mathrm{B}(\mathrm{ad}(H)(E_\alpha), \theta(E_\alpha)) \\ &= \mathrm{B}(E_\alpha, \mathrm{ad}(H)(\theta(E_\alpha))) \\ &= \mathrm{B}(E_\alpha, \theta(\mathrm{ad}(H)(E_\alpha))) \\ &= \mathrm{B}(E_\alpha, \theta(\alpha(H)E_\alpha)) \\ &= \overline{\alpha(H)}\mathrm{B}(E_\alpha, \theta(E_\alpha)) \\ &= -\overline{\alpha(H)}(E_\alpha, E_\alpha), \end{aligned}$$

Hence we have that

$$\alpha(H)(E_\alpha, E_\alpha) = -\overline{\alpha(H)}(E_\alpha, E_\alpha)$$

and since $(E_\alpha, E_\alpha) > 0$, we must have that $\alpha(H) = -\overline{\alpha(H)}$, which implies that $\alpha(H)$ is purely imaginary. This proves the proposition since $\alpha \in R$ and $H \in \mathfrak{h}$ were chosen arbitrarily.

□

Lemma 4.9. [4] *The conjugation θ maps $\mathfrak{g}_\alpha$ into $\mathfrak{g}_{-\alpha}$, for all $\alpha \in R \cup \{0\}$.*

Proof. We begin by noting that this is obvious for $\alpha = 0$, since $\mathfrak{g}_0 = \mathfrak{h}^{\mathbb{C}}$ and clearly $\theta(H_1 + iH_2) = H_1 - iH_2 \in \mathfrak{h}^{\mathbb{C}}$ for all $H_1, H_2 \in \mathfrak{h}$. For each $\alpha \in R$ we choose basis elements $E_\alpha \in \mathfrak{g}_\alpha$. Let H be any element in $\mathfrak{h}^{\mathbb{C}}$. Then there exist $H_1, H_2 \in \mathfrak{h}$, such that

$$H = H_1 + iH_2.$$

Since θ maps $\mathfrak{h}^{\mathbb{C}}$ into itself, we have that

$$[\theta(H), E_\alpha] = \alpha(\theta(H))E_\alpha.$$

Applying θ to both sides of the above the equation gives us

$$\theta([\theta(H), E_\alpha]) = \theta(\alpha(\theta(H))E_\alpha)$$

or equivalently

$$[H, \theta(E_\alpha)] = \overline{\alpha(\theta(H))}\theta(E_\alpha).$$

Since $\theta(H) = \theta(H_1 + iH_2) = H_1 - iH_2$, we get that

$$\alpha(\theta(H)) = \alpha(H_1 - iH_2) = \alpha(H_1) - i\alpha(H_2).$$

From Proposition 4.8 we know that α takes purely imaginary values on $\mathfrak{h}$, so $-i\alpha(H_2) \in \mathbb{R}$ and $\alpha(H_1) \in i\mathbb{R}$. We therefore get that

$$\overline{\alpha(\theta(H))} = -i\alpha(H_2) - \alpha(H_1) = -(\alpha(H_1) + i\alpha(H_2)) = -\alpha(H).$$

This implies that

$$[H, \theta(E_\alpha)] = \overline{\alpha(\theta(H))}\theta(E_\alpha) = -\alpha(H)\theta(E_\alpha).$$

Since we can do this for any $H \in \mathfrak{h}^{\mathbb{C}}$, it follows that $\theta(E_\alpha) \in \mathfrak{g}_{-\alpha}$. □

By a theorem in [13], every generalized flag manifold admits a G-invariant complex structure. For a proof of this theorem see [5] page 501-502. We may therefore assume that there exists a G-invariant complex structure J on M. Then we obtain $\mathfrak{a}^+$ as before which satisfies (4.1). Since $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}} \subset \mathfrak{a}^+$ and $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$, there exists a root system R_{A^+} with respect to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$, such that

$$\mathfrak{a}^+ = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_{A^+}} \mathfrak{g}_\alpha.$$

It is clear that $R_K \subset R_{A^+}$, since $\mathfrak{k}^{\mathbb{C}} \subset \mathfrak{a}^+$. We define $\Delta^+ = R_{A^+} \backslash R_K$. Then

$$\mathfrak{a}^+ = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Next let $\Delta^- = \{\alpha \in R | -\alpha \in \Delta^+\}$ and define

$$\mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

It is easy to see from Lemma 4.9 that $\theta(\mathfrak{a}^+) = \mathfrak{a}^-$, which implies $\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^+ + \mathfrak{a}^-$ and $\mathfrak{a}^+ \cap \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}}$. We have the following result:

Lemma 4.10. [12] *Let R_K,Δ^+ and Δ^- be as above. Then the following holds:*

$$R = R_K \cup \Delta^+ \cup \Delta^- \quad \textit{disjoint union} \tag{4.2}$$

$$\textit{If } \alpha \in R_K \cup \Delta^+,\ \beta \in \Delta^+ \textit{ and } \alpha + \beta \in R,\ \textit{then } \alpha + \beta \in \Delta^+ \tag{4.3}$$

Moreover, the subset Δ^+ is uniquely determined by $\mathfrak{a}^+$.

Proof. Let us start by proving (4.2). By definition $\Delta^+ \cap R_K = \emptyset$. Suppose $\alpha \in \Delta^- \cap R_K$. Then $-\alpha \in \Delta^+ \cap R_K$, where we used the fact that R_K is a root system. Clearly this contradicts the fact that Δ^+ and R_K are disjoint. Hence, we must have that $\Delta^- \cap R_K = \emptyset$.

Suppose now that $\alpha \in \Delta^+ \cap \Delta^-$. Then $\mathfrak{g}_\alpha \subset \mathfrak{a}^+ \cap \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}}$, which is false since $\alpha \notin R_K$. The sets R_K, Δ^+ and Δ^- must therefore be disjoint. We note that

$$\begin{aligned}
\mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha = \mathfrak{g}^{\mathbb{C}} & \\
&= \mathfrak{a}^+ + \mathfrak{a}^- \\
&= (\mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha) + (\mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha) \\
&= \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha \oplus \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.
\end{aligned}$$

From these calculations we see that the number of roots in $R_K \cup \Delta^+ \cup \Delta^-$ must be the same as the number of roots in R. This, together with the fact that $R_K \cup \Delta^+ \cup \Delta^- \subseteq R$, implies that $R = R_K \cup \Delta^+ \cup \Delta^-$. This proves that (4.2) holds.

Assume next that $\alpha \in R_K \cup \Delta^+$, $\beta \in \Delta^+$ and that $\alpha + \beta \in R$. We know from Chapter 1 that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta},$$

and since $\mathfrak{g}_\alpha, \mathfrak{g}_\beta \subset \mathfrak{a}^+$ and $\mathfrak{a}^+$ is a subalgebra, we have that $\mathfrak{g}_{\alpha+\beta} \in \mathfrak{a}^+$. This implies that $\alpha+\beta \in R_K$ or $\alpha + \beta \in \Delta^+$. Assume first that $\alpha \in R_K$, and assume for a contradiction that $\alpha + \beta = \gamma \in R_K$. Then $\beta = \gamma - \alpha$. We know that $\alpha \in R_K$ implies $-\alpha \in R_K$, and since $\gamma - \alpha = \beta \in \Delta^+$ we see in particular that $\gamma - \alpha$ is a root. Since $-\alpha, \gamma \in R_K$, and $-\alpha + \gamma \in R$, we must therefore have that $\beta = -\alpha + \gamma \in R_K$. But by (4.2), $R_K \cap \Delta^+ = \emptyset$. So the assumption that $\alpha + \beta \in R_K$ must be false. It is clear that we can apply the same reasoning to Δ^-, i.e.

$$\alpha + \beta \in \Delta^- \quad \text{for } \alpha \in R_K,\ \beta \in \Delta^-. \tag{4.4}$$

Now let $\alpha \in \Delta^+$. We still have that $\alpha+\beta \in R_K$ or $\alpha+\beta \in \Delta^+$. We again assume for a contradiction that $\alpha + \beta = \gamma \in R_K$. Then $\gamma - \alpha \in \Delta^-$, by (4.4), since $-\alpha \in \Delta^-$. But this contradicts the fact that $\gamma - \alpha = \beta \in \Delta^+$, since $\Delta^+ \cap \Delta^- = \emptyset$. So the assumption that $\alpha + \beta \in R_K$ must be false. This proves (4.3).

Suppose that Γ^+ is another subset of R which satisfies

$$\mathfrak{a}^+ = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{\beta \in \Gamma^+} \mathfrak{g}_\beta.$$

The sets Δ^+ and Γ^+ must have the same number of elements, if not then the two decompositions of $\mathfrak{a}^+$ would have different dimensions, which is impossible. Let $\beta \in \Gamma^+$ and assume $\beta \notin \Delta^+$. Then $\beta \in \Delta^-$ by (4.2). But this means that

$$\mathfrak{g}_\beta \subset \mathfrak{a}^+ \cap \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}}$$

which is impossible since $\beta \notin R_K$ by (4.2). So we must have that $\beta \in \Delta^+$. But this means that $\Gamma^+ \subseteq \Delta^+$ and since the two sets have the same number of elements they must in fact be equal. □

Now assume that we are given a subset Δ^+ of R that satisfies (4.2) and (4.3). Then we define

$$\mathfrak{a}^+ = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

We have the following result.

Lemma 4.11. [12] *The subspace* $\mathfrak{a}^+ \subset \mathfrak{g}^{\mathbb{C}}$ *is a subalgebra of* $\mathfrak{g}^{\mathbb{C}}$*, which satisfies* (4.1)*, that is*

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^+ + \theta(\mathfrak{a}^+), \quad \mathfrak{a}^+ \cap \theta(\mathfrak{a}^+) = \mathfrak{k}^{\mathbb{C}}.$$

Proof. We let

$$\mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}} \oplus \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

It then follows directly from (4.2) and Lemma 4.9 that

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{a}^+ + \mathfrak{a}^-, \quad \mathfrak{a}^+ \cap \mathfrak{a}^- = \mathfrak{k}^{\mathbb{C}} \text{ and } \theta(\mathfrak{a}^+) = \mathfrak{a}^-.$$

We now show that $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Let $Z_1 = X_1 + Y_1$, $Z_2 = X_2 + Y_2 \in \mathfrak{a}^+$, with $X_1, X_2 \in \mathfrak{k}^{\mathbb{C}}$ and $Y_1, Y_2 \in \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$. Then

$$[Z_1, Z_2] = [X_1, X_2] + [X_1, Y_2] + [Y_1, X_2] + [Y_1, Y_2].$$

Since $\mathfrak{k}^{\mathbb{C}}$ is a subalgebra, $[X_1, X_2] \in \mathfrak{k}^{\mathbb{C}}$. Next we consider $[Y_1, X_2]$. We have that

$$Y_1 = \sum_{\alpha \in \Delta^+} a_\alpha E_\alpha$$
$$X_2 = H + \sum_{\beta \in R_K} b_\beta E_\beta,$$

where $H \in \mathfrak{h}^{\mathbb{C}}$, $E_\alpha \in \mathfrak{g}_\alpha$. From Chapter 1, we know that $[E_\alpha, H] \in \mathfrak{g}_\alpha$, and $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta}$. If $\beta + \alpha \in R$, then we know from (4.3) that $\beta + \alpha \in \Delta^+$. Using this it is clear that

$$[Y_1, X_2] = \sum_{\alpha \in \Delta^+} a_\alpha [E_\alpha, H] + \sum_{\alpha \in \Delta^+} \sum_{\beta \in R_K} a_\alpha b_\beta [E_\alpha, E_\beta] \in \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha.$$

Using (4.3) in the same way as above, it is easy to see that the other two terms in $[Z_1, Z_2]$ belongs to $\sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$ as well. So $[Z_1, Z_2] \in \mathfrak{a}^+$, and since Z_1, Z_2 was arbitrarily chosen, it follows that $\mathfrak{a}^+$ is a subalgebra of $\mathfrak{g}^{\mathbb{C}}$. □

We may now prove the following theorem.

Theorem 4.12. [12] *There is a bijection between the set of* G*-invariant complex structures* J *on* M *and the set of subsets* Δ^+ *of* R *satisfying* (4.2) *and* (4.3).

Proof. [12] By Lemma 4.10 we have a well-defined correspondence between subalgebras which satisfies (4.1) and subsets which satisfies (4.2) and (4.3). It is clear that this correspondence is injective. From Lemma 4.11 we get that this correspondence is surjective as well. The theorem then follows from Theorem 4.7. □

4.2 Weyl Chambers

In this section we introduce the Weyl chambers. These are certain subsets of the Lie algebra $\mathfrak{g}$ and we shall see that there is a one-to-one correspondence between Weyl chambers and subsets Δ^+ of R.

As before let $M = G/K$ be a generalized flag manifold with root space decomposition

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha$$

and

$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha.$$

We define

$$\mathfrak{t} = Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h},$$

where $Z(\mathfrak{k}^{\mathbb{C}}) = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [X, Y] = 0 \text{ for all } Y \in \mathfrak{k}^{\mathbb{C}}\}$. We then have the following lemma:

Lemma 4.13. [3] *The subspace* $\mathfrak{t}$ *satisfies*

$$\mathfrak{t} = \{X \in \mathfrak{h} \mid \alpha(X) = 0 \text{ for all } \alpha \in R_K\}.$$

Proof. [3] We see that

$$\begin{aligned}\mathfrak{t} &= Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h} \\ &= \{X \in \mathfrak{g}^{\mathbb{C}} \mid [X,Y] = 0 \text{ for all } Y \in \mathfrak{k}^{\mathbb{C}}\} \cap \mathfrak{h} \\ &= \{X \in \mathfrak{g}^{\mathbb{C}} \mid [X,\mathfrak{h}^{\mathbb{C}}] = 0,\ [X,E_\alpha] = 0 \text{ for all } \alpha \in R_K\} \cap \mathfrak{h} \\ &= \{X \in \mathfrak{h}^{\mathbb{C}} \mid \alpha(X)E_\alpha = 0 \text{ for all } \alpha \in R_K\} \cap \mathfrak{h} \\ &= \{X \in \mathfrak{h} \mid \alpha(X) = 0 \text{ for all } \alpha \in R_K\}\end{aligned}$$

where we have used that if $[X,H] = 0$ for all $H \in \mathfrak{h}^{\mathbb{C}}$, then $X \in \mathfrak{g}_0$ by definition and $\mathfrak{g}_0 = \mathfrak{h}^{\mathbb{C}}$ since $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. $\square$

Let $\Delta = R \backslash R_K$ and let

$$L_\alpha = \{X \in \mathfrak{t} \mid \alpha(X) = 0\} \quad \text{for } \alpha \in \Delta,$$

and

$$\mathfrak{t}' = \mathfrak{t} \backslash \cup_{\alpha \in \Delta} L_\alpha.$$

The connected components in $\mathfrak{t}'$ are called **Weyl chambers**.

Let C be a Weyl chamber. Let $Z \in C$ and suppose that $i\alpha(Z) > 0$ for some $\alpha \in \Delta$. Assume that there exists $X \in C$ such that $i\alpha(X) < 0$. Since C is connected there exists a path $\gamma : [0,1] \to C$ such that $\gamma(0) = Z$ and $\gamma(1) = X$. The root α is in the dual of $\mathfrak{h}^{\mathbb{C}}$, which implies that α is continuous. By the intermediate value theorem there must exist $\epsilon \in [0,1]$ such that $\alpha(\gamma(\epsilon)) = 0$. However, this implies that $\gamma(\epsilon) \notin C$ which is a contradiction. So if $i\alpha(Z) > 0$ for some $Z \in C$, then $i\alpha(Z) > 0$ for all $Z \in C$.

Theorem 4.14. [4] *There is a bijection between the set of Weyl chambers in* $\mathfrak{t}'$ *and the set of subsets of* R *satisfying* (4.2) *and* (4.3).

Proof. Let C be a Weyl chamber and let $Z \in C$. We define

$$\Delta^+ = \{\alpha \in \Delta \mid i\alpha(Z) > 0\}.$$

We show that Δ^+ satisfies (4.2) and (4.3). By definition, $\Delta = R\backslash R_K$, so $\Delta \cap R_K = \emptyset$. Moreover,

$$\Delta^- = \{\alpha \in R \mid -\alpha \in \Delta^+\} = \{\alpha \in \Delta \mid i\alpha(Z) < 0\}.$$

It follows that $\Delta = \Delta^+ \cup \Delta^-$, since $i\alpha(Z)$ is either greater than 0 or less than 0 for all $\alpha \in \Delta$. It is also clear that $\Delta^+ \cap \Delta^- = \emptyset$. This implies that

$$R = R_K \cup \Delta = R \cup \Delta^+ \cup \Delta^- \quad \text{disjoint union.}$$

So (4.2) is satisfied.

To check (4.3) we let $\alpha \in R_K$ and $\beta \in \Delta^+$ and assume that $\alpha + \beta \in R$. Then $\alpha(Z) = 0$ since

$$Z \in C \subset \mathfrak{t} = \{X \in \mathfrak{h} \mid \alpha(X) = 0 \text{ for all } \alpha \in R_K\}.$$

Hence

$$i(\alpha+\beta)(Z) = i\alpha(Z) + i\beta(Z) = i\beta(Z) > 0,$$

so $\alpha + \beta \in \Delta^+$. Next we let $\alpha \in \Delta^+$ as well and assume that $\alpha + \beta \in R$. Then clearly

$$i(\alpha+\beta)(Z) = i\alpha(Z) + i\beta(Z) > 0,$$

so $\alpha + \beta \in \Delta^+$. Since we did this for arbitrary $\alpha \in R_K \cup \Delta^+$ and $\beta \in \Delta^+$, it follows that (4.3) holds as well.

We have just constructed a function

$$\Phi : \text{the set of Weyl chambers} \to \text{the subsets of } \Delta \text{ satisfying (4.2) and (4.3).}$$

We now define

$$\Psi : \text{the subsets of } \Delta \text{ satisfying (4.2) and (4.3)} \to \text{the set of Weyl chambers}$$

by $\Psi(\Delta^+) = \{X \in \mathfrak{t}' \mid i\alpha(X) > 0 \text{ for all } \alpha \in \Delta^+\}$. We need to show that $\Psi(\Delta^+)$ is nonempty for any given Δ^+ which satisfies (4.2) and (4.3). In order to do this we let R_K^+ be a set of positive roots in R_K. We then let $R^+ = R_K^+ \cup \Delta^+$. It is easy to see using (4.2), (4.3) and the fact that R_K^+ is a set of positive roots in R_K, that R^+ is a set of positive roots in R. By Proposition 1.7, we may choose a set

$$\Pi = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$$

of simple roots in R^+. Since $R^+ = R_K^+ \cup \Delta^+$, a simple root α_i belongs to either R_K^+ or Δ^+. Now let $V = \operatorname{span}_{\mathbb{R}}(R)$. Then Π is a basis of V. It is clear that $V \subseteq (\mathfrak{h}^{\mathbb{C}})^*$. The vector space $W := V^*$ is of dimension n, so W is isomorphic to $\mathbb{R}^n$, with the isomorphism F given by

$$F(X) = (\alpha_1(X), \alpha_2(X), \dots, \alpha_n(X)).$$

We may assume that the simple roots are ordered so that $\alpha_i \in \Delta^+$ for $1 \leq i \leq m$ for some $1 < m < n$ and $\alpha_i \in R_K^+$ for $m < i \leq n$. Let $Y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ be such that $y_i > 0$ for all $1 \leq i \leq m$ and $y_i = 0$ for $m < i \leq n$. Since F is an isomorphism, there exists $X \in W$ such that $F(X) = Y$, or equivalently

$$(\alpha_1(X), \alpha_2(X), \dots, \alpha_m(X), \alpha_{m+1}(X), \dots, \alpha_n(X)) = (y_1, y_2, \dots, y_m, 0, \dots, 0).$$

Every element in R_K can be written as a linear combination of $\alpha_{m+1}, \dots \alpha_n$, so it follows that $\alpha(X) = 0$ for all $\alpha \in R_K$. Recall that every root takes purely imaginary values on $\mathfrak{h}$. If we combine this with the fact that $\alpha_i(X) > 0$ for all $1 \leq i \leq m$ we must have that $X = iH$ for some $H \in \mathfrak{h}$. Let $\alpha \in \Delta^+$. From Proposition 1.7 we know that α can be written as

$$\alpha = \sum_{i=1}^{n} a_i \alpha_i$$

where $a_i \in \mathbb{Z}^+$ and $a_i > 0$ for at least one $1 \leq i \leq m$. Using this, we see that

$$i\alpha(H) = \alpha(X) = \sum_{i=1}^{n} a_i \alpha_i(X) > 0.$$

We have now shown that the element $H \in \mathfrak{h}$ satisfies $\alpha(H) = 0$ for all $\alpha \in R_K$ and that $i\alpha(H) > 0$ for all $\alpha \in \Delta^+$. This implies that $H \in \mathfrak{t}$ and that $H \in \Psi(\Delta^+)$, which shows that $\Psi(\Delta^+)$ is nonempty.

We see that

$$(1-t)X + tY \in \Psi(\Delta^+)$$

for all $X, Y \in \Psi(\Delta^+)$ and all $t \in [0,1]$. From this we see that $\Psi(\Delta^+)$ is a connected subset of $\mathfrak{t}'$, so $\Psi(\Delta^+) \subseteq D$, where D is a Weyl chamber. Let $\alpha \in \Delta^+$ and let $X \in D$. Since $i\alpha(Z) > 0$ for all $Z \in \Psi(\Delta^+)$, it follows that $i\alpha(X) > 0$ since D is connected. This holds for all $\alpha \in \Delta^+$ and all $X \in D$ which implies that $D \subseteq \Psi(\Delta^+)$. We therefore have that $\Psi(\Delta^+) = D$.

The following calculations show that Ψ is an inverse to Φ.

$$\begin{aligned}\Phi(\Psi(\Delta^+)) &= \{\alpha \in \Delta \mid i\alpha(Z) > 0 \text{ for } Z \in \Psi(\Delta^+)\} \\ &= \{\alpha \in \Delta \mid i\alpha(Z) > 0 \text{ for } Z \in \mathfrak{t}' \text{ such that } i\beta(Z) > 0 \text{ for all } \beta \in \Delta^+\}\end{aligned}$$

We see directly that $\Delta^+ \subseteq \Phi(\Psi(\Delta^+))$ and since these two sets must have the same number of elements, it follows that $\Phi(\Psi(\Delta^+)) = \Delta^+$.

$$\Psi(\Phi(C)) = \{X \in \mathfrak{t}' \mid i\alpha(X) > 0 \text{ for all } \alpha \in \Phi(C)\}$$

$$= \{X \in \mathfrak{t}' \mid i\alpha(X) > 0 \text{ for all } \alpha \in \Delta \text{ such that } i\alpha(Z) > 0 \text{ for all } Z \in C\}.$$

It is clear that $C \subseteq \Psi(\Phi(C))$ and since $\Psi(\Phi(C))$ is a Weyl chamber, we must have that $C = \Psi(\Phi(C))$. So Φ is a bijection between the set of Weyl chambers and the set of subsets of Δ satisfying (4.2) and (4.3). $\square$

Combining Theorems 4.12 and 4.14 gives us the following result.

Theorem 4.15. [4] *There is bijection between the set of G-invariant complex structures on M and the set of Weyl chambers in $\mathfrak{t}'$.*

4.3 The Construction of the Kähler Metrics

In this section we will see that for each G-invariant complex structure on a generalized flag manifold M, there exists a family of G-invariant Kähler metrics on M.

Let $M = G/K$ be a generalized flag manifold and let J be a G-invariant complex structure on M. As before we have

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R} \mathfrak{g}_\alpha,$$
$$\mathfrak{k}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \sum_{\alpha \in R_K} \mathfrak{g}_\alpha$$

and we let $\Delta = R \backslash R_K$. We then define

$$\mathfrak{m}^{\mathbb{C}} = \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha.$$

It follows directly that $\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\mathbb{C}}$.

Before moving on with the construction of the metrics, we state and prove some preliminary results.

Lemma 4.16. *Let $(\mathfrak{k}^{\mathbb{C}})^\perp = \{X \in \mathfrak{g}^{\mathbb{C}} \mid B(X, Y) = 0$ for all $Y \in \mathfrak{k}^{\mathbb{C}}\}$. Then $\mathfrak{m}^{\mathbb{C}} = (\mathfrak{k}^{\mathbb{C}})^\perp$.*

Proof. Recall from Chapter 1 that $\mathrm{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha, \beta \in R \cup \{0\}$ such that $\alpha + \beta \neq 0$. Let $\alpha \in \Delta$ and $\beta \in R_K \cup \{0\}$. Then $\alpha + \beta \neq 0$, so we have that $\mathrm{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$. Since every element in $\mathfrak{k}^{\mathbb{C}}$ is a linear combination of elements in $\mathfrak{g}_\alpha$, $\alpha \in R_K \cup \{0\}$ and every element in $\mathfrak{m}^{\mathbb{C}}$ is a linear combination of elements in $\mathfrak{g}_\alpha$, $\alpha \in \Delta$, it follows that

$$\mathfrak{m}^{\mathbb{C}} \subseteq (\mathfrak{k}^{\mathbb{C}})^\perp.$$

Conversely, suppose that $Z = Z_1 + Z_2 \in (\mathfrak{k}^{\mathbb{C}})^\perp$, where $Z_1 \in \mathfrak{k}^{\mathbb{C}}$ and $Z_2 \in \mathfrak{m}^{\mathbb{C}}$. Then for all $X \in \mathfrak{k}^{\mathbb{C}}$ we have

$$0 = \mathrm{B}(Z, X) = \mathrm{B}(Z_1, X) + \mathrm{B}(Z_2, X) = \mathrm{B}(Z_1, X)$$

where we used $Z_2 \in \mathfrak{m}^{\mathbb{C}} \subseteq (\mathfrak{k}^{\mathbb{C}})^\perp$. But this implies that for any element $Y = Y_1 + Y_2 \in \mathfrak{g}^{\mathbb{C}}$, where $Y_1 \in \mathfrak{k}^{\mathbb{C}}$ and $Y_2 \in \mathfrak{m}^{\mathbb{C}}$, we have

$$\mathrm{B}(Z_1, Y) = \mathrm{B}(Z_1, Y_1) + \mathrm{B}(Z_1, Y_2) = 0.$$

This contradicts the nondegeneracy of B on $\mathfrak{g}^{\mathbb{C}}$ unless $Z_1 = 0$. This means that $Z \in \mathfrak{m}^{\mathbb{C}}$ and since $Z \in (\mathfrak{k}^{\mathbb{C}})^\perp$ was chosen arbitrarily, it follows that

$$(\mathfrak{k}^{\mathbb{C}})^\perp \subseteq \mathfrak{m}^{\mathbb{C}}.$$

$\square$

Proposition 4.17. [4] *For all $\alpha \in R$ we may choose basis elements $E_\alpha \in \mathfrak{g}_\alpha$, such that $\theta(E_\alpha) = -E_{-\alpha}$ and $[E_\alpha, E_{-\alpha}] = H_\alpha$, for all $\alpha \in R$.*

Proof. [4] For each $\alpha \in R$ we choose basis elements $E_\alpha \in \mathfrak{g}_\alpha$, such that

$$[E_\alpha, E_{-\alpha}] = H_\alpha,\ B(E_\alpha, E_{-\alpha}) = 1,$$

for all $\alpha \in R$. We know from Proposition 1.6 that such a choice of basis elements is possible. From Lemma 4.9 we know that $\theta(E_\alpha) \in \mathfrak{g}_{-\alpha}$ and since the $\mathfrak{g}_\alpha$ are of dimension 1, we must have that

$$\theta(E_\alpha) = c_\alpha E_{-\alpha}$$

for some $c_\alpha \in \mathbb{C}$.

Let us begin by considering $\theta(H_\alpha)$. For any $H = H_1 + iH_2 \in \mathfrak{h}^{\mathbb{C}}$, we have

$$\mathrm{B}(H_\alpha, \theta(H)) = \alpha(\theta(H)) = \alpha(H_1) - i\alpha(H_2),$$

which implies

$$\overline{\mathrm{B}(H_\alpha, \theta(H))} = -i\alpha(H_2) - \alpha(H_1) = -\alpha(H) = \mathrm{B}(-H_\alpha, H). \tag{4.5}$$

As in Section 2 of Chapter 1, we see that

$$\mathrm{ad}(\theta(X))(Y) = (\theta \circ \mathrm{ad}(X) \circ \theta)(Y),$$

and if we combine this with the fact that θ is conjugate-linear, we get that

$$\begin{aligned}
\mathrm{B}(\theta(X), \theta(Y)) &= \mathrm{Tr}(\theta \circ \mathrm{ad}(X) \circ \mathrm{ad}(Y) \circ \theta) \\
&= \overline{\mathrm{Tr}(\mathrm{ad}(Y) \circ \theta \circ \theta \circ \mathrm{ad}(X))} \\
&= \overline{\mathrm{B}(X, Y)}.
\end{aligned}$$

This implies that

$$\overline{\mathrm{B}(H_\alpha, \theta(H))} = \mathrm{B}(\theta(H_\alpha), \theta(\theta(H))) = \mathrm{B}(\theta(H_\alpha), H).$$

Using this expression for $\overline{\mathrm{B}(H_\alpha, \theta(H))}$ in (4.5), we obtain

$$\mathrm{B}(\theta(H_\alpha), H) = \mathrm{B}(-H_\alpha, H).$$

Since H was chosen arbitrarily, this holds for all $H \in \mathfrak{h}^{\mathbb{C}}$. Using that B is non-degenerate on $\mathfrak{h}^{\mathbb{C}}$ we get that

$$\theta(H_\alpha) = -H_\alpha, \text{ for all } \alpha \in R. \tag{4.6}$$

We may use (4.6) to obtain the following:

$$\begin{aligned}
-H_\alpha &= \theta(H_\alpha) \\
&= \theta([E_\alpha, E_{-\alpha}]) \\
&= [\theta(E_\alpha), \theta(E_{-\alpha})] \\
&= [c_\alpha E_{-\alpha}, c_{-\alpha} E_\alpha] \\
&= -c_\alpha c_{-\alpha}[E_\alpha, E_{-\alpha}] \\
&= -c_\alpha c_{-\alpha} H_\alpha.
\end{aligned}$$

This implies that $c_\alpha c_{-\alpha} = 1$. Moreover, we have the following calculation

$$\begin{aligned}
E_\alpha &= \theta(\theta(E_\alpha)) \\
&= \theta(c_\alpha E_{-\alpha}) \\
&= \overline{c_\alpha}\theta(E_{-\alpha}) \\
&= \overline{c_\alpha} c_{-\alpha} E_\alpha,
\end{aligned}$$

which implies that $\overline{c_\alpha} c_{-\alpha} = 1$.

So the coefficients c_α satisfy

$$\begin{cases} c_\alpha c_{-\alpha} &= 1 \\ \overline{c_\alpha} c_{-\alpha} &= 1 \end{cases}$$

and it is easy to see that this implies that $c_\alpha \in \mathbb{R}$, for all $\alpha \in R$.

Now define for each $\alpha \in R$,

$$E'_\alpha = \frac{1}{|c_\alpha|^{\frac{1}{2}}} E_\alpha.$$

Then

$$[E'_\alpha, E'_{-\alpha}] = \frac{1}{|c_\alpha c_{-\alpha}|^{\frac{1}{2}}}[E_\alpha, E_{-\alpha}] = H_\alpha$$

and

$$\begin{aligned}\theta(E'_\alpha) &= \frac{1}{|c_\alpha|^{\frac{1}{2}}}\theta(E_\alpha)\\ &= \frac{c_\alpha}{|c_\alpha|^{\frac{1}{2}}}E_{-\alpha}\\ &= \frac{c_\alpha}{|c_\alpha c_\alpha c_{-\alpha}|^{\frac{1}{2}}}E_{-\alpha}\\ &= \frac{c_\alpha}{|c_\alpha|}E'_{-\alpha}\\ &= \mathrm{sign}(c_\alpha)E'_{-\alpha}.\end{aligned}$$

So we have now found elements $E'_\alpha \in \mathfrak{g}_\alpha$, such that

$$\theta(E'_\alpha) = \pm E'_{-\alpha},\ [E'_\alpha, E'_{-\alpha}] = H_\alpha$$

We now drop the prime $'$ and simply call these elements E_α.

Assume that there exists $\alpha \in R$ such that $\theta(E_\alpha) = E_{-\alpha}$. We want to show that this assumption leads to a contradiction. In order to do this we determine

$$\mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) = \{Z \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} \mid \theta(Z) = Z\ \}.$$

Let $Z = X + Y$, where $X = aE_\alpha$, $Y = bE_{-\alpha}$, $a, b \in \mathbb{C}$ and $Z \neq 0$. Applying θ to Z gives us

$$\begin{aligned}\theta(Z) &= \theta(X) + \theta(Y)\\ &= \overline{a}\theta(E_\alpha) + \overline{b}\theta(E_{-\alpha})\\ &= \overline{a}E_{-\alpha} + \overline{b}E_\alpha.\end{aligned}$$

So $\theta(Z) = Z$ if and only if

$$\overline{a}E_{-\alpha} + \overline{b}E_\alpha = aE_\alpha + bE_{-\alpha}$$

which is equivalent to $b = \overline{a}$. This implies that $\theta(Z) = Z$ if and only if

$$Z = aE_\alpha + \overline{a}E_{-\alpha}$$

for some $a \in \mathbb{C}$. Now recall that $\mathrm{B}(E_\alpha, E_{-\alpha}) = 1$ and $\mathrm{B}(E_\alpha, E_\alpha) = \mathrm{B}(E_{-\alpha}, E_{-\alpha}) = 0$. We then see that

$$\mathrm{B}(Z, Z) = \mathrm{B}(aE_\alpha + \overline{a}E_{-\alpha}, aE_\alpha + \overline{a}E_{-\alpha}) = 2|a|^2 > 0.$$

This means that B is positive definite on $\mathfrak{g} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$. However, since $\mathfrak{g}$ is compact and semisimple, B is negative definite on $\mathfrak{g}$. Hence we have obtained a contradiction and we conclude that there cannot exist any $\alpha \in R$ such that $\theta(E_\alpha) = E_{-\alpha}$. So the only possibility is that

$$\theta(E_\alpha) = -E_{-\alpha} \text{ for all } \alpha \in R$$

which is exactly what we wanted to show. □

We let $\mathfrak{m} = \{X \in \mathfrak{m}^{\mathbb{C}} \mid \theta(X) = X\}$. Then it is clear that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. From Lemma 4.16 we see that $\mathfrak{m} = (\mathfrak{k})^\perp$.

Lemma 4.18. *The Killing form B is nondegenerate on $\mathfrak{m}$.*

Proof. Assume that there exist $X \in \mathfrak{m}$ such that $\mathrm{B}(X, Y) = 0$ for all $Y \in \mathfrak{m}$. Since $B(X, Y) = 0$ for all $Y \in \mathfrak{k}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, we get that for $Y = Y_1 + Y_2 \in \mathfrak{g}$ where $Y_1 \in \mathfrak{k}$ and $Y_2 \in \mathfrak{m}$,

$$\mathrm{B}(X, Y) = \mathrm{B}(X, Y_1) + \mathrm{B}(X, Y_2) = 0.$$

The killing form B is nondegenerate on $\mathfrak{g}$ since $\mathfrak{g}$ is semisimple. We must therefore have that $X = 0$, which implies that B is nondegenerate on $\mathfrak{m}$. □

Since $\mathrm{Ad}(k)$ is an automorphism for all $k \in K$ we have that

$$\mathrm{B}(\mathrm{Ad}(k)(X), \mathrm{Ad}(k)(Y)) = \mathrm{B}(X, Y)$$

for all $X, Y \in \mathfrak{g}$ and all $k \in K$. So for $X \in \mathfrak{m}$ and $Y \in \mathfrak{k}$,

$$\mathrm{B}(\mathrm{Ad}(k)(X), Y) = \mathrm{B}(X, \mathrm{Ad}(k^{-1})(Y)) = 0,$$

which implies that $\mathrm{Ad}(k)(X) \in \mathfrak{m}$ for all $k \in K$ and all $X \in \mathfrak{m}$. So the decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

is reductive.

Suppose now that $M = G/K$ is a Kähler manifold with G-invariant Kähler metric g. This means that g is Hermitian and $\omega(X, Y) = g(JX, Y)$ is closed. Let I_0 be the $\mathrm{Ad}(K)$-invariant complex structure on $\mathfrak{m}$ which corresponds to J and $\langle , \rangle$ the $\mathrm{Ad}(K)$-invariant scalar product on $\mathfrak{m}$ which corresponds to g. Recall from Proposition 3.13 that g is Hermitian if and only if $\langle I_0(X), I_0(Y)\rangle = \langle X, Y\rangle$ for all $X, Y \in \mathfrak{m}$. Also recall the 2-form

$$\omega_0(X, Y) = \langle (I_0(X), Y\rangle$$

on $\mathfrak{m}$ and the result that ω is closed if and only if

$$\omega_0([X, Y]_{\mathfrak{m}}, Z) + \omega_0([Y, Z]_{\mathfrak{m}}, X) + \omega_0([Z, X]_{\mathfrak{m}}, Y) = 0, \tag{4.7}$$

for all $X, Y, Z \in \mathfrak{m}$.

Since B is nondegenerate on $\mathfrak{m}$, there exists a linear map $\phi : \mathfrak{m} \to \mathfrak{m}$, such that

$$\omega_0(X, Y) = \mathrm{B}(\phi(X), Y),$$

for all $X, Y \in \mathfrak{m}$. By using that $\omega_0(X, Y) = -\omega_0(Y, X)$, we see that

$$\mathrm{B}(\phi(X), Y) = \omega_0(X, Y) = -\omega_0(Y, X) = -\mathrm{B}(\phi(Y), X) = -\mathrm{B}(X, \phi(Y)).$$

So ϕ is skewsymmetric with respect to B. We extend ϕ to $\mathfrak{g}$ by letting $\phi(X) = 0$ for $X \in \mathfrak{k}$. We will continue to call this extension ϕ.

This is a good place to write out what we intend to do. We will begin by showing that ϕ is a derivation on $\mathfrak{g}$. From Proposition 1.14 this implies that $\phi = \mathrm{ad}(Z_\phi)$, for some $Z_\phi \in \mathfrak{g}$. We can then obtain an expression for $\langle , \rangle$ acting on the basis elements of $\mathfrak{m}$, which depends on the roots in Δ, the set Δ^+ and the element Z_ϕ. Using that $\langle , \rangle$ is an inner product we will see that Z_ϕ must belong to the Weyl chamber which corresponds to J.

Proposition 4.19. [4] *The linear map $\phi : \mathfrak{g} \to \mathfrak{g}$ is a derivation of $\mathfrak{g}$.*

Proof. [4] Our plan is to show that

$$\mathrm{B}(\phi([X, Y]), Z) = \mathrm{B}([\phi(X), Y], Z) + \mathrm{B}([X, \phi(Y)], Z),$$

for all $X, Y, Z \in \mathfrak{g}$ and then use the fact that B is nondegenerate on $\mathfrak{g}$.

Let $X, Y, Z \in \mathfrak{m}$. We rewrite (4.7) by using that $\omega_0(X, Y) = \mathrm{B}(\phi(X), Y)$:

$$\mathrm{B}(\phi([X, Y]_{\mathfrak{m}}), Z) + \mathrm{B}(\phi([Y, Z]_{\mathfrak{m}}), X) + \mathrm{B}(\phi([Z, X]_{\mathfrak{m}}), Y) = 0$$

which is equivalent with

$$\mathrm{B}(\phi([X,Y]_{\mathfrak{m}}),Z) = -\mathrm{B}(\phi([Y,Z]_{\mathfrak{m}}),X) - \mathrm{B}(\phi([Z,X]_{\mathfrak{m}}),Y),$$

and

$$\begin{aligned}
-\mathrm{B}(\phi([Y,Z]_{\mathfrak{m}}),X) - \mathrm{B}(\phi([Z,X]_{\mathfrak{m}}),Y) &= \mathrm{B}([Y,Z]_{\mathfrak{m}},\phi(X)) + \mathrm{B}([Z,X]_{\mathfrak{m}},\phi(Y)) \\
&= \mathrm{B}([Y,Z],\phi(X)) + \mathrm{B}([Z,X],\phi(Y)) \\
&= \mathrm{B}(Y,[Z,\phi(X)]) + \mathrm{B}(Z,[X,\phi(Y)]) \\
&= \mathrm{B}([Z,\phi(X)],Y) + \mathrm{B}([X,\phi(Y)],Z) \\
&= \mathrm{B}(Z,[\phi(X),Y]) + \mathrm{B}([X,\phi(Y)],Z).
\end{aligned}$$

Combining the above equalities we see that

$$\mathrm{B}(\phi([X,Y]),Z) = \mathrm{B}(\phi([X,Y]_{\mathfrak{m}}),Z) = \mathrm{B}([\phi(X),Y],Z) + \mathrm{B}([X,\phi(Y)],Z) \tag{4.8}$$

for all $X,Y,Z \in \mathfrak{m}$. We continue to let $X,Y \in \mathfrak{m}$ but now let $Z \in \mathfrak{k}$. Then

$$\begin{aligned}
\mathrm{B}([\phi(X),Y],Z) + \mathrm{B}([X,\phi(Y)],Z) &= \mathrm{B}(\phi(X),[Y,Z]) - \mathrm{B}([\phi(Y),X],Z) \\
&= \mathrm{B}(\phi(X),[Y,Z]) + \mathrm{B}(\phi(Y),[Z,X]) \\
&= \omega_0(X,[Y,Z]) + \omega_0(Y,[Z,X]) \\
&= 0 \\
&= \mathrm{B}(\phi([X,Y]_{\mathfrak{m}}),Z) \\
&= \mathrm{B}(\phi([X,Y]),Z),
\end{aligned}$$

where we used Corollary 3.16. So

$$\mathrm{B}(\phi([X,Y]),Z) = \mathrm{B}([\phi(X),Y],Z) + \mathrm{B}([X,\phi(Y)],Z), \tag{4.9}$$

for all $X,Y \in \mathfrak{m}$ and $Z \in \mathfrak{k}$. Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ it follows from (4.8) and (4.9) that

$$\mathrm{B}(\phi([X,Y]),Z) = \mathrm{B}([\phi(X),Y],Z) + \mathrm{B}([X,\phi(Y)],Z) \tag{4.10}$$

for all $X,Y \in \mathfrak{m}$ and $Z \in \mathfrak{g}$.

The identity $\omega_0([Z,X],Y) = -\omega_0(X,[Z,Y])$ for $X,Y \in \mathfrak{m}$ and $Z \in \mathfrak{k}$, translates into

$$\mathrm{B}(\phi([Z,X]),Y) = -\mathrm{B}(\phi(X),[Z,Y]).$$

Using this we obtain

$$\begin{aligned}
\mathrm{B}(\phi([Z,X]),Y) &= -\mathrm{B}(\phi(X),[Z,Y]) \\
&= -\mathrm{B}(Z,[Y,\phi(X)]) \\
&= -\mathrm{B}([Y,\phi(X)],Z) \\
&= \mathrm{B}([Z,\phi(X)],Y)
\end{aligned}$$

for $X,Y \in \mathfrak{m}$ and $Z \in \mathfrak{k}$. So

$$\mathrm{B}(\phi([Z,X]),Y) = \mathrm{B}([Z,\phi(X)],Y), \tag{4.11}$$

for all $X,Y \in \mathfrak{m}$ and all $Z \in \mathfrak{k}$. This equality actually holds for all $Y \in \mathfrak{g}$, since

$$\begin{aligned}
\mathrm{B}(\phi([Z,X]),Y) &= \mathrm{B}(\phi([Z,X]),Y_{\mathfrak{m}}) \\
&= \mathrm{B}([Z,\phi(X)],Y_{\mathfrak{m}}) \\
&= \mathrm{B}([Z,\phi(X)],Y).
\end{aligned}$$

Now for the general case we let $X,Y,Z \in \mathfrak{g}$ and write $X = X_1 + X_2$, $Y = Y_1 + Y_2$, where $X_1, Y_1 \in \mathfrak{k}$ and $X_2, Y_2 \in \mathfrak{m}$. We then get, using (4.10) and (4.11), that

$$\mathrm{B}(\phi([X,Y]),Z) = \mathrm{B}(\phi([X_1+X_2,Y_1+Y_2],Z))$$

$$
\begin{aligned}
&= \mathrm{B}(\phi([X_1, Y_1]), Z) + \mathrm{B}(\phi([X_1, Y_2]), Z) + \mathrm{B}(\phi([X_2, Y_1]), Z) \\
&\quad + \mathrm{B}(\phi([X_2, Y_2]), Z) \\
&= \mathrm{B}([X_1, \phi(Y_2)], Z) + \mathrm{B}([\phi(X_2), Y_1], Z) + \mathrm{B}([\phi(X_2), Y_2], Z) \\
&\quad + \mathrm{B}([X_2, \phi(Y_2)], Z) \\
&= \mathrm{B}([X_1 + X_2, \phi(Y_2)], Z) + \mathrm{B}([\phi(X_2), Y_1 + Y_2], Z) = \\
&= \mathrm{B}([\phi(X), Y], Z) + \mathrm{B}([X, \phi(Y)], Z).
\end{aligned}
$$

Using that the Killing form is nondegenerate on $\mathfrak{g}$ we get

$$\phi([X, Y]) = [\phi(X), Y] + [X, \phi(Y)]$$

for all $X, Y \in \mathfrak{g}$. □

From Proposition 1.14 we know that there exists a unique $Z_\phi \in \mathfrak{g}$, such that $\phi = \mathrm{ad}(Z_\phi)$. Hence we may write

$$\omega_0(X, Y) = \mathrm{B}(\phi(X), Y) = \mathrm{B}(\mathrm{ad}(Z_\phi)(X), Y) = \mathrm{B}(Z_\phi, [X, Y]),$$

for all $X, Y \in \mathfrak{m}$.

Proposition 4.20. [4] *The element Z_ϕ is in the subspace $\mathfrak{t}$.*

Proof. Recall that $\mathfrak{t} = \mathrm{Z}(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h}$ and that

$$\mathfrak{h}^{\mathbb{C}} = \mathfrak{g}_0 = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [X, H] = 0 \text{ for all } H \in \mathfrak{h}^{\mathbb{C}}\}.$$

That $Z_\phi \in Z(\mathfrak{k}^{\mathbb{C}})$ follows directly from the fact that $\phi(X) = 0$ for all $X \in \mathfrak{k}$ and $\phi(X) = \mathrm{ad}(Z_\phi)(X)$. In particular we have that Z_ϕ commutes with every element in $\mathfrak{h}^{\mathbb{C}}$, since $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{k}^{\mathbb{C}}$. This implies that $Z_\phi \in \mathfrak{h}^{\mathbb{C}}$. We know that $Z_\phi \in \mathfrak{g}$ and we just saw that $Z_\phi \in \mathfrak{h}^{\mathbb{C}}$ so we must necessarily have that $Z_\phi \in \mathfrak{h}^{\mathbb{C}} \cap \mathfrak{g} = \mathfrak{h}$. It follows directly from this that $Z_\phi \in Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h} = \mathfrak{t}$. □

We are now ready to prove the following theorem:

Theorem 4.21. [4] *Given a generalized flag manifold M, with G-invariant complex structure J and G-invariant Kähler metric g, the corresponding $Ad(K)$-invariant scalar product $\langle , \rangle$ on $\mathfrak{m}$ must satisfy*

$$\langle E_\alpha, E_\beta \rangle = \begin{cases} \frac{1}{\epsilon_\alpha} \alpha(Z_\phi) \ \textit{if } \beta = -\alpha \\ 0 \ \textit{if } \beta \neq -\alpha, \end{cases}$$

where $\epsilon_\alpha = \pm i$ for $\alpha \in \Delta^{\pm}$. Moreover, the element Z_ϕ belongs to the Weyl chamber that corresponds to J.

Proof. [4] Let Δ^+ be the subset of Δ corresponding to J, which satisfies (4.2) and (4.3). We let

$$\mathfrak{m}^+ = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \ \mathfrak{m}^- = \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Since $\mathfrak{m}^{\pm}$ satisfies $\mathfrak{a}^{\pm} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^{\pm}$ we know from Lemma 4.4 that

$$\mathfrak{m}^{\pm} = \{X \in \mathfrak{m}^{\mathbb{C}} \mid I_0(X) = \pm iX \ \}.$$

Let E_α, for $\alpha \in \Delta$, be the generators for the $\mathfrak{g}_\alpha$'s, which satisfies

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \ \theta(E_\alpha) = -E_{-\alpha}.$$

We have that $I_0(E_\alpha) = \epsilon_\alpha E_\alpha$, where

$$\epsilon_\alpha = \begin{cases} i \text{ if } \alpha \in \Delta^+ \\ -i \text{ if } \alpha \in \Delta^- \end{cases}$$

Let $\alpha, \beta \in \Delta$. By definition,

$$\omega_0(E_\alpha, E_\beta) = \langle I_0(E_\alpha), E_\beta \rangle = \epsilon_\alpha \langle E_\alpha, E_\beta \rangle. \tag{4.12}$$

However we showed earlier that $\omega_0(X, Y) = \mathrm{B}(Z_\phi, [X, Y])$ for $X, Y \in \mathfrak{m}^{\mathbb{C}}$, so

$$\omega_0(E_\alpha, E_\beta) = \mathrm{B}(Z_\phi, [E_\alpha, E_\beta]).$$

Recall from Chapter 1 that

$$\mathrm{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \text{ if } \alpha + \beta \neq 0 \text{ for } \alpha, \beta \in R \cup \{0\}.$$

We also know from Chapter 1 that $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta}$. Since $Z_\phi \in \mathfrak{t} \subset \mathfrak{h}^{\mathbb{C}} = \mathfrak{g}_0$, we have that

$$\mathrm{B}(Z_\phi, [E_\alpha, E_\beta]) = 0 \text{ if } \alpha + \beta \neq 0,$$

which implies that $\omega_0(E_\alpha, E_\beta) = 0$ if $\beta \neq -\alpha$. Let us now consider the case when $\beta = -\alpha$. In this case

$$\begin{aligned}\omega_0(E_\alpha, E_{-\alpha}) &= \mathrm{B}(Z_\phi, [E_\alpha, E_{-\alpha}]) \\ &= \mathrm{B}(Z_\phi, H_\alpha) \\ &= \alpha(Z_\phi).\end{aligned}$$

We have now shown the following:

$$\omega_0(E_\alpha, E_\beta) = \begin{cases} \alpha(Z_\phi) \text{ if } \beta = -\alpha \\ 0 \text{ if } \beta \neq -\alpha \end{cases} \tag{4.13}$$

From (4.12) we get that

$$\langle E_\alpha, E_\beta \rangle = \frac{1}{\epsilon_\alpha} \omega_0(E_\alpha, E_\beta).$$

We combine this with (4.13) to obtain the following:

$$\langle E_\alpha, E_\beta \rangle = \begin{cases} \frac{1}{\epsilon_\alpha} \alpha(Z_\phi) \text{ if } \beta = -\alpha \\ 0 \text{ if } \beta \neq -\alpha. \end{cases} \tag{4.14}$$

By assumption $\langle , \rangle$ is a scalar product on $\mathfrak{m}$, so if we let

$$\langle X, Y \rangle^{\mathbb{C}} = \langle X, \theta(Y) \rangle$$

for $X, Y \in \mathfrak{m}^{\mathbb{C}}$, then $\langle , \rangle^{\mathbb{C}}$ will be an inner product on $\mathfrak{m}^{\mathbb{C}}$. Let $\alpha \in \Delta^+$. Then $E_\alpha \neq 0$, so

$$\langle E_\alpha, E_\alpha \rangle^{\mathbb{C}} > 0.$$

If we recall that $\theta(E_\alpha) = -E_{-\alpha}$, we see that

$$-\langle E_\alpha, E_{-\alpha} \rangle = \langle E_\alpha, \theta(E_\alpha) \rangle = \langle E_\alpha, E_\alpha \rangle^{\mathbb{C}} > 0.$$

Using (4.14) we get

$$-\frac{1}{\epsilon_\alpha} \alpha(Z_\phi) > 0, \tag{4.15}$$

since we assumed $\alpha \in \Delta^+$ we know that $\epsilon_\alpha = i$. So (4.15) is equivalent to

$$i\alpha(Z_\phi) > 0$$

and since $\alpha \in \Delta^+$ was chosen arbitrarily, this holds for all $\alpha \in \Delta^+$. We know from Proposition 4.20 that $Z_\phi \in \mathfrak{t}$ and we just showed that $i\alpha(Z_\phi) > 0$ for all $\alpha \in \Delta^+$. This implies that Z_ϕ belongs to some Weyl chamber. Because of the inequalities $i\alpha(Z_\phi) > 0$ for all $\alpha \in \Delta^+$, it is clear that this Weyl chamber must be precisely the one that corresponds to Δ^+.

□

Suppose now that we are only given a generalized flag manifold M with a G-invariant complex structure J. We would then like to show that there exists a G-invariant Kähler metric on M. In order to do this we will construct an $\mathrm{Ad}(K)$-invariant inner product $\langle , \rangle$ on $\mathfrak{m}$ which satisfies

$$\langle I_0(X), I_0(Y)\rangle = \langle X, Y\rangle$$

for all $X, Y \in \mathfrak{m}$ and such that the corresponding 2-form ω_0, is closed i.e. satisfies (4.7). By our previous results, this implies the existence of a G-invariant Kähler metric on M.

We continue to let Δ^+ be the subset of $\Delta = R \backslash R_K$ which corresponds to J and we also let $\mathfrak{m}^{\mathbb{C}}$, $\mathfrak{m}$ and $\mathfrak{m}^{\pm}$ be defined as before. Let Z_J be any element in the Weyl chamber corresponding to Δ^+. By using that $\mathfrak{m} = \{X \in \mathfrak{m}^{\mathbb{C}} \mid \theta(X) = X\}$, we get that every $X \in \mathfrak{m}$ can be written as

$$X = \sum_{\alpha\in\Delta^+} a_\alpha E_\alpha - \overline{a_\alpha} E_{-\alpha}$$

where $a_\alpha \in \mathbb{C}$. We begin by defining a bilinear form $\langle , \rangle$ on the basis elements E_α, for $\alpha, \beta \in \Delta$. Bases on our results from Theorem 4.21 we define $\langle , \rangle$ in the following way:

$$\langle E_\alpha, E_\beta\rangle = \begin{cases} \frac{1}{\epsilon_\alpha}\alpha(Z_J) \text{ if } \beta = -\alpha \\ 0 \text{ if } \beta \neq -\alpha \end{cases},$$

where $\epsilon_\alpha = \pm i$ for $\alpha \in \Delta^{\pm}$.

Proposition 4.22. [4] *The bilinear form $\langle , \rangle$ is an inner product on $\mathfrak{m}$.*

Proof. We begin by noting that

$$\langle E_\alpha, E_{-\alpha}\rangle = \frac{1}{\epsilon_\alpha}\alpha(Z_J) \in \mathbb{R}$$

for all $\alpha \in \Delta$, since $i\alpha(Z_J) > 0$ for all $\alpha \in \Delta^+$ and $i\alpha(Z_J) < 0$ for all $\alpha \in \Delta^-$. This is because we chose Z_J to be in the Weyl chamber corresponding to Δ^+. Moreover,

$$\langle E_{-\alpha}, E_\alpha\rangle = \frac{1}{\epsilon_{-\alpha}}(-\alpha(Z_J)) = \frac{1}{\epsilon_\alpha}\alpha(Z_J) = \langle E_\alpha, E_{-\alpha}\rangle,$$

where we used that $-\epsilon_{-\alpha} = \epsilon_\alpha$. Next we let

$$X = \sum_{\alpha\in\Delta^+} a_\alpha E_\alpha - \overline{a_\alpha} E_{-\alpha},\ Y = \sum_{\beta\in\Delta^+} b_\beta E_\beta - \overline{b_\beta} E_{-\beta}$$

be two elements in $\mathfrak{m}$. Then

$$\begin{aligned}
\langle X, Y\rangle &= \langle \sum_{\alpha\in\Delta^+} a_\alpha E_\alpha - \overline{a_\alpha} E_{-\alpha}, \sum_{\beta\in\Delta^+} b_\beta E_\beta - \overline{b_\beta} E_{-\beta}\rangle \\
&= \sum_{\alpha\in\Delta^+} (-a_\alpha \overline{b_\alpha}\langle E_\alpha, E_{-\alpha}\rangle - \overline{a_\alpha} b_\alpha \langle E_{-\alpha}, E_\alpha\rangle) \\
&= \sum_{\alpha\in\Delta^+} -(a_\alpha\overline{b_\alpha} + \overline{a_\alpha} b_\alpha)\langle E_\alpha, E_{-\alpha}\rangle \\
&= \sum_{\alpha\in\Delta^+} -(a_\alpha\overline{b_\alpha} + \overline{a_\alpha} b_\alpha)\frac{1}{\epsilon_\alpha}\alpha(Z_J) \\
&= \sum_{\alpha\in\Delta^+} (a_\alpha\overline{b_\alpha} + \overline{a_\alpha} b_\alpha) i\alpha(Z_J).
\end{aligned}$$

First we note that

$$\overline{a_\alpha\overline{b_\alpha} + \overline{a_\alpha} b_\alpha} = \overline{a_\alpha} b_\alpha + a_\alpha \overline{b_\alpha}$$

which implies that $a_\alpha \overline{b_\alpha} + \overline{a_\alpha} b_\alpha \in \mathbb{R}$ for all $\alpha \in \Delta^+$ and we already know that $i\alpha(Z_J) \in \mathbb{R}$. It follows that $\langle , \rangle$ is real valued on $\mathfrak{m} \times \mathfrak{m}$. If we let $Y = X$ we see that

$$\begin{aligned} \langle X, X \rangle &= \sum_{\alpha \in \Delta^+} (a_\alpha \overline{a_\alpha} + \overline{a_\alpha} a_\alpha) i\alpha(Z_J) \\ &= \sum_{\alpha \in \Delta^+} 2|a_\alpha|^2 i\alpha(Z_J). \end{aligned}$$

It is clear that $|a_\alpha|^2 \geq 0$ and $i\alpha(Z_J) > 0$ for all $\alpha \in \Delta^+$. Hence

$$\langle X, X \rangle \geq 0$$

with equality if and only if $a_\alpha = 0$ for all $\alpha \in \Delta^+$, which is equivalent to $X = 0$. So $\langle , \rangle$ is bilinear, positive definite and it is obvious that it is symmetric, since $\langle E_\alpha, E_\beta \rangle = \langle E_\beta, E_\alpha \rangle$, for all $\alpha, \beta \in \Delta$. So $\langle , \rangle$ is an inner product on $\mathfrak{m}$. □

We directly move on with the next proposition.

Proposition 4.23. [4] *The inner product $\langle , \rangle$ satisfies*

$$\langle I_0(X), I_0(Y)) \rangle = \langle X, Y \rangle$$

for all $X, Y \in \mathfrak{m}$.

Proof. We know that $I_0(E_\alpha) = \pm i E_\alpha$, depending on whether $\alpha \in \Delta^\pm$. So for $\beta \neq -\alpha$

$$\langle I_0(E_\alpha), I_0(E_\beta) \rangle = \langle \pm i E_\alpha, \pm i E_\beta \rangle = 0$$

and for $\beta = -\alpha$ we have

$$\langle I_0(E_\alpha), I_0(E_{-\alpha}) \rangle = \langle \pm i E_\alpha, \mp i E_{-\alpha} \rangle = \langle E_\alpha, E_{-\alpha} \rangle.$$

This means that $\langle I_0(X), I_0(Y) \rangle = \langle X, Y \rangle$ whenever $X = E_\alpha$, $Y = E_\beta$, for $\alpha, \beta \in \Delta$. Let

$$X = \sum_{\alpha \in \Delta} a_\alpha E_\alpha, \; Y = \sum_{\beta \in \Delta} b_\beta E_\alpha$$

be two elements in $\mathfrak{m}$. Then

$$\begin{aligned} \langle I_0(X), I_0(Y) \rangle &= \sum_{\alpha, \beta \in \Delta} a_\alpha b_\beta \langle I_0(E_\alpha), I_0(E_\beta) \rangle \\ &= \sum_{\alpha, \beta \in \Delta} a_\alpha b_\beta \langle E_\alpha, E_\beta \rangle \\ &= \langle X, Y \rangle. \end{aligned}$$

□

We proceed by showing the $\mathrm{Ad}(K)$-invariance.

Proposition 4.24. [4] *The inner product $\langle , \rangle$ is $Ad(K)$-invariant.*

Proof. We begin by noting that for $\alpha \in \Delta$ we have

$$\begin{aligned} \langle E_\alpha, E_{-\alpha} \rangle &= \frac{1}{\epsilon_\alpha} \alpha(Z_J) \\ &= -\epsilon_\alpha \mathrm{B}(H_\alpha, Z_J) \\ &= -\epsilon_\alpha \mathrm{B}([E_\alpha, E_{-\alpha}], Z_J) \\ &= -\mathrm{B}([\epsilon_\alpha E_\alpha, E_{-\alpha}], Z_J) \\ &= -\mathrm{B}([I_0(E_\alpha), E_{-\alpha}], Z_J). \end{aligned}$$

Recall that $\mathrm{B}(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ for $\alpha, \beta \in R \cup \{0\}$ such that $\alpha + \beta \neq 0$. This means that for $\beta \neq -\alpha$ we have that

$$-\mathrm{B}([I_0(E_\alpha), E_\beta], Z_J) = -\epsilon_\alpha \mathrm{B}([E_\alpha, E_\beta], Z_J) = 0 = \langle E_\alpha, E_\beta \rangle,$$

where we also used that $[E_\alpha, E_\beta] \in \mathfrak{g}_{\alpha+\beta}$ and that $Z_J \in \mathfrak{t} \subset \mathfrak{h}^{\mathbb{C}} = \mathfrak{g}_0$. These calculations imply that

$$\langle E_\alpha, E_\beta \rangle = -\mathrm{B}([I_0(E_\alpha), E_\beta], Z_J).$$

For all $\alpha, \beta \in \Delta$. Now let

$$X = \sum_{\alpha \in \Delta} a_\alpha E_\alpha, \;\; Y = \sum_{\beta \in \Delta} b_\beta E_\beta$$

be two elements in $\mathfrak{m}$. Then

$$\begin{aligned}
-\mathrm{B}([I_0(X), Y], Z_J) &= -\mathrm{B}([I_0(\sum_{\alpha \in \Delta} a_\alpha E_\alpha), \sum_{\beta \in \Delta} b_\beta E_\beta], Z_J) \\
&= \sum_{\alpha, \beta \in \Delta} a_\alpha b_\beta (-\mathrm{B}([I_0(E_\alpha), E_\beta], Z_J)) \\
&= \sum_{\alpha, \beta \in \Delta} a_\alpha b_\beta \langle E_\alpha, E_\beta \rangle \\
&= \langle X, Y \rangle.
\end{aligned}$$

Hence we have that

$$\langle X, Y \rangle = -\mathrm{B}([I_0(X), Y], Z_J)$$

for all $X, Y \in \mathfrak{m}$.

By construction Z_J belongs to the Weyl chamber corresponding to Δ^+, so in particular

$$Z_J \in \mathfrak{t} = Z(\mathfrak{k}^{\mathbb{C}}) \cap \mathfrak{h}.$$

It is easy to see that $Z(\mathfrak{k})$ is the Lie algebra of $C(K)$ and from this we get that $\exp(tZ_J) \in C(K)$ for all $t \in \mathbb{R}$. Using this we see that for $k \in K$

$$\mathrm{Ad}(k)(Z_J) = \frac{d}{dt}(I_k(\exp(tZ_J)))|_{t=0} = \frac{d}{dt}(\exp(tZ_J))|_{t=0} = Z_J.$$

We also know that I_0 is $\mathrm{Ad}(K)$-invariant, that is $I_0(\mathrm{Ad}(k)(X)) = \mathrm{Ad}(k)(I_0(X))$, for all $k \in K$ and all $X \in \mathfrak{m}$. Recall that $\mathrm{Ad}(k)$ is a Lie algebra automorphism for all $k \in K$. If we combine this with Lemma 1.12, the $\mathrm{Ad}(K)$-invariance of I_0 and the fact that $\mathrm{Ad}(k)(Z_J) = Z_J$ for all $k \in K$, we obtain the following identity:

$$\begin{aligned}
\mathrm{B}([I_0(\mathrm{Ad}(k)(X)), \mathrm{Ad}(k)(Y)], Z_J) &= \mathrm{B}([\mathrm{Ad}(k)(I_0(X)), \mathrm{Ad}(k)(Y)], Z_J) \\
&= \mathrm{B}(\mathrm{Ad}(k)([I_0(X), Y]), Z_J) \\
&= \mathrm{B}([I_0(X), Y], \mathrm{Ad}(k^{-1})(Z_J)) \\
&= \mathrm{B}([I_0(X), Y], Z_J),
\end{aligned}$$

for all $k \in K$ and all $X, Y \in \mathfrak{m}$. But this implies that

$$\begin{aligned}
\langle \mathrm{Ad}(k)(X), \mathrm{Ad}(k)(Y) \rangle &= -\mathrm{B}([I_0(\mathrm{Ad}(k)(X)), \mathrm{Ad}(k)(Y)], Z_J) \\
&= -\mathrm{B}([I_0(X), Y], Z_J) \\
&= \langle X, Y \rangle
\end{aligned}$$

for all $k \in K$ and all $X, Y \in \mathfrak{m}$. This proves that the inner product $\langle , \rangle$ is $\mathrm{Ad}(K)$-invariant.

□

Finally we will show that ω_0 satisfies (4.7). In order to do this we need an additional result. Recall the number $N_{\alpha,\beta}$ defined for $\alpha, \beta \in R$ such that $\alpha + \beta \neq 0$ by

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}$$

Lemma 4.25. [9] *If $\alpha, \beta, \gamma \in R$ satisfies $\alpha + \beta + \gamma = 0$ then*

$$N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha}.$$

Proof. [9] Using the Jacobi identity and the assumption that $\alpha + \beta + \gamma = 0$, we see that

$$\begin{aligned} 0 =& [E_\alpha, [E_\beta, E_\gamma]] + [E_\gamma, [E_\alpha, E_\beta]] + [E_\beta, [E_\gamma, E_\alpha]] \\ &= [E_\alpha, N_{\beta,\gamma} E_{\beta+\gamma}] + [E_\gamma, N_{\alpha,\beta} E_{\alpha+\beta}] + [E_\beta, N_{\gamma,\alpha} E_{\gamma+\alpha}] \\ &= N_{\beta,\gamma} H_\alpha + N_{\alpha,\beta} H_\gamma + N_{\gamma,\alpha} H_\beta \end{aligned}$$

So

$$N_{\beta,\gamma} H_\alpha + N_{\alpha,\beta} H_\gamma + N_{\gamma,\alpha} H_\beta = 0 \tag{4.16}$$

By definition H_γ is the unique element in $\mathfrak{h}^{\mathbb{C}}$ such that $\mathrm{B}(H_\gamma, H) = \gamma(H)$ for all $H \in \mathfrak{h}^{\mathbb{C}}$. But

$$\gamma(H) = -(\alpha + \beta)(H) = -\mathrm{B}(H_\alpha, H) - \mathrm{B}(H_\beta, H) = \mathrm{B}(-(H_\alpha + H_\beta), H).$$

By uniqueness we must therefore have that $-H_\gamma = H_\alpha + H_\beta$. If we use this in (4.16) we get

$$(N_{\beta,\gamma} - N_{\alpha,\beta}) H_\alpha + (N_{\gamma,\alpha} - N_{\alpha,\beta}) H_\beta = 0.$$

Suppose that the vectors H_α and H_β are not linearly independent. Then there exists some complex number c such that $H_\beta = c H_\alpha$. Hence

$$\beta(H) = \mathrm{B}(H_\beta, H) = \mathrm{B}(c H_\alpha, H) = c\alpha(H)$$

for all $H \in \mathfrak{h}^{\mathbb{C}}$ and therefore $\beta = c\alpha$. But $c\alpha \notin R$ unless $c = -1, 0, 1$. If $c = 1$ then $\alpha = \beta$ so we have $2\alpha + \gamma = 0$ or equivalently $\gamma = -2\alpha$. But $-2\alpha \notin R$ so $c \neq 1$. If $c = 0$ we get $\beta = 0$ and 0 is not a root so $c \neq 0$. If $c = -1$ we get $\gamma = 0$ which is also a contradiction. So there cannot exist such a number c. This means that

$$(N_{\beta,\gamma} - N_{\alpha,\beta}) H_\alpha + (N_{\gamma,\alpha} - N_{\alpha,\beta}) H_\beta = 0$$

if and only if

$$N_{\beta,\gamma} = N_{\alpha,\beta} = N_{\gamma,\alpha}.$$

□

Proposition 4.26. [4] *The bilinear form ω_0 is closed.*

Proof. Let $\alpha, \beta, \gamma \in \Delta$ and consider

$$\omega_0([E_\alpha, E_\beta]_{\mathfrak{m}}, E_\gamma) + \omega_0([E_\beta, E_\gamma]_{\mathfrak{m}}, E_\alpha) + \omega_0([E_\gamma, E_\alpha]_{\mathfrak{m}}, E_\beta). \tag{4.17}$$

Suppose first that $\alpha + \beta \notin R$. Then $[E_\alpha, E_\beta] = 0$ or $[E_\alpha, E_\beta] = H_\alpha$. In either case $[E_\alpha, E_\beta]_{\mathfrak{m}} = 0$, which implies that $\omega_0([E_\alpha, E_\beta]_{\mathfrak{m}}, E_\gamma) = 0$.

We now suppose that $\alpha + \beta \in R$. In the case when $\alpha + \beta \in R_K$ we clearly have that

$$[E_\alpha, E_\beta]_{\mathfrak{m}} = N_{\alpha,\beta} (E_{\alpha+\beta})_{\mathfrak{m}} = 0.$$

If $\alpha + \beta \in \Delta$ we instead get that

$$\begin{aligned} \omega_0([E_\alpha, E_\beta]_{\mathfrak{m}}, E_\gamma) &= \omega_0(N_{\alpha,\beta} (E_{\alpha+\beta})_{\mathfrak{m}}, E_\gamma) \\ &= \omega_0(N_{\alpha,\beta} E_{\alpha+\beta}, E_\gamma) \\ &= \langle N_{\alpha,\beta} I_0(E_{\alpha+\beta}), E_\gamma \rangle \\ &= \langle N_{\alpha,\beta} \epsilon_{\alpha+\beta} E_{\alpha+\beta}, E_\gamma \rangle \\ &= \begin{cases} N_{\alpha,\beta}(\alpha + \beta)(Z_J) & \text{if } \alpha + \beta + \gamma = 0 \\ 0 & \text{otherwise .} \end{cases} \end{aligned}$$

We cannot have that $\alpha+\beta+\gamma=0$ when $\alpha+\beta\notin\Delta$, because this would imply that $\gamma\notin\Delta$. If we combine this with the fact that $\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma)=0$ when $\alpha+\beta\notin\Delta$, we get that

$$\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma)=\begin{cases}N_{\alpha,\beta}(\alpha+\beta)(Z_J) & \text{if } \alpha+\beta+\gamma=0\\ 0 & \text{otherwise}\end{cases}$$

for all $\alpha,\beta,\gamma\in\Delta$.

Clearly the other two terms in (4.17) can also be expressed in this way.

$$\omega_0([E_\beta,E_\gamma]_{\mathfrak{m}},E_\alpha)=\begin{cases}N_{\beta,\gamma}(\beta+\gamma)(Z_J) & \text{if } \alpha+\beta+\gamma=0\\ 0 & \text{otherwise}\end{cases}$$
$$\omega_0([E_\gamma,E_\alpha]_{\mathfrak{m}},E_\beta)=\begin{cases}N_{\gamma,\alpha}(\gamma+\alpha)(Z_J) & \text{if } \alpha+\beta+\gamma=0\\ 0 & \text{otherwise}\end{cases}.$$

It then follows that (4.17) vanishes when $\alpha+\beta+\gamma\neq 0$. We now assume $\alpha+\beta+\gamma=0$. In this case we know from Lemma 4.25 that $N_{\alpha,\beta}=N_{\beta,\gamma}=N_{\gamma,\alpha}$. Using this we see that

$$\begin{aligned}&\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma)+\omega_0([E_\beta,E_\gamma]_{\mathfrak{m}},E_\alpha)+\omega_0([E_\gamma,E_\alpha]_{\mathfrak{m}},E_\beta)\\ &\quad=N_{\alpha,\beta}(\alpha+\beta)(Z_J)+N_{\beta+\gamma}(\beta+\gamma)(Z_J)+N_{\gamma,\alpha}(\gamma+\alpha)(Z_J)\\ &\quad=N_{\alpha,\beta}(\alpha+\beta+\beta+\gamma+\gamma+\alpha)(Z_J)\\ &\quad=N_{\alpha,\beta}((\alpha+\beta+\gamma)(Z_J)+(\alpha+\beta+\gamma)(Z_J))\\ &\quad=0,\end{aligned}$$

since $\alpha+\beta+\gamma=0$.

We now consider the general case, so let

$$X=\sum_{\alpha\in\Delta}a_\alpha E_\alpha,\ Y=\sum_{\beta\in\Delta}b_\beta E_\beta,\ Z=\sum_{\gamma\in\Delta}c_\gamma E_\gamma$$

be elements in $\mathfrak{m}$. Then

$$[X,Y]_{\mathfrak{m}}=\sum_{\alpha,\beta\in\Delta}a_\alpha b_\beta[E_\alpha,E_\beta]_{\mathfrak{m}}$$

so

$$\begin{aligned}\omega_0([X,Y]_{\mathfrak{m}},Z)&=\omega_0(\sum_{\alpha,\beta\in\Delta}a_\alpha b_\beta[E_\alpha,E_\beta]_{\mathfrak{m}},\sum_{\gamma\in\Delta}c_\gamma E_\gamma)\\ &=\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma).\end{aligned}$$

In the same way we get

$$\begin{aligned}\omega_0([Y,Z]_{\mathfrak{m}},X)&=\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\beta,E_\gamma]_{\mathfrak{m}},E_\alpha)\\ \omega_0([Z,X]_{\mathfrak{m}},Y)&=\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\gamma,E_\alpha]_{\mathfrak{m}},E_\beta).\end{aligned}$$

This means that

$$\begin{aligned}&\omega_0([X,Y]_{\mathfrak{m}},Z)+\omega_0([Y,Z]_{\mathfrak{m}},X)+\omega_0([Z,X]_{\mathfrak{m}},Y)\\ &=\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma)+\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\beta,E_\gamma]_{\mathfrak{m}},E_\alpha)\\ &\quad+\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\beta c_\gamma\omega_0([E_\gamma,E_\alpha]_{\mathfrak{m}},E_\beta)\\ &=\sum_{\alpha,\beta,\gamma\in\Delta}a_\alpha b_\alpha c_\alpha(\omega_0([E_\alpha,E_\beta]_{\mathfrak{m}},E_\gamma)+\omega_0([E_\beta,E_\gamma]_{\mathfrak{m}},E_\alpha)+\omega_0([E_\gamma,E_\alpha]_{\mathfrak{m}},E_\beta))\end{aligned}$$

$= 0.$

Since X, Y, Z were chosen arbitrarily, this concludes the proof.

□

Combining these propositions with our earlier results, we may prove the following theorem.

Theorem 4.27. [1], [4] *Given a generalized flag manifold M with a G-invariant complex structure J there exists a G-invariant Kähler metric on M.*

Proof. By Propositions 4.22, 4.23, 4.24 and 4.26 the scalar product $\langle , \rangle$ is $\mathrm{Ad}(K)$-invariant, satisfies $\langle I_0(X), I_0(Y) \rangle = \langle X, Y \rangle$ for all $X, Y \in \mathfrak{m}$ and the corresponding bilinear form ω_0 is closed. By Propositions 2.12, 3.13 and 3.17 this implies the existence of a G-invariant Kähler metric on M.

Remark 4.28. Note that this metric is not uniquely defined since Z_J was chosen arbitrarily from the Weyl chamber corresponding to J. So we could have chosen another element W_J in this Weyl chamber and we would have then ended up with a different metric.

□

Appendix A

Differentiable Manifolds

Let M be a differentiable manifold. Following [8] we will always assume that M is Hausdorff and second-countable.

Lemma A.1. [14] *Let M and N be two differentiable manifolds. If $\varphi : M \to N$ is a smooth, bijective and everywhere non-singular map, then φ is a diffeomorphism.*

Proof. [14] Since φ is assumed to be smooth, bijective and everywhere non-singular we see that φ is a diffeomorphism if and only if $d\varphi_x$ is surjective for all $x \in M$. Assume for a contradiction that this is not the case, so there exists some $x \in M$ such that $d\varphi_x$ is not surjective. Since φ was assumed to be everywhere non-singular this implies

$$m = \dim(M) < \dim(N) = n.$$

Let (V, y) be a chart on N such that $y(V) = \mathbb{R}^n$. Let $\{(U_i, x_i) \mid i = 1, 2 \ldots\}$ be a countable atlas on M. Since M is assumed to be second-countable it is possible to find such an atlas.

Consider the function $y \circ \varphi$. This function has range $\mathbb{R}^n$ since φ was assumed to be surjective. For each i $x_i(U_i)$ is an open subset of $\mathbb{R}^m$. Let λ^n be the Lebesgue measure on $\mathbb{R}^n$. Then, since $m < n$ and $x_i(U_i) \subset \mathbb{R}^m \subset \mathbb{R}^n$, it follows that $\lambda^n(x_i(U_i)) = 0$. Since φ was assumed to be smooth, the map

$$y \circ \varphi \circ x_i^{-1} : x_i(U_i) \subset \mathbb{R}^n \to \mathbb{R}^n$$

is smooth. Since smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^n$ takes sets of measure 0 to sets of measure 0, we have

$$\lambda^n((y \circ \varphi \circ x_i^{-1})(x_i(U_i))) = 0.$$

Since $M = \cup_{n=1}^{\infty} U_i$ we have that

$$\begin{aligned} \mathbb{R}^n &= (y \circ \varphi)(M) \\ &= \cup_{n=1}^{\infty} (y \circ \varphi)(U_i) \\ &= \cup_{n=1}^{\infty} (y \circ \varphi \circ x_i^{-1})(x_i(U_i)). \end{aligned}$$

But this implies that

$$\lambda^n(\mathbb{R}^n) = \lambda^n(\cup_{n=1}^{\infty} (y \circ \varphi \circ x_i^{-1})(x_i(U_i))) \leq \sum_{n=1}^{\infty} \lambda^n((y \circ \varphi \circ x_i^{-1})(x_i(U_i))) = 0$$

which is clearly false. $\square$

Bibliography

[1] Alekseevsky, D.V. and Perelomov, A.M. *Invariant Kähler-Einstein metrics on compact homogeneous spaces*, Funktsional. Anal. i Prilozhen., 20(3); Engl. transl. in Funct. Anal. Appl., 20(3), 171–182, (1986).

[2] A. Arvanitoyeorgos, *An Introduction to Lie Groups and the Geometry of Homogeneous Spaces*, Student Mathematical Library **22**, AMS (2003).

[3] A. Arvanitoyeorgos, *New Invariant Einstein Metrics On Generalized Flag Manifolds*, Trans. Amer. Math. Soc. **337** (1993), pp. 981–995.

[4] M. Bordemann, M. Forger, H. Römer, *Homogeneous Kähler manifolds: paving the way towards supersymmetric sigma-models*, Commun. Math. Phys. **102** (1986), 605–647.

[5] A. Borel and F. Hirzebruch, *Characteristic classes and homogeneous spaces I*, Amer. J. Math. **80** (1958), 458-538.

[6] D. Bump *Lie groups*, Graduate Texts in Mathematics, vol. **225**. Springer, (2004)

[7] A. Fröhlicher, *Zur DifferentialGeometrie der Komplexen Structuren*, Math. Ann. **129** (1955), 50-95.

[8] S. Gudmundsson, *An Introduction to Riemannian Geometry*, Lund University (2012). http://www.matematik.lu.se/matematiklu/personal/sigma/Riemann.pdf

[9] S. Helgason, *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press (1978).

[10] A W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser (2002).

[11] S. Kobayashi, K. Nomizu, *Foundations of Differential Geometry* vol. II, John Wiley and Sons (1969).

[12] M. Nishiyama, *Classification of invariant complex structures on irreducible compact simply connected coset spaces*, Osaka J. Math. **21** (1984), 39-58.

[13] H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math, vol. 76 (1954), pp. 1-32.

[14] F. Warner, *Foundations of Differential Manifolds and Lie Groups*, Springer-Verlag, New York (1983).

[15] K. Yang, *Almost complex homogeneous spaces and their submanifolds*, World Scientific (1987).

[16] K. Yano, M. Kon, *Structures on Manifolds*, Series in Pure Mathematics **3**, World Scientific (1984).

Master's Theses in Mathematical Sciences 2013:E17
ISSN 1404-6342

LUNFMA-3077-2013

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lth.se/