Biharmonic Functions on Spheres and Hyperbolic Spaces

Jennifer Pai
Master’s thesis
2019:E53
Abstract

This Master’s thesis is a study of the recent constructions of proper biharmonic complex-valued functions from the hyperbolic spaces $\mathbb{H}^n$ and the spheres $S^n$ to $\mathbb{C}$ of any dimension $n \geq 2$ in [10] and from $SU(2)$ to $\mathbb{C}$ in [8].

Since the $n$-dimensional hyperbolic space can be modeled in several different ways, we will study the construction of explicit proper $r$-harmonic functions on the half-space model $\mathbb{H}^n$ and use an isometry to translate these functions onto the hyperboloid model $\mathcal{H}^n$. Then we will use a duality principle to translate those functions onto the $n$-dimensional sphere $S^n$.

The first proper biharmonic functions, from open subsets of the classical compact simple Lie group $SU(2)$ have been constructed in [9] as quotients of linear combinations of the matrix coefficients for the standard irreducible respresentation $\pi_1$ of $SU(2)$. In chapter 5 we study the constructions of new proper biharmonic functions as rational functions in the matrix coefficients of the irreducible representations $\pi_2, \pi_3,$ and $\pi_4$ of $SU(2)$, respectively.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.
Acknowledgments

I would like to thank my supervisor Sigmundur Gudmundsson for his patience and guidance throughout this project. Thank you to my family for loving and supporting me especially during this thesis period. Finally, thank you to my friends in Lund and abroad for all the encouragement the past two years to help get me through it. Thank you and love you all! ☽

Jennifer Pai
# Contents

1. Introduction 1
2. The Hyperbolic Upper-Half Space $\mathbb{H}^n$ 5
3. The Hyperbolic One-Sheeted Hyperboloid $\mathcal{H}^n$ 9
4. The $n$-Dimensional Sphere $\mathbb{S}^n$ 13
5. The Special Unitary Group $\text{SU}(2)$ 17
   5.1 The Standard Irreducible Representation $\pi_1$ of $\text{SU}(2)$ 18
   5.2 The Irreducible Representation $\pi_2$ of $\text{SU}(2)$ 21
   5.3 The Irreducible Representation $\pi_3$ of $\text{SU}(2)$ 22
   5.4 The Irreducible Representation $\pi_4$ of $\text{SU}(2)$ 22
   5.5 The Irreducible Representation $\pi_n$ of $\text{SU}(2)$ 23

Bibliography 25
Chapter 1

Introduction

We will assume that the reader has an understanding of the lecture notes *An Introduction to Riemannian Geometry* [11]. We will construct recent proper biharmonic complex-valued functions from the hyperbolic spaces $\mathbb{H}^n$ and the spheres $S^n$ to $\mathbb{C}$ of any dimension $n \geq 2$ and we will need the following definitions.

**Definition 1.1.** [11] For positive integers $m, n \in \mathbb{Z}^+$ with $m \geq n$ a differentiable map $\phi : M^m \to N^n$ between two manifolds is said to be a *submersion* if for each $p \in M$ the differential $d\phi_p : T_p M \to T_{\phi(p)} N$ is surjective.

**Definition 1.2.** [11] Let $(M, g)$ be a Riemannian manifold. Then the *gradient* of a smooth function $f : (M, g) \to \mathbb{R}$ is the vector field $\nabla f$ characterized by $g(\nabla f, E) = df(E), \quad (x \in M, E \in T_x M)$.

In local coordinates, it has the expression

$$\nabla f = \left( \sum_{i,j=1}^m g^{ij} \frac{\partial f}{\partial x^j} \right) \frac{\partial}{\partial x^i}.$$  

Now let $(M, g)$ be a smooth $m$-dimensional manifold equipped with a Riemannian metric $g$. We complexify the tangent bundle $TM$ of $M$ to $T^\mathbb{C}M$ and extend the metric $g$ to a complex-bilinear form on $T^\mathbb{C}M$. Then the gradient $\nabla f$ of a complex-valued function $f : (M, g) \to \mathbb{C}$ is a section of $T^\mathbb{C}M$.

We generalize the Laplace operator to functions on (semi)-Riemannian manifolds and we have the *Laplace-Beltrami operator* applied onto a complex-valued function defined below.

**Definition 1.3.** [10] The well-known linear *Laplace-Beltrami operator* (alt. tension field) $\tau$ on $(M, g)$ acts locally on $f$ as follows

$$\tau(f) = \text{div}(\nabla f) = \sum_{i,j=1}^m \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^j} \left( g^{ij} \sqrt{|g|} \frac{\partial f}{\partial x^i} \right). \quad (1.1)$$

**Proposition 1.4.** [10] For two complex-valued functions $f, h : (M, g) \to \mathbb{C}$, we have the following well-known relation:

$$\tau(f \cdot h) = \tau(f) \cdot h + 2 \cdot \kappa(f, h) + f \cdot \tau(h),$$

where the conformality operator $\kappa$ is given by

$$\kappa(f, h) = g(\nabla f, \nabla h).$$
Now we use this generalized Laplace operator $\tau$ to define the notions of harmonicity as follows.

**Definition 1.5.** [10] For an integer $r > 0$ the *iterated Laplace-Beltrami operator* $\tau^r$ is given by

$$\tau^0(f) = f \text{ and } \tau^r(f) = \tau(\tau^{(r-1)}(f)).$$

We say that a complex-valued function $f : (M, g) \to \mathbb{C}$ is

(a) $r$-harmonic if $\tau^r(f) = 0$, and

(b) proper $r$-harmonic if $\tau^r(f) = 0$ and $\tau^{(r-1)}(f)$ does not vanish identically.

The harmonic functions are $1$-harmonic for $r = 1$ and the biharmonic functions are the $2$-harmonic ones. We will be looking at the upper-half space and hyperboloid models of the $n$-dimensional hyperbolic space $\mathbb{H}^n$ and the $n$-dimensional sphere $S^n$.

**Definition 1.6.** [5] [12] A smooth map $\phi : (M, g) \to (N, h)$ between Riemannian manifolds is called a harmonic morphism if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset $V$ of $N$ with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$.

We can think of the composition $f \circ \phi$ as the pull-back $\phi^* f$ of $f$ and we see that a harmonic morphism is a smooth map which pulls back (local) harmonic functions to harmonic functions. In other words it pulls back germs of harmonic functions to germs of harmonic functions.

Above we defined the tension field $\tau$ of a function from a (semi)-Riemannian manifold to the complex plane and now we will define a tension field applied on a smooth map between two Riemannian manifolds.

**Definition 1.7.** [1] Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. The tension field of $\phi$ is the section $\tau(\phi) \in C^\infty(\phi^{-1}TN)$, the pull-back bundle of $\phi^{-1}TN$, defined by

$$\tau(\phi) = \text{div } d\phi = \text{trace } \nabla d\phi = \sum_{i=1}^m \nabla d\phi(e_i, e_i),$$

where $\{e_i\}$ is any local orthonormal frame of the tangent bundle $TM$ of $M$.

**Proposition 1.8.** [1] (Composition Law) The tension field of the composition of two maps $\phi : M \to N$ and $\psi : N \to P$ is given by

$$\tau(\psi \circ \phi) = d\psi(\tau(\phi)) + \text{trace } \nabla d\psi(d\phi, d\phi)$$

Here

$$\text{trace } \nabla d\psi(d\phi, d\phi) = \sum_{i=1}^m \nabla d\psi(d\phi(e_i), d\phi(e_i)),$$

where $\{e_i\}$ is a local orthonormal frame.

Now we will use the definition below of horizontally weak conformality to give a characterization of a harmonic morphism as proven by Fuglede and Ishihara.

**Definition 1.9.** [5] [12] Let $\phi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds, and let $x \in M$. Then $\phi$ is said to be horizontally weakly conformal at $x$ if either
(i) \( d\phi_x = 0 \), or 
(ii) \( d\phi_x \) is surjective and there exists a positive real number \( \lambda(x) > 0 \) such that 
\[
h(d\phi_x(X), d\phi_x(Y)) = \lambda^2(x)g(X, Y),
\]
for \( X, Y \in \ker d\phi_x \). The function \( \lambda \) is called the dilation of \( \phi \) at \( x \). The map \( \phi \) is called horizontally (weakly conformal) if it is horizontally weakly conformal at every point of \( M \).

**Theorem 1.10.** (Characterization) (Fuglede 1978 [5], Ishihara 1979 [12]) A smooth map \( \phi : M \to N \) between Riemannian manifolds is a harmonic morphism if and only if \( \phi \) is both harmonic and horizontally weakly conformal.

The following are important relations between \( r \)-harmonic functions and harmonic morphisms developed in [9].

**Proposition 1.11.** [9] Let \( \pi : (\hat{M}, \hat{g}) \to (M, g) \) be a submersive harmonic morphism from a semi-Riemannian manifold \( (\hat{M}, \hat{g}) \) to a Riemannian manifold \( (M, g) \). Further let \( f : (M, g) \to \mathbb{C} \) be a smooth function and \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) be the composition \( \hat{f} = f \circ \pi \). If \( \lambda : \hat{M} \to \mathbb{R}^+ \) is the dilation of \( \pi \) then the tension field satisfies 
\[
\tau(f) \circ \pi = \lambda^{-2} \tau(\hat{f}) \text{ and } \tau^r(f) \circ \pi = \lambda^{-2} \tau(\lambda^{-2} \tau^{r-1}(f))
\]
for all positive integers \( r \geq 2 \).

**Proof.** Since \( \pi \) is a harmonic morphism, we have from Theorem 1.10 that \( \pi \) is a horizontally conformal, harmonic map. Let \( \{e_i\} \) be a local orthonormal frame of \( T\hat{M} \). The dilation \( \lambda \) gives the following relation
\[
\lambda^2 g(e_i, e_i) = \hat{g}(d\pi(e_i), d\pi(e_i)).
\]
Now let \( d\pi e_i = \lambda \hat{e}_i \) where \( |\hat{e}_i| = 1 \). Following Proposition 1.10, Definition 1.7, and the harmonicity of \( \pi \), we have 
\[
\tau(\hat{f}) = \tau(f \circ \pi) = \text{trace} \nabla df(d\pi, d\pi) + df(\tau(\pi))
\]
\[
= \sum_{i=1}^m \nabla df(d\pi(e_i), d\pi(e_i))
\]
\[
= \sum_{i=1}^m \nabla df(\lambda \hat{e}_i, \lambda \hat{e}_i)
\]
\[
= \lambda^2 \sum_{i=1}^m \nabla df(\hat{e}_i, \hat{e}_i) \circ \pi
\]
\[
= \lambda^2 \tau(f) \circ \pi.
\]
For the second statement, set \( h = \tau(f) \) and \( \hat{h} = \lambda^{-2} \circ \tau(\hat{f}) \). Then 
\[
h \circ \pi = \tau(f) \circ \pi = \lambda^{-2} \tau(\hat{f}) = \hat{h}
\]
and it follows from the first statement that 
\[
\tau \left( \lambda^{-2} \circ \tau(\hat{f}) \right) = \tau(\hat{h}) = \tau(h \circ \pi) = \lambda^2 \tau(h) \circ \pi = \lambda^2 \tau^2(f) \circ \pi
\]
and we have
\[ \tau^2(f) \circ \pi = \lambda^{-2} \tau \left( \lambda^{-2} \tau(\hat{f}) \right). \]
The rest follows by induction. \(\square\)

**Corollary 1.12.** [9] Let \( \pi : (\hat{M}, \hat{g}) \to (M, g) \) be a submersive harmonic morphism, from a semi-Riemannian manifold \( (\hat{M}, \hat{g}) \) to a Riemannian manifold \( (M, g) \), with constant dilation. Further, let \( f : (M, g) \to \mathbb{C} \) be a smooth function and \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) be the composition \( \hat{f} = f \circ \pi \). Then the following statements are equivalent.

(a) \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) is proper \( r \)-harmonic.
(b) \( f : (M, g) \to \mathbb{C} \) is proper \( r \)-harmonic.

**Proof.** From Proposition 1.11 and the linearity of \( \tau \), we have for any positive integer \( r \),
\[ \tau^r(f) \circ \pi = \lambda^{-2} \tau(\lambda^{-2} \tau^{r-1}(\hat{f})) = \lambda^{-2r} \tau^r(\hat{f}). \]
Then \( \hat{f} \) is proper \( r \)-harmonic if and only if \( f \) is proper \( r \)-harmonic. \(\square\)
Chapter 2

The Hyperbolic Upper-Half Space \( \mathbb{H}^n \)

In this chapter we construct complex-valued proper \( r \)-harmonic functions on the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \) for any \( r \geq 1 \) and \( n \geq 2 \). We model \( \mathbb{H}^n \) as the hyperbolic upper-half space i.e. the differentiable manifold
\[
\mathbb{H}^n = \{(t, x) \mid t \in \mathbb{R}^+ \text{ and } x \in \mathbb{R}^{n-1}\}
\]
equipped with its standard Riemannian metric \( ds^2 \) satisfying
\[
\frac{1}{t^2} \cdot (dt^2 + dx_1^2 + \cdots + dx_{n-1}^2).
\]

Then from Equation (1.1), we have that the Laplace-Beltrami operator satisfies
\[
\tau(f) = \sum_{i,j=1}^{n} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( g^{ij} \sqrt{|g|} \frac{\partial f}{\partial x_i} \right)
= \frac{1}{\sqrt{t^{2n}}} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} (t^{2-n} \frac{\partial f}{\partial x_i})
= t^n \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left( (t^{2-n} \cdot \frac{\partial f}{\partial x_i}) \right)
= t^n \left( \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_{n-1}^2} + t^2 \cdot \frac{\partial^2 f}{\partial t^2} \right) + (2-n)t^{1-n} \frac{\partial f}{\partial t}
= t^2 \cdot \left( \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_{n-1}^2} \right) + t^2 \cdot \frac{\partial^2 f}{\partial t^2} - (n-2) \cdot t \cdot \frac{\partial f}{\partial t}.
\]

Now we will use the above definition of the Laplace-Beltrami operator on the upper half-space to construct complex-valued harmonic functions on \( \mathbb{H}^n \).

**Theorem 2.1.** [10] Let the \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \) be modelled as the upper-half space i.e. \( \mathbb{H}^n = \mathbb{R}^+ \times \mathbb{R}^{n-1} \). Let \( h : \mathbb{R}^{n-1} \to \mathbb{C} \) be a non-constant function harmonic with respect to the Euclidean metric on \( \mathbb{R}^{n-1} \) and \( p_1 : \mathbb{R}^+ \to \mathbb{C} \) be differentiable. Then the function \( f_1 : \mathbb{H}^n \to \mathbb{C} \) defined by
\[
f_1(t, x) = p_1(t) \cdot h(x)
\]
is harmonic on $\mathbb{H}^n$ if and only if $p_1$ is of the form $p_1(t) = a_1 + b_1 \cdot t^{n-1}$, for some constants $a_1, b_1 \in \mathbb{C}$.

**Proof.** We assume that $h : \mathbb{R}^{n-1} \to \mathbb{C}$ is a harmonic function with respect to the Euclidean metric on $\mathbb{R}^{n-1}$ i.e.

$$\frac{\partial^2 h}{\partial x_1^2} + \cdots + \frac{\partial^2 h}{\partial x_{n-1}^2} = 0.$$ 

Then by Proposition 1.4, the tension field $\tau(f_1)$ satisfies

$$\tau(f_1) = \tau(p_1(t) \cdot h(x)) = \tau(p_1(t)) \cdot h(x) + g(\nabla p_1, \nabla h) + p_1 \cdot \tau(h) = t^2 \cdot h(x) \cdot \frac{\partial^2 p_1}{\partial t^2} - (n-2) \cdot t \cdot h(x) \cdot \frac{\partial p_1}{\partial t}.$$ 

Thus $f_1 : \mathbb{H}^n \to \mathbb{C}$ is harmonic if and only if $\tau(p_1) = 0$ since $h(x) \neq 0$ and is non-constant. Then we multiply by $t^{-n}$ on both sides and we have

$$t^{-n} \cdot \tau(p_1) = t^{2-n} \cdot \frac{\partial^2 p_1}{\partial t^2} + (2-n) \cdot t^{1-n} \cdot \frac{\partial p_1}{\partial t} = \frac{\partial}{\partial t} (t^{2-n} \cdot \frac{\partial p_1}{\partial t}) = 0.$$ 

Now we integrate and there exists a complex constant $b_1$ such that

$$\frac{\partial p_1}{\partial t} = b_1(n-1) \cdot t^{n-2}$$

and integrating again gives

$$p_1(t) = a_1 + b_1 \cdot t^{n-1}$$

for some constant $a_1 \in \mathbb{C}$. \hfill \qed

Now from this double integration, we can define the integral operator $I_n$, which we will use to construct proper $r$-harmonic functions on $\mathbb{H}^n$.

**Definition 2.2.** [10] Let $p : \mathbb{R}^+ \to \mathbb{R}$ be a continuous function. Then we define the integral operator $I_n$ by

$$I_n(p)(t) = \int t^{-n} \cdot (\int t^{-n} \cdot p(t)dt + \alpha)dt + \beta,$$

where $\alpha, \beta \in \mathbb{C}$ are undetermined constants.

**Theorem 2.3.** [10] Let the $n$-dimensional hyperbolic space $\mathbb{H}^n$ be modelled as the upper-half space i.e. $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{R}^{n-1}$. Let $h : \mathbb{R}^{n-1} \to \mathbb{C}$ be a non-constant function harmonic with respect to the Euclidean metric on $\mathbb{R}^{n-1}$ and $p_r : \mathbb{R}^+ \to \mathbb{C}$ be given by

$$p_r(t) = (a_r + b_r \cdot t^{n-1}) \cdot \log(t)^{r-1}$$

where $(a_r, b_r) \in \mathbb{C}^2$ is non-zero and $r \geq 1$. Then the function $f_r : \mathbb{H}^n \to \mathbb{C}$ with

$$f_r(t, x) = p_r(t) \cdot h(x)$$

is proper $r$-harmonic on $\mathbb{H}^n$. 

6
Proof. We will prove the statement for \( r = 1 \) and \( r = 2 \), then the reader can repeat the process and the result will follow by induction. We are assuming that \( h : \mathbb{R}^{n-1} \to \mathbb{C} \) is a harmonic function with respect to the Euclidean metric on \( \mathbb{R}^{n-1} \) so for each \( k \in \mathbb{Z}^+ \), the tension field satisfies
\[
\tau^k(f_1(x, t)) = h(x) \cdot \tau^k(p_r(t))
\]

We have seen in Theorem 2.1 that if \( p_0(t) = 0 \), then \( p_1(t) = I_n(p_0)(t) \) is of the form
\[
p_1(t) = I_n(0)(t) = \int t^{n-2} \cdot (\int t^{-n} \cdot \alpha dt + \beta) dt + \beta
= \int (t^{n-2} \cdot \alpha) dt + \beta
= \frac{\alpha}{n-1} t^{n-1} + \beta
= a_1 + b_1 t^{n-1}
\]
where \( a_1, b_1 \) are constants in \( \mathbb{C} \). Then we apply Theorem 2.1 and we have that \( f_1(t, x) = p_1(t) \cdot h(x) \) is proper 1-harmonic iff \( (a, b) \in \mathbb{C}^2 \) is nonzero. Then we have \( I_n(p_1)(t) = q_{21}(t) + q_{22}(t) \) where
\[
q_{21}(t) = \frac{1}{(n-2)^2} \cdot (\alpha(n-1)^2 + (n-1)\beta - b_1)t^{n-1})
q_{22}(t) = -\frac{1}{n-1}(a_1 - b_1 t^{n-1}) \log t
\]
and we can write \( q_{21}(t) = \frac{1}{(n-1)^2}(a_1 + b_1 t^{n-1}) \) for constants \( a_1, b_1 \in \mathbb{C} \), thus \( \tau(q_{21}) = 0 \).

We have
\[
\frac{\partial q_{22}}{\partial t} = -\frac{a_1}{t(n-1)} + \frac{b_1 t^{n-2}}{n-1} + b_1 t^{n-1} \log t
\]
\[
\frac{\partial^2 q_{22}}{\partial t^2} = \frac{a_1}{t^2(n-1)} + \frac{b_1(n-2) t^{n-3}}{n-1} + b_1(n-2) t^{n-3} \log t + b_1 t^{n-3}
\]
which gives
\[
\tau(q_{22}) = t^2 \left( \frac{a_1}{t^2(n-1)} + \frac{b_1(n-2) t^{n-3}}{n-1} + b_1(n-2) t^{n-3} \log t + b_1 t^{n-3} \right)
- (n-2)t (-\frac{a_1}{t(n-1)} + \frac{b_1 t^{n-2}}{n-1} + b_1 t^{n-1} \log t)
= a_2 + b_2 t^{n-1}
\]
for constants \( a_2, b_2 \in \mathbb{C} \).

From earlier we have that \( \tau(q_{22}) = a_2 + b_2 t^{n-1} \) is proper 1-harmonic for any non-zero \( (a_2, b_2) \in \mathbb{C}^2 \). This implies that the function
\[
p_2(t) = (a_2 + b_2 \cdot t^{n-1}) \cdot \log t
\]
is proper 2-harmonic as well. This immediately tells us that \( f_2(t, x) = h(x) \cdot p_2(t) \) is proper 2-harmonic on \( \mathbb{H}^n \) if and only if \( (a_2, b_2) \in \mathbb{C}^2 \) is non-zero. \( \square \)
Example 2.4. [10] Let $h : \mathbb{R}^3 \to \mathbb{C}$ be a non-constant function harmonic with respect to the Euclidean metric on $\mathbb{R}^3$ and $p_2 : \mathbb{R}^+ \to \mathbb{C}$ be given by

$$p_2(t) = (a_2 + b_2 \cdot t^3) \cdot \log(t),$$

where $(a_2, b_2) \in \mathbb{C}^2$ is non-zero. Then by Theorem 2.3 we have that the function $f_2 : \mathbb{H}^4 \to \mathbb{C}$ with

$$f_2(t, x) = p_2(t) \cdot h(x)$$

is proper biharmonic on $\mathbb{H}^4$. 
Chapter 3

The Hyperbolic One-Sheeted Hyperboloid $\mathcal{H}^n$

In this chapter we translate the complex-valued proper $r$-harmonic functions constructed on the upper-half space model $\mathbb{H}^n$ onto the one-sheeted hyperboloid model $\mathcal{H}^n$ of the $n$-dimensional hyperbolic space.

Let $M^{n+1}$ be the standard $(n + 1)$-dimensional Minkowski space equipped with its Lorentzian metric

$$(x, y)_L = -x_0y_0 + \sum_{k=1}^{n} x_ky_k.$$ 

The open set

$$U^{n+1} = \{ y \in M^{n+1} \mid (y, y)_L < 0 \}$$

is bounded by the light cone and contains the $n$-dimensional hyperbolic space

$$\mathcal{H}^n = \{ (y_0, y_1, \ldots, y_n) \in M^{n+1} \mid (y, y)_L = -1 \}.$$ 

Let $\pi : U^{n+1} \rightarrow \mathcal{H}^n$ be the radial projection given by

$$\pi : y \mapsto \frac{y}{\sqrt{-(y, y)_L}}.$$ 

This is a well-known submersive harmonic morphism with dilation $\lambda^{-2}(y) = -|y|_L^2$, see Lemma 4.1 from [6]. Then we have the following version of Proposition 1.11.

**Proposition 3.1.** [10] Let $\pi : U^{n+1} \rightarrow \mathcal{H}^n$ be the submersive harmonic morphism given by

$$\pi : y \mapsto \frac{y}{\sqrt{-(y, y)_L}}.$$ 

Further let $f : \mathcal{H}^n \rightarrow \mathbb{C}$ be a smooth function and $\hat{f} : U^{n+1} \rightarrow \mathbb{C}$ be the composition $\hat{f} = f \circ \pi$. Then the tension field of $f$ satisfies

$$\tau(f) \circ \pi = -|y|_L^2 \cdot \tau(\hat{f}) \quad \text{and} \quad \tau^r(f) \circ \pi = -|y|_L^2 \cdot \tau(-|y|_L \cdot \tau^{(r-1)}(\hat{f}))$$

for all positive integers $r \geq 2$.

**Proof.** Our dilation is $\lambda^{-2} = -|y|_L^2$ and we apply Proposition 1.11 and we have

$$\tau(f) \circ \pi = \lambda^{-2} \tau(\hat{f}).$$
\[ \tau'(f) \circ \pi = \lambda^{-2} \tau(\lambda^{-2} \tau^{-1}(\hat{f})) = -|y|_L^2 \cdot \tau(-|y|_L^2 \cdot \tau^{-1}(\hat{f})). \]

\[ \square(\hat{f}) = -\frac{\partial^2 \hat{f}}{\partial y_0^2} + \frac{\partial^2 \hat{f}}{\partial y_1^2} + \cdots + \frac{\partial^2 \hat{f}}{\partial y_n^2}. \]

**Example 3.2.** We will use the well-known isometry \( \Psi : (H^n, ds_L^2) \to (\mathbb{H}^n, ds^2) \) from [2] between the two different models of the \( n \)-dimensional hyperbolic space defined below,

\[ \Psi : (y_0, y_1, \ldots, y_n) \mapsto 2 \cdot \left( \frac{1}{y_0 + y_1}, \frac{y_2}{y_0 + y_1}, \ldots, \frac{y_n}{y_0 + y_1} \right). \]

The composition \( \Phi = \Psi \circ \pi : (U^{n+1}, ds_L^2) \to (\mathbb{H}^n, ds^2) \) is defined as

\[ \Phi(y) = \Psi(\sqrt{-y_0 y_L} \frac{y_1}{\sqrt{-y_0 y_L}}, \ldots, \frac{y_n}{\sqrt{-y_0 y_L}}) = \left( \frac{2 \sqrt{y_0^2 - y_1^2 - \cdots - y_n^2}}{y_0 + y_1}, \frac{2y_2}{y_0 + y_1}, \ldots, \frac{2y_n}{y_0 + y_1} \right). \]

Now we have the following result corresponding to Theorem 2.3.

**Theorem 3.3.** [10] Let the \( n \)-dimensional hyperbolic space \( H^n \) be modelled as the one-sheeted hyperboloid in the Minkowski space \( M^{n+1} \). Let \( h : \mathbb{R}^{n-1} \to \mathbb{C} \) be a non-constant function harmonic with respect to the Euclidean metric on \( \mathbb{R}^{n-1} \) and \( p_r : \mathbb{R}^+ \to \mathbb{C} \) be given by

\[ p_r(t) = (a_r + b_r \cdot t^{n-1}) \cdot \log(t)^r, \]

where \((a_r, b_r) \in \mathbb{C}^2\) is non-zero and \( r \geq 1 \). Then the function \( f : U^{n+1} \to \mathbb{C} \) with

\[ f(y_0, y_1, \ldots, y_n) = p_r(\frac{2 \sqrt{y_0^2 - y_1^2 - \cdots - y_n^2}}{y_0 + y_1}) \cdot h(\frac{2y_2}{y_0 + y_1}, \ldots, \frac{2y_n}{y_0 + y_1}) \]

induces a proper \( r \)-harmonic function on \( H^n \).

**Proof.** We will use the isometry \( \Psi : (H^n, ds_L^2) \to (\mathbb{H}^n, ds^2) \) defined above and \( f_r : \mathbb{H}^n \to \mathbb{C} \) with

\[ f_r(t, x) = p_r(t) \cdot h(x) \]

from Theorem 2.3 to construct proper \( r \)-harmonic functions on \( H^n \). Let \( \hat{f}_r : H^n \to \mathbb{C} \) be the composition of the two, \( \hat{f}_r = f_r \circ \Psi \). We have

\[ \hat{f}_r(y_0, y_1, \ldots, y_n) = f_r(\Psi(y_0, y_1, \ldots, y_n)) \]

\[ = f_r(2 \frac{2y_2}{y_0 + y_1}, \frac{2y_2}{y_0 + y_1}, \ldots, \frac{2y_n}{y_0 + y_1}) \]

\[ = p_r(\frac{2}{y_0 + y_1}) \cdot h(\frac{2y_2}{y_0 + y_1}, \ldots, \frac{2y_n}{y_0 + y_1}). \]
Since $f_r$ is $r$-harmonic by Theorem 2.3 and we have that $\Psi$ is an isometry, $\hat{f}_r$ is proper $r$-harmonic for non-zero $(a_r, b_r) \in \mathbb{C}$ as well. We have the submersive harmonic morphism $\pi : U^{n+1} \to \mathcal{H}^n$ given by

$$\pi : y \mapsto \frac{y}{\sqrt{-(y, y)_L}}$$

and let $f : U^{n+1} \to \mathbb{C}$ be the composition of the two, $f = \hat{f}_r \circ \pi$, defined as

$$f(y_0, y_1, \ldots, y_n) = p_r\left(\frac{2\sqrt{y_0^2 - y_1^2 \cdots - y_n^2}}{y_0 + y_1}\right) \cdot h\left(\frac{2y_2}{y_0 + y_1}, \ldots, \frac{2y_n}{y_0 + y_1}\right).$$

From Proposition 3.1 we have

$$\Box(\hat{f}) \circ \pi = -|y|_{L}^2 \cdot \Box(f) \quad and \quad \Box^r(\hat{f}) \circ \pi = -|y|_{L}^2 \cdot \Box(-|y|_{L}^2 \cdot \Box^{(r-1)}(f)).$$

Then by the proper $r$-harmonicity of $\hat{f}_r$ and Corollary 1.12, we have that $f$ is proper $r$-harmonic on $U^{n+1}$. \hfill \Box
Chapter 4

The $n$-Dimensional Sphere $S^n$

In this chapter we will use a duality principle to translate the bi-harmonic functions constructed on $H^n$ to bi-harmonic functions on $S^n$.

Let the $(n+1)$-dimensional real vector space $\mathbb{R}^{n+1}$ be equipped with the standard Euclidean scalar product satisfying

$$(x, y) = x_1y_1 + \cdots + x_ny_n + x_{n+1}y_{n+1}.$$ 

The $n$-dimensional round unit sphere $S^n$ in $\mathbb{R}^{n+1}$ is defined by

$$S^n = \{(y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1} \mid y_1^2 + y_2^2 + \cdots + y_{n+1}^2 = 1\}.$$

The radial projection $\pi : \mathbb{R}^{n+1}\setminus\{0\} \to S^n$ with

$$\pi : y \mapsto \frac{y}{|y|}.$$

This is a well-known submersive harmonic morphism and its dilation satisfies $\lambda^{-2}(y) = |y|^2$, see Lemma 4.1 in [6]. Now we have the following version of Proposition 1.11.

**Proposition 4.1.** [10] Let $\pi : \mathbb{R}^{n+1}\setminus\{0\} \to S^n$ be the submersive harmonic morphism given by

$$\pi : y \mapsto \frac{y}{|y|}.$$

Further let $W$ be an open subset of $S^n$, $f : W \to \mathbb{C}$ be a smooth function and $\hat{f} : \pi^{-1}(W) \subset \mathbb{R}^{n+1}\setminus\{0\} \to \mathbb{C}$ be the composition $\hat{f} = f \circ \pi$. Then the tension fields of $f$ satisfy

$$\tau(f) \circ \pi = |y|^2 \cdot \tau(\hat{f}) \quad \text{and} \quad \tau^2(f) \circ \pi = |y|^2 \cdot \tau(|y|^2 \cdot \tau^{r-1}(\hat{f}))$$

for all positive integers $r \geq 2$.

**Proof.** Our dilation is $\lambda^{-2} = |y|^2$ and we apply Proposition 1.11 and obtain

$$\tau(f) \circ \pi = \lambda^{-2} \tau(\hat{f}) = |y|^2 \cdot \tau(\hat{f})$$

$$\tau^r(f) \circ \pi = \lambda^{-2} \tau(\lambda^{-2} \tau^{r-1}(\hat{f})) = |y|^2 \cdot \tau(|y|^2 \cdot \tau^{r-1}(\hat{f})).$$

$\square$
In the \((n+1)\)-dimensional Euclidean space \(\mathbb{R}^{n+1}\), the tension field is the classical Laplace operator \(\Delta\) given by

\[
\Delta(f) = \frac{\partial^2 f}{\partial y_1^2} + \frac{\partial^2 f}{\partial y_2^2} + \cdots + \frac{\partial^2 f}{\partial y_{n+1}^2}.
\]

Let \(G\) be a non-compact semi-simple Lie group with the Cartan decomposition \(\mathfrak{g} = \mathfrak{k} + \mathfrak{p}\) of the Lie algebra of \(G\) where \(\mathfrak{k}\) is the Lie algebra of a maximal compact subgroup \(K\). Let \(G^\mathbb{C}\) denote the complexification of \(G\) and \(U\) be the compact subgroup of \(G^\mathbb{C}\) with Lie algebra \(u = \mathfrak{k} + i\mathfrak{p}\). Let \(G^\mathbb{C}\) and its subgroups be equipped with a left-invariant semi-Riemannian metric which is a multiple of the Killing form by a negative constant. Then the subgroup \(U\) of \(G^\mathbb{C}\) is Riemannian and \(G\) is semi-Riemannian.

Let \(f : W \to \mathbb{C}\) be a real analytic function from an open subset \(W\) of \(G\). Then \(f\) extends uniquely to a holomorphic function \(f^\mathbb{C} : W^\mathbb{C} \to \mathbb{C}\) from some open subset \(W^\mathbb{C}\) of \(G^\mathbb{C}\). By restricting this to \(U \cap W^\mathbb{C}\), we obtain a real analytic function \(f^* : W^* \to \mathbb{C}\) on some open subset \(W^*\) of \(U\). The function \(f^*\) is called the dual function of \(f\).

**Example 4.2.** We know that \(S^n\) is the compact dual of \(\mathcal{H}^n\). We can define a function from \(S^n \to \mathbb{C}\) as

\[
f(x_1, \ldots, x_n, x_{n+1}) = x_1 + \cdots + x_n + x_{n+1},
\]

then it has a dual function \(\mathcal{H}^n \to \mathbb{C}\) defined as

\[
f^*(x_1, \ldots, x_n, x_{n+1}) = ix_1 + \cdots + x_n + x_{n+1}.
\]

Now we can relate the dual function to the property of harmonicity with a well-known duality principle.

**Theorem 4.3.** [9] (Duality Principle) A complex-valued function \(f : W \to \mathbb{C}\) is proper \(r\)-harmonic if and only if its dual \(f^* : W^* \to \mathbb{C}\) is proper \(r\)-harmonic.

We now have the following result corresponding to Theorem 2.3.

**Theorem 4.4.** [10] Let \(S^n\) be the round unit sphere in the standard \((n+1)\)-dimensional Euclidean space \(\mathbb{R}^{n+1}\). Let \(h : \mathbb{R}^{n-1} \to \mathbb{C}\) be a non-constant function harmonic with respect to the Euclidean metric on \(\mathbb{R}^{n-1}\) and \(p_r : \mathbb{R}^+ \to \mathbb{C}\) be given by

\[
p_r(t) = (a_r + b_r \cdot t^{n-1}) \cdot \log(t)^r-1,
\]

where \((a_r, b_r) \in \mathbb{C}\) is non-zero and \(r \geq 1\). Then the function \(f_r : W \to \mathbb{C}\) defined on an open subset \(W\) of \(S^n\) with

\[
f_r(y_1, \ldots, y_n, y_{n+1}) = p_r^*(\frac{2|y|}{y_2 + i \cdot y_1}) \cdot h^*(\frac{2y_3}{y_2 + i \cdot y_1}, \ldots, \frac{2y_{n+1}}{y_2 + i \cdot y_1})
\]

is proper \(r\)-harmonic. Here \(p_r^*\) and \(h^*\) are some local complex analytic extensions of \(p_r\) and \(h\), respectively.

**Proof.** Let \(f^* : \mathcal{H}^n \to \mathbb{C}\) be defined as

\[
f^*(y_1, \ldots, y_n, y_{n+1}) = p_r(\frac{2|y|}{y_1 + y_2}) \cdot h(\frac{2y_3}{y_1 + y_2}, \ldots, \frac{2y_{n+1}}{y_1 + y_2}).
\]
This is proper $r$-harmonic on $\mathcal{H}^n$ by Theorem 3.3. Since $\mathbb{S}^n$ is the dual of $\mathcal{H}^n$, we have the dual function $f_r : \mathbb{S}^n \to \mathbb{C}$ which is proper $r$-harmonic on $\mathbb{S}^n$ by Theorem 4.3. It is defined as

$$f_r(y_1, \ldots, y_n) = p_r^*(\frac{2|y|}{y_2 + i \cdot y_1}) \cdot h^*(\frac{2y_3}{y_2 + i \cdot y_1}, \ldots, \frac{2y_{n+1}}{y_2 + i \cdot y_1}).$$

$\square$
Chapter 5

The Special Unitary Group SU(2)

In this chapter we study the special unitary group $SU(2)$ and its higher finite-dimensional irreducible representations $\pi_n$. We will describe the construction of proper bi-harmonic functions on $SU(n)$ in terms of matrix coefficients of the standard irreducible representation $\pi_1$ of $SU(2)$ in full detail. Then we cite [8] to extend the construction to its finite-dimensional irreducible representations $\pi_2, \pi_3, \pi_4$, and present a conjecture for the extension to $\pi_n$.

**Definition 5.1.** [4] A finite dimensional representation of a Lie group on $V$ is a continuous homomorphism $\pi : G \to Aut V$, where $V$ is a vector space.

Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ of left-invariant vector fields on $G$. Then a Euclidean scalar product $g$ on $\mathfrak{g}$ induces a left-invariant Riemannian metric on the group $G$ and turns it into a homogeneous Riemannian manifold. If $Z$ is a left invariant vector field on $G$ and $f : U \to \mathbb{C}$ is a complex-valued function defined locally on $G$, then the first and second order derivatives satisfy

$$Z(f)(p) = \frac{d}{ds}[f(p \cdot \exp(sZ))]|_{s=0},$$

$$Z^2(f)(p) = \frac{d^2}{ds^2}[f(p \cdot \exp(sZ))]|_{s=0}.$$  

From now on we assume that $G$ is a subgroup of the complex general linear group

$$GL_n(\mathbb{C}) = \{ A \in \mathbb{C}^{n \times n} | \det A \neq 0 \}$$

equipped with its standard Riemannian metric. This is induced by the Euclidean scalar product on the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ given by

$$g(Z, W) = \Re \text{trace}(ZW^*).$$

We use the Koszul formula for the Levi-Civita connection $\nabla$ on $GL_n(\mathbb{C})$ and see that

$$g(\nabla_Z Z, W) = g([W, Z], Z)$$

$$= \Re \text{trace}(WZ - ZW)Z^*$$

$$= \Re \text{trace} W(ZZ^* - Z^* Z)^*$$

$$= g([Z, Z^*], W).$$

17
Let $[Z, Z^\ast]_g$ be the orthogonal projection of the bracket $[Z, Z^\ast]$ onto the subalgebra $g$ of $\mathfrak{gl}_n(\mathbb{C})$. Then the above calculations show that

$$\nabla_Z Z = [Z, Z^\ast]_g.$$ 

This implies that the tension field $\tau(f)$ and the conformality operator $\kappa(f, h)$ are given by

$$\tau(f) = \sum_{Z \in \mathcal{B}} Z^2(f) - [Z, Z^\ast]_g(f) \quad \text{and} \quad \kappa(f, h) = \sum_{Z \in \mathcal{B}} Z(f)Z(h),$$

where $\mathcal{B}$ is any orthonormal basis for the Lie algebra $g$.

For $1 \leq i, j \leq n$ we shall by $E_{ij}$ denote the element of $\mathfrak{gl}_n(\mathbb{R})$ satisfying $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and by $D_t$ the diagonal matrices

$$D_t = E_{tt}.$$ 

For $1 \leq r < s \leq n$ let $X_{rs}$ and $Y_{rs}$ be the matrices satisfying

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}).$$

With the above notation, we have the following matrix identities from [7]

$$\sum_{r<s} X_{rs}^2 = \frac{(n-1)}{2} I_n, \quad \sum_{r<s} Y_{rs}^2 = -\frac{(n-1)}{2} I_n, \quad \sum_{t=1}^n D_t^2 = I_n,$$

$$\sum_{r<s} X_{rs} E_{jl} X_{rs}^t = \frac{1}{2} (E_{lj} + \delta_{lj}(I_n - 2E_{lj})), \quad \sum_{r<s} Y_{rs} E_{jl} Y_{rs}^t = -\frac{1}{2} (E_{lj} - \delta_{lj}I_n),$$

$$\sum_{t=1}^n D_t E_{jl} D_t^t = \delta_{jl} E_{lj},$$

where $X^t_{rs}, Y^t_{rs}$, and $D_t^t$ are the transpose matrices.

### 5.1 The Standard Irreducible Representation $\pi_1$ of $SU(2)$

In this section we describe known proper biharmonic functions on the special unitary group $SU(n)$. They are quotients of first order homogeneous polynomials in the matrix coefficients of the standard irreducible representation $\pi_1$ of $SU(2)$.

The unitary group $U(n)$ is the compact subgroup of $GL_n(\mathbb{C})$ given by

$$U(n) = \{ z \in GL_n(\mathbb{C}) \mid z \cdot z^\ast = I_n \}$$

with its standard matrix representation

$$\pi^1 = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\
 z_{21} & z_{22} & \cdots & z_{2n} \\
 \vdots & \vdots & \ddots & \vdots \\
 z_{n1} & z_{n2} & \cdots & z_{nn} \end{bmatrix}.$$
The circle group $S^1 = \{ e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R} \}$ acts on the unitary group $U(n)$ by multiplication

$$(e^{i\theta}, z) \mapsto e^{i\theta}z$$

and the orbit space of this action is the special unitary group

$$\text{SU}(n) = \{ z \in U(n) \mid \det z = 1 \}.$$ 

The natural projection $\pi : U(n) \to \text{SU}(n)$ is a harmonic morphism with constant dilation $\lambda \equiv 1$.

The Lie algebra $\mathfrak{u}(n)$ of the unitary group $U(n)$ satisfies

$$\mathfrak{u}(n) = \{ Z \in \mathbb{C}^{n \times n} \mid Z + Z^* = 0 \}$$

and for this we have the canonical orthonormal basis

$$\{ Y_{rs}, iX_{rs} \mid 1 \leq r < s \leq n \} \cup \{ iD_t \mid t = 1, \ldots, n \}.$$ 

The tension field $\tau(f)$ and the conformality operator $\kappa(f, h)$ are given by

$$\tau(f) = \sum_{Z \in \mathcal{B}} Z^2(f) \quad \text{and} \quad \kappa(f, h) = \sum_{Z \in \mathcal{B}} Z(f)Z(h),$$

where $\mathcal{B}$ is any orthonormal basis for the Lie algebra $\mathfrak{u}(n)$.

Now by means of a direct computation, we have the following basic result.

**Lemma 5.2.** [7] For $1 \leq j, \alpha \leq n$, let $z_{ja} : U(n) \to \mathbb{C}$ be the complex-valued matrix coefficients of the standard representation of $U(n)$ given by

$$z_{ja} : z \mapsto e_j \cdot z \cdot e_{\alpha}^t,$$

where $\{ e_1, \ldots, e_n \}$ is the canonical basis for $\mathbb{C}^n$. Then the following relations hold

$$\tau(z_{ja}) = -nz_{ja} \quad \text{and} \quad \kappa(z_{ja}, z_{\alpha \beta}) = -z_{\alpha \beta}z_{ja}. \quad (5.1)$$

**Proof.** Let $Z$ be an element of the Lie algebra $\mathfrak{u}(n)$. Then we have

$$Z(z_{ja}) = e_j \cdot z \cdot Z \cdot e_{\alpha}^t \quad \text{and} \quad Z^2(z_{ja}) = e_j \cdot z \cdot Z^2 \cdot e_{\alpha}^t.$$ 

Then we use the matrix identities from above and obtain

$$\tau(z_{ja}) = \sum_{r<s} Y_{rs}^2(z_{ja}) + \sum_{r<s} (iX_{rs})^2(z_{ja}) + \sum_{t=1}^n (iD_t)^2(z_{ja})$$

$$= \sum_{r<s} e_j \cdot z \cdot Y_{rs}^2 \cdot e_{\alpha}^t + \sum_{r<s} e_j \cdot z \cdot -X_{rs}^2 \cdot e_{\alpha}^t + \sum_{t=1}^n e_j \cdot z \cdot (-D_t^2) \cdot e_{\alpha}^t$$

$$= -\frac{(n-1)}{2} z_{ja} - \frac{(n-1)}{2} z_{ja} - x_{ja}$$

$$= -nz_{ja}$$

and

$$\kappa(z_{ja}, z_{\alpha \beta}) = \sum_{r<s} e_j \cdot z \cdot Y_{rs} \cdot e_{\alpha}^t \cdot e_{k} \cdot Y_{rs} \cdot z^t \cdot e_{\beta}^t + \sum_{r<s} e_j \cdot z \cdot iX_{rs} \cdot e_{\alpha}^t \cdot e_{k} \cdot iX_{rs} \cdot z^t \cdot e_{\beta}^t.$$
+ \sum_{t=1}^{n} e_j \cdot z \cdot iD_t \cdot e'_\alpha \cdot e_k \cdot iD'_t \cdot z' \cdot e'_\beta
\]

\[= e_j \cdot z \left( \sum_{r<s} Y_{rs} \cdot E_{ak} \cdot Y'_{rs} - \sum_{r<s} X_{rs} \cdot E_{ak} \cdot X'_{rs} - \sum_{t=1}^{n} D_t \cdot E_{ak} D'_t \right) \cdot z' \cdot e'_\beta
\]

\[= e_j \cdot z \left( -\frac{1}{2} E_{ak} + \frac{1}{2} \delta_{ka} I_n - \frac{1}{2} E_{ka} - \frac{1}{2} \delta_{ka} I_n + E_{ka} \delta_{ka} - \delta_{ak} E_{ka} \right) \cdot z' \cdot e'_\beta
\]

\[= e_j \cdot z \cdot -E_{ak} \cdot z' \cdot e'_\beta
\]

\[= -z_{ja} z_{k\beta}
\]

The next result describes the first known proper biharmonic functions from the unitary group $SU(n)$ [9].

**Proposition 5.3.** [9] Let $p,q \in \mathbb{C}^n$ be linearly independent and $P,Q : U(n) \to \mathbb{C}$ be the complex-valued functions on the unitary group given by

\[P(z) = \sum_{j=1}^{n} p_j z_{ja} \quad \text{and} \quad Q(z) = \sum_{k=1}^{n} q_k z_{k\beta}.
\]

Further, let the rational function $f(z) = P(z)/Q(z)$ be defined on the open and dense subset $W_Q = \{ z \in U(n) \mid Q(z) \neq 0 \}$ of the unitary group. Then the following is true.

(a) The function $f$ is harmonic if and only if $\alpha = \beta$.

(b) The function $f$ is proper biharmonic if and only if $\alpha \neq \beta$.

The corresponding statements hold for the function induced on $SU(n)$.

**Proof.** Let $f$ be the quotient $f = P/Q$. Then we have

\[\tau(f) = \sum_{X \in \mathcal{B}} X^2(f) = \sum_{X \in \mathcal{B}} X \left( \frac{X(P)Q - X(Q)P}{Q^2} \right)
\]

\[= \sum_{X \in \mathcal{B}} \frac{X(X(P)Q - X(Q)P) \cdot Q^2 - X(Q^2) \cdot (X(P)Q - X(Q)P)}{Q^4}
\]

\[= \sum_{X \in \mathcal{B}} \frac{Q^2(X^2(P)Q - X^2(Q)P) - 2Q(X(P)Q - X(Q)P)}{Q^4}
\]

\[= \sum_{X \in \mathcal{B}} \frac{Q^3 X^2(P) - QX^2(Q)P - 2QX(P) + 2PX(Q)}{Q^3}
\]

\[= \frac{Q^2 \tau(P) - PQ \tau(Q) - 2Q\kappa(P,Q) + 2P\kappa(Q,Q)}{Q^3}
\]

and this gives

\[Q^3 \tau(f) = Q^2 \tau(P) - 2Q\kappa(P,Q) + 2P\kappa(Q,Q) - PQ \tau(Q).
\]
It follows from Lemma 5.2 that $P, Q : \mathbb{U}(n) \to \mathbb{C}$ are eigenfunctions of the Laplace-Beltrami operator $\tau$ and that $\kappa(P, Q) = -Q^2$. This implies

$$\tau(f) = \frac{Q^2}{Q^3} \tau(P) - \frac{2Q}{Q^3} \kappa(P, Q) + \frac{2P}{Q^3} \kappa(Q, Q) - \frac{PQ}{Q^3} \tau(Q)$$

$$= \frac{1}{Q} \tau(P) - \frac{P}{Q^2} \tau(Q) - 2Q^{-2} \kappa(P, Q) - 2f$$

$$= -\frac{1}{Q} \sum_{j=1}^{n} p_j n z_{j\alpha} + \frac{P}{Q^2} \sum_{j=1}^{n} q_k n z_{k\beta} - 2Q^{-2} \kappa(P, Q) - 2f$$

$$= -\frac{nP}{Q} + \frac{nPQ}{Q^2} - 2Q^{-2} \kappa(P, Q) - 2f.$$

This gives

$$\tau(f) = -2f - 2\kappa(P, Q)Q^{-2}. \quad (5.2)$$

Then we have

$$\kappa(P, Q) = -\sum_{jk} p_j q_k z_{j\alpha} z_{k\beta},$$

and equation 5.2 tells us that $f$ is harmonic if and only if $\kappa(P, Q) = -PQ$. Since $p, q \in \mathbb{C}^n$ are linearly independent, this holds if and only if $\alpha = \beta$.

If we now assume that $\alpha \neq \beta$ then, again using Lemma 5.2, we have

$$\tau(\kappa(P, Q)) = 2PQ - 2n \kappa(P, Q),$$

$$\kappa(\kappa(P, Q), Q^{-2}) = 4\kappa(P, Q)Q^{-2},$$

$$\tau(Q^{-2}) = 2(n - 3)Q^{-2}.$$

We prove that $f$ is proper bi-harmonic by computing the bitension field $\tau^2(f)$ using the equalities above.

$$\tau^2(f) = -2\tau(f) - 2\tau(\kappa(P, Q)Q^{-2})$$

$$= 4f + 4\kappa(P, Q)Q^{-2} - 2\tau(\kappa(P, Q)Q^{-2})$$

$$= 4f + 4\kappa(P, Q)Q^{-2} - 2\tau(\kappa(P, Q))Q^{-2} - 4\kappa(\kappa(P, Q), Q^{-2}) - 2\kappa(P, Q)\tau(Q^{-2})$$

$$= 4f - 4f + (4 + 4n - 4n - 16 + 12)\kappa(P, Q)Q^{-2}$$

$$= 0.$$

Then this holds for $\text{SU}(n)$ because $f$ is invariant under the action of $\mathbb{S}^1$ on $\mathbb{U}(n)$. \hfill \Box

### 5.2 The Irreducible Representation $\pi_2$ of $\text{SU}(2)$

In this section we describe how the construction of Proposition 5.3 for $\text{SU}(2)$ can be extended to its 3-dimensional irreducible representation $\pi_2$ of $\mathbb{U}(2)$ given by the following matrix

$$\pi_2 = \begin{bmatrix}
  z_{11}^2 & z_{11}z_{12} & z_{12}^2 \\
  2z_{11}z_{21} & z_{11}z_{22} + z_{12}z_{21} & 2z_{12}z_{22} \\
  z_{21}^2 & z_{21}z_{22} & z_{22}^2
\end{bmatrix}.$$
Theorem 5.4. \cite{8} Let $p, q \in \mathbb{C}^3$ and $P, Q : U(2) \to \mathbb{C}$ be the complex-valued functions on the unitary group given by

$$P(z) = \sum_{j=1}^{3} p_j \pi_j^2 \quad \text{and} \quad Q_\beta(z) = \sum_{k=1}^{3} q_k \pi_k^2.$$ 

Let the rational functions $f(z) = P(z)/Q(z)$ be defined on the open and dense subset $W_Q = \{ z \in U(2) \mid Q(z) \neq 0 \}$ of the unitary group. If $\alpha \neq \beta$ then the function $f$ is proper biharmonic if $p_2 q_3 - p_3 q_2 \neq 0$ and

$$p_1 q_3^2 = q_4 (2p_2 q_3 - p_3 q_2), \quad q_1 q_3 = q_2^2.$$ 

The corresponding statement holds for the function induced on $SU(2)$.

**Proof.** See proof in \cite{8}. \hfill \Box

5.3 The Irreducible Representation $\pi_3$ of $SU(2)$.

In this section we describe how the construction of Theorem 5.4 can be extended to the 4-dimensional irreducible representation $\pi_3$ of $U(2)$ given by the matrix

$$\pi_3 = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$ 

Theorem 5.5. \cite{8} Let $p, q \in \mathbb{C}^4$ and $P, Q : U(2) \to \mathbb{C}$ be the complex-valued functions on the unitary group given by

$$P(z) = \sum_{j=1}^{4} p_j \pi_j^3 \quad \text{and} \quad Q(z) = \sum_{k=1}^{4} q_k \pi_k^3.$$ 

Let the rational function $f(z) = P(z)/Q(z)$ be defined on the open and dense subset $W_Q = \{ z \in U(2) \mid Q(z) \neq 0 \}$ of the unitary group. If $\alpha \neq \beta$ then the function $f$ is proper biharmonic if $p_3 q_4 - p_4 q_3 \neq 0$ and

$$p_1 q_4^3 = q_3 (3p_3 q_4 - 2p_4 q_3), \quad q_1 q_4^2 = q_3^2,$$

$$p_2 q_4^2 = q_3 (2p_3 q_4 - p_4 q_3), \quad q_2 q_4 = q_3^2.$$ 

The corresponding statement holds for the function induced on $SU(2)$.

**Proof.** See proof in \cite{8}. \hfill \Box

5.4 The Irreducible Representation $\pi_4$ of $SU(2)$

In this section we describe how the construction of Theorem 5.4 for $SU(2)$ can be extended to its 5-dimensional irreducible representation $\pi_4$ of $U(2)$ given by the matrix

$$\pi_4 = \begin{bmatrix} \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$ 

22
Theorem 5.6. [8] Let \( p, q \in \mathbb{C}^5 \) and \( P, Q : U(2) \rightarrow \mathbb{C} \) be the complex-valued functions on the unitary group given by

\[
P(z) = \sum_{j=1}^{5} p_j \pi_{j\alpha}^4 \quad \text{and} \quad Q(z) = \sum_{k=1}^{5} q_k \pi_{k\beta}^4.
\]

Let the rational function \( f(z) = P(z)/Q(z) \) be defined on the open and dense subset \( W_Q = \{ z \in U(2) \ | \ Q(z) \neq 0 \} \) of the unitary group. If \( \alpha \neq \beta \) then the function is proper biharmonic if \( p_4 q_5 \neq p_5 q_4 \) and

\[
\begin{align*}
p_{14}^3 &= q_4^3(4p_{45}^3 - 3p_{54}^3), \quad q_4 q_5^3 = q_4^4, \nonumber \\
p_{24}^3 &= q_4^2(3p_{45}^3 - 2p_{54}^3), \quad q_2 q_5^2 = q_4^3, \nonumber \\
p_{34}^2 &= q_4(2p_{45}^3 - p_{54}^3), \quad q_3 q_5 = q_4^2. \nonumber
\end{align*}
\]

The corresponding statement holds for the function induced on \( SU(2) \).

Proof. See proof in [8]. \( \square \)

5.5 The Irreducible Representation \( \pi_n \) of \( SU(2) \)

In this section we discuss the general \((n+1)\)-dimensional irreducible representation \( \pi_n \) of \( SU(2) \).

Conjecture 5.7. [8] For \( n > 1 \), let \( p, q \in \mathbb{C}^{n+1} \) and \( P, Q : U(2) \rightarrow \mathbb{C} \) be the complex-valued functions on the unitary group given by

\[
P(z) = \sum_{j=1}^{n+1} p_j \pi_{j\alpha}^n \quad \text{and} \quad Q(z) = \sum_{k=1}^{n+1} q_k \pi_{k\beta}^n.
\]

Let the rational function \( f(z) = P(z)/Q(z) \) be defined on the open and dense subset \( W_Q = \{ z \in U(2) \ | \ Q(z) \neq 0 \} \) of the unitary group. If \( \alpha \neq \beta \) then the function \( f \) is proper biharmonic if \( p_n q_{n+1} - p_{n+1} q_n \neq 0 \) and

\[
\begin{align*}
p_{1n}^{n-1} &= q_n^{n-1}(n \cdot p_n q_{n+1} - (n-1) \cdot p_{n+1} q_n), \quad q_1 q_{n+1}^{n-1} = q_n^n, \nonumber \\
p_{2n}^{n-2} &= q_n^{n-2}((n-1) \cdot p_n q_{n+1} - (n-2) \cdot p_{n+1} q_n), \quad q_2 q_{n+1}^{n-2} = q_n^{n-1}, \nonumber \\
&\quad \vdots \nonumber \\
p_{nn}^1 &= q_n(2 \cdot p_n q_{n+1} - 1 \cdot p_{n+1} q_n), \quad p_{n-1} q_{n+1}^2 = q_n^2. \nonumber
\end{align*}
\]

The corresponding statement holds for the function induced on \( SU(2) \).
Bibliography


