WILLMORE SURFACES

JOEL PERSSON
Master’s thesis
2003:E9

Lund University
Centre for Mathematical Sciences
Mathematics
Abstract

The main aim of this Master's dissertation is to investigate the interesting connection between minimal surfaces in the 3-dimensional sphere $S^3$ and Willmore surfaces in the Euclidean space $E^3$. Willmore surfaces $\Sigma$ in $E^3$ are stationary immersions for the so called Willmore functional

$$\mathcal{W} = \int_{\Sigma} H^2 \cdot dA,$$

where $H$ is the mean curvature of $\Sigma$. These can be characterized as the solutions of the heavily non-linear partial differential equation

$$\Delta H + 2H(H^2 - K) = 0,$$

where $K$ is the Gaussian curvature of $\Sigma$. Special cases of Willmore surfaces are the minimal surfaces with $H \equiv 0$.

We build up the necessary technical tools to deal with the problem. We prove that the Willmore functional is conformally invariant. Using calculus of variation we deduce the Euler-Lagrange equation for the problem. Finally we show that the problem actually has a solution on compact oriented surfaces of any genus $g \geq 0$.

It has been my firm intention throughout this work to give references to stated results. Any statement or proof without references is considered to be too well-known for a reference to be given.
Acknowledgements

I would like to thank everyone who has helped me with my work and in particular my supervisor Sigmundur Gudmundsson for his support and guidance through this project. I am very grateful for his encouragement during the whole time as well as interesting discussion on various subjects.

Joel Persson
Contents

Introduction 3
Chapter 1. The Willmore Functional in $E^3$ 5
Chapter 2. The Pull-Back Bundle 9
Chapter 3. The Willmore Functional in $(M^3, g)$ 15
  1. The Tangent and Normal Bundles 15
  2. The Conformal Invariance 18
Chapter 4. Variational Methods 21
  1. Minimal Submanifolds 21
  2. The Laplace Operator in Vector Bundles 23
  3. The Euler-Lagrange Equation 25
Bibliography 33
Introduction

In this Master’s dissertation we study a generalization of minimal surfaces called Willmore surfaces. For a natural number $n \in \mathbb{N}$ and a smooth compact orientable surface $\Sigma$ of genus $n$, let $\mathcal{I}(n)$ be the set of smooth immersions $f : \Sigma \to M^3$ of the surface $\Sigma$ into a 3-dimensional Riemannian manifold $(M, g)$. On $\mathcal{I}(n)$ the Willmore functional $\mathcal{W} : \mathcal{I}(n) \to \mathbb{R}_0^+$ is defined by

$$\mathcal{W}(f) = \int_{\Sigma} (H^2 + K)dA,$$

where $H : \Sigma \to \mathbb{R}$ and $K : \Sigma \to \mathbb{R}$ are the mean curvature of $\Sigma$ and the sectional curvature in $M^3$ evaluated at $df(T\Sigma)$, respectively. The study of this functional was proposed by T. Willmore in [16] from 1965. Critical points of the functional $\mathcal{W}$ are called Willmore surfaces. In the case $M^3 = \mathbb{R}^3$ that is $\mathbb{R}^3$ with the Euclidean metric the subject was first considered by G. Thomson in [11] from 1923 and W. Blaschke in [1] from 1929. They also noticed that $\mathcal{W}$ is invariant under conformal transformations on $\mathbb{R}^3$. This was later proved explicitly by J. H. White in [14]. One way to characterize the critical points of $\mathcal{W}$ is to calculate its first variational formula. This leads to the non-linear partial differential equation

$$\Delta H + 2H(H^2 - K) = 0.$$

This was actually known to G. Thomson and W. Blaschke. A more generalized version was considered by J. Weiner in [13] allowing a more general receiving space then $\mathbb{R}^3$ as well as surfaces with boundary. Since $\mathcal{W}$ is invariant under conformal transformations of $\mathbb{R}^3$ and by a result of Lawson in [7] stating that there are minimal surfaces of arbitrary genus embedded in $S^3$, we can conclude from (2) that there are compact embedded Willmore surfaces in $\mathbb{R}^3$ of arbitrary genus.
CHAPTER 1

The Willmore Functional in $E^3$

In this chapter we will by $E^3$ denote the 3-dimensional Euclidean space i.e. $\mathbb{R}^3$ equipped with the standard Euclidean metric.

Proposition 1.1. Let $f : \Sigma \to E^3$ be an immersion of a compact surface $\Sigma$ into $E^3$. Then the Willmore functional $\mathcal{W}(f)$ satisfies

$$\mathcal{W}(f) \geq 4\pi$$

with equality if and only if $\Sigma$ is embedded as the standard round sphere $S^2$ in $E^3$.

Proof. [17] Let $\Sigma^+_0$ be the part of $\Sigma$ with non-negative Gaussian curvature. Taking an affine hyperplane not intersecting $\Sigma$ and translate it in its normal direction towards the surface, any of the first points $p \in \Sigma$ touching the plane has non-negative curvature. This means that the Gauss map $N : \Sigma \to S^2$ restricted to $\Sigma^+_0$ is surjective and hence

$$\int_{\Sigma^+_0} KdA \geq 4\pi.$$ 

We have

$$H^2 - K = \frac{1}{4}(k_1 - k_2)^2 \geq 0$$

and we find

$$\int_{\Sigma} H^2 dA \geq \int_{\Sigma^+_0} H^2 dA \geq \int_{\Sigma^+_0} KdA \geq 4\pi.$$ 

We also see that equality is obtained if and only if $k_1 = k_2$ for each point $p$. This is only the case when $\Sigma$ is embedded as a sphere [15].

Willmore Conjecture 1.2. [16] Let $f : T^2 \to E^3$ be an immersion of the 2-dimensional torus $T^2$ into $E^3$. Then the Willmore functional of $f$ satisfies

$$\mathcal{W}(f) \geq 2\pi^2.$$ 

This conjecture is still open but has been proved in several special cases. Equality is actually obtained for the torus in $E^3$ with generating circles of radii of the ratio $1 : \sqrt{2}$ We will now consider the case when the $T^2$ is a "tube" embedded in $E^3$. For this we will need Fenchel's theorem.
**Theorem 1.3.** Let $\gamma$ be a simple closed curve in $E^3$ with curvature $\kappa$. Then

$$\int_{\gamma} |\kappa| ds \geq 2\pi$$

with equality if and only if $\gamma$ is a plane convex curve.

**Proof.** [3]. Take a tube $\Sigma$ of radius $r$ around $\gamma$ given by

$$x(s, v) = \gamma(s) + r(n(s) \cos v + b(s) \sin v) \quad s \in [0, l] \text{ and } v \in [0, 2\pi],$$

where $n$ and $b$ is the normal and bi-normal to the curve and $l$ is the length of the curve. By calculating the coefficients of the first and second fundamental forms we find

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-\kappa \cdot \cos v}{r(1 - r\kappa \cos v)}$$

and

$$dA = \sqrt{EG - F^2} ds dv = r(1 - r\kappa \cos v) ds dv.$$

So for $\kappa = 0$ we have $K = 0$ and also $K \geq 0$ for $v \in [\pi/2, 3\pi/2]$ Integrating over $\Sigma^+_0$ yields

$$\int_{\Sigma^+_0} K dA \geq - \int_{\gamma} |\kappa| ds \int_{\pi/2}^{3\pi/2} \cos v dv = 2 \int_{\gamma} |\kappa| ds$$

and by Proposition (1.1) we have

$$\int_{\gamma} |\kappa| ds \geq 2\pi.$$

Left is to prove that equality is obtained if and only if $\gamma$ is a plane convex curve. For every fixed $s \in [0, l]$ we get a circle on the tube which the Gauss-map maps bijectively onto a great circle $\Gamma_s$ on $S^2$. Let $\Gamma^+_s$ be the closed half-circle given by the points of $\Gamma_s$ which correspond to points on the original circle on the tube with positive Gaussian curvature. Assuming that $\gamma$ is a plane convex curve then all semicircles $\Gamma^+_s$ have the same end points $p, q$ for every $s \in [0, l]$. By convexity of $\gamma$ we have $\Gamma^+_s \cap \Gamma^+_q = \{p, q\}$. Hence in this case the area of the image of the Gauss normal map is actually $4\pi$ and

$$\int_{\gamma} |\kappa| ds = 2\pi.$$

To prove the converse suppose $\int_{\gamma} |\kappa| ds = 2\pi$ then $\int_{\Sigma^+_0} K = 4\pi$. All semicircles must have the same end points $p$ and $q$, or there are two distinct great circles $\Gamma_s$ and $\Gamma_q$ arbitrarily close to each other with corresponding semicircles with distinct endpoints $p$ and $q$ that intersects in two antipodal points. One of these must then correspond to two points with a positive Gaussian curvature. Thus we have two points on the tube with positive Gaussian curvature which maps...
to the same point on $S^2$. Since the Gauss normal map is locally a diffeomorphism at each point and also each point of $S^2$ is the image of some point of $\Sigma_0^+$ we get

$$\int_{\Sigma_0^+} K dA > 4\pi$$

giving a contradiction.

The intersection of the bi-normal $b$ of $\gamma$ with $\Sigma$ gives the points of zero Gaussian curvature. Thus the bi-normal must be parallel to the vector $p - q$ and $\gamma$ lies in a plane normal to $p - q$. For convexity note that

$$\int_{\gamma} \kappa \cdot ds \geq 2\pi$$

holds for any closed curve $\gamma$ with positive orientation number thus in our case $\kappa = |\kappa|$.

\[\square\]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{torus.png}
\caption{Torus of ratio of the radii 1 : $\sqrt{2}$.}
\end{figure}

\textbf{Proposition 1.4.} [10] Let $\Sigma$ be a torus in $E^3$ given by moving a circle in $E^3$ around a smooth closed curve $\gamma$ in such a way that the normal of the circle is parallel to the normal of the normal plane of the curve at each point and the center of the circle always is a point on $\gamma$. Then

$$\int_{\Sigma} H^2 dA \geq 2\pi,$$

with equality if and only if $\gamma$ is a circle and ratio of the radii is 1 : $\sqrt{2}$. 
Proof. [17] $\Sigma$ can be considered as tube with radius $r$ as in the proof above. The mean curvature $H$ is given by

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)},$$

with the coefficients from the fundamental forms of the surface $\Sigma$. In our case we get

$$H = \frac{(1 - 2r \kappa \cos \nu)}{2r(1 - r \kappa \cos \nu)}$$

and

$$\int_{\Sigma} H^2 = \int_{0}^{l} \int_{0}^{2\pi} \frac{(1 - 2r \kappa \cos \nu)^2}{4r(1 - r \kappa \cos \nu)} d\nu ds.$$

Evaluating the inner integral yields

$$\int_{\Sigma} H^2 = \pi \int_{0}^{l} \frac{1}{2r \sqrt{1 - r^2 \kappa^2}} ds = \frac{\pi}{2} \int_{0}^{l} \frac{|\kappa|}{\sqrt{1 - r^2 \kappa^2}} ds.$$

Now $|\kappa| \sqrt{1 - r^2 \kappa^2}$ attains its minimum 1/2 when $\kappa r = 1/\sqrt{2}$. We are left with

$$\int_{\Sigma} H^2 \geq \pi \int_{0}^{l} |\kappa| ds.$$

By Fenchel’s theorem we finally get

$$\int_{\Sigma} H^2 dA \geq 2\pi^2$$

with equality if and only if $\kappa r = 1/\sqrt{2}$ i.e. when $\Sigma$ is the anchor ring with radii of ratio $1/\sqrt{2}$. \qed
CHAPTER 2

The Pull-Back Bundle

Throughout this chapter we assume $\tilde{M}$ and $M$ to be smooth manifolds of dimension $\tilde{m}$ and $m$, respectively. Let $f : \tilde{M} \to M$ be a smooth map from $\tilde{M}$ to $M$. By $f^*(TM)$ we denote the pull-back bundle

$$f^*(TM) = \{(p, v) \in \tilde{M}, v \in T_{f(p)}M\}$$

over $\tilde{M}$ via $f$. Then $f^*(TM)$ is a smooth $m$-dimensional vector bundle which fiber at a point $p \in \tilde{M}$ is the vector space $T_{f(p)}M$. The following diagram illustrates the situation.

$$\begin{array}{ccc}
\tilde{M} & \xrightarrow{f} & M \\
\pi^* & \downarrow & \pi \\
\tilde{M} & \xrightarrow{\pi} & M
\end{array}$$

Here $\pi^*$ is the projection map of the pull-back bundle $f^*(TM)$ given by $\pi^* : (p, v) \mapsto p$ and $\pi$ is the projection map of the tangent bundle $TM$ of $M$. Note that a vector field $X \in C^\infty(TM)$ induces a section $X^* \in C^\infty(f^*(TM))$ of the pull-back bundle given by

$$X^*_p = X_{f(p)}$$

for each $p \in \tilde{M}$. This means that a local frame $\{e_i\}$ for $TM$ induces a local frame $\{e^*_i\}$ for the pull-back bundle $f^*(TM)$.

DEFINITION 2.1. A smooth vector bundle $(E, M, \pi)$ is said to be Riemannian if there exists a smooth map

$$b : C^\infty(E) \otimes C^\infty(E) \to C^\infty(M)$$

and a connection

$$\nabla^E : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$$

on $E$ which is compatible with $b$. That is

$$X(b(\psi, \varphi)) = b(\nabla^E_X\psi, \varphi) + b(\psi, \nabla^E_X\varphi)$$

for any vector field $X \in C^\infty(TM)$ on $M$ and smooth sections $\psi, \varphi \in C^\infty(E)$ of the bundle $E$. The map $b$ is called a metric on the bundle $E$. The curvature $R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$ of the connection $\nabla^E$ is defined by
A Riemannian metric $g$ on a smooth manifold $M$ induces the Levi-Civita connection $\nabla$ on the tangent bundle $(TM, M, \pi)$ of $M$. By the fundamental theorem of Riemannian geometry we know that $g$ and $\nabla$ turn the tangent bundle $TM$ into a Riemannian vector bundle over $M$ since we for any vector fields $X, Y, Z \in C^\infty(TM)$ have that

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

We will now use $g$ and $\nabla$ on $M$ to make the pull-back bundle $f^*(TM)$ into a Riemannian vector bundle over $\tilde{M}$.

**Proposition 2.2.** Let $(M, g)$ be a Riemannian manifold and $f : \tilde{M} \to M$ be a smooth map from $\tilde{M}$ to $M$. Then the Levi-Civita connection $\nabla$ on $(M, g)$ induces a unique connection $\nabla^*$ on the pull-back bundle $f^*(TM)$ defined by

$$\nabla^*_X (\varphi) = (\nabla_{df(\tilde{Y})} \varphi)^*$$

for $\tilde{X} \in C^\infty(T\tilde{M})$ and $\varphi \in C^\infty(TM)$.

**Proof.** It is clear that $\nabla^*$ is a connection. To prove uniqueness consider a local orthogonal frame $\{e_i\}$ for $TM$. Then given a local section $\tilde{\varphi}$ of $f^*(TM)$ we have $\tilde{\varphi} = \sum_i \alpha_i e_i^*$ for some local functions $\alpha_i$. Then

$$\nabla^*_X \tilde{\varphi} = \nabla^*_X (\sum_i \alpha_i e_i^*)$$

$$= \sum_i X(\alpha_i) e_i^* + \sum_i \alpha_i (\nabla^*_X e_i^*).$$

To show that the expression above is independent of the choice of frame $\{e_i\}$ consider another frame $\{f_j\}$ for $TM$. We have

$$\tilde{\varphi} = \sum_j \beta_j f_j^*$$

and

$$f_j^* = \sum_i x_{ij} e_i^*$$

for some functions $\beta_j, x_{ij}$ locally defined on $\tilde{M}$. Then

$$\nabla^*_X (\sum_j \beta_j f_j^*)$$

$$= \sum_j (X(\beta_j) f_j^* + \beta_j \nabla^*_X f_j^*)$$

$$= \sum_j \{X(\beta_j) \sum_i x_{ij} e_i^* + \beta_j \nabla^*_X (\sum_i x_{ij} e_i^*)\}$$

$$= \sum_i \left\{ \sum_j \{X(\beta_j) x_{ij} e_i^* + \beta_j X(x_{ij}) e_i^* + \beta_j x_{ij} \nabla^*_X e_i^*\} \right\}$$
\[= \sum_i \{ \bar{X}(\sum_j \beta_j x_j^e) e^*_i + \alpha_i \nabla^*_X e^*_i \} \]

\[= \nabla^*_X (\alpha_i e^*_i). \]

Thus the connection \( \nabla^* \) is unique and independent of the choice of the local frame. \( \square \)

**Proposition 2.3.** Let \((M, g)\) be a Riemannian manifold and further let \(f: \bar{M} \to M\) be a smooth map from \(\bar{M}\) to \(M\). Then the pull-back connection \( \nabla^* \) on the pull-back bundle \( f^*(TM) \) satisfies

\[ \nabla^*_X (df(Y))^* - \nabla^*_Y (df(X))^* = (df([X, Y]))^*. \]

**Proof.** We define \( T : C^\infty(T\bar{M}) \times C^\infty(T\bar{M}) \to C^\infty(f^*(TM)) \) by

\[ T(\bar{X}, \bar{Y}) = \nabla^*_a (df(\bar{Y}))^* - \nabla^*_a (df(\bar{X}))^* - (df([\bar{X}, \bar{Y}]))^*. \]

Let \( a \in C^\infty(\bar{M}) \) be a function on \(\bar{M}\), then

\[ T(a\bar{X}, a\bar{Y}) = \nabla^*_a (df(\bar{Y}))^* - \nabla^*_a (df(\bar{X}))^* = - (df([\bar{X}, \bar{Y}]))^*. \]

By skew-symmetry of \( T \) we also have \( T(\bar{X}, a\bar{Y}) = aT(\bar{X}, \bar{Y}) \) and hence \( T \) is a tensor field. Let \( p \in \bar{M} \) be an arbitrary point and consider around \( p \) and \( f(p) \in M \) two local systems of coordinates \( \{x_i\} \) and \( \{y_j\} \). Since \( T \) is torsion-free it is enough to show that \( T(\partial/\partial x_i, \partial/\partial x_j) = 0 \). We have

\[ df(\partial/\partial x_i) = \sum_k \partial f_k/\partial x_i \cdot \partial/\partial y_k \]

so

\[ T(\partial/\partial x_i, \partial/\partial x_j) = \sum_k \{ \nabla^*_a (\partial f_k/\partial x_i)^* - \nabla^*_a (\partial f_k/\partial x_j)^* \} \]

\[ = \sum_k \{ \partial^2 f_k/\partial x_i \partial x_j \cdot (\partial/\partial y_k)^* + \partial f_k/\partial x_j \cdot \nabla^*_a (\partial^2 f_k/\partial x_i \partial x_j)^* \} \]
\[
\begin{align*}
- \frac{\partial^2 f_k}{\partial x_i \partial y_k} \cdot (\frac{\partial f_k}{\partial y_k})^* & - \frac{\partial f_k}{\partial x_i} \cdot \nabla^* \frac{\partial}{\partial y_k} (\frac{\partial}{\partial x_j})^* \\
= \sum_{r, k} \left\{ \frac{\partial f_k}{\partial x_i} \frac{\partial f_r}{\partial x_i} \left( \nabla^* \frac{\partial}{\partial y_k} \right)^* - \frac{\partial f_k}{\partial x_i} \frac{\partial f_r}{\partial x_i} \left( \nabla^* \frac{\partial}{\partial y_r} \right)^* \right\} \\
= \sum_{r, k} \frac{\partial f_k}{\partial x_i} \frac{\partial f_r}{\partial x_i} \left[ \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_r} \right]^* \\
= 0.
\end{align*}
\]

We have shown that
\[
\nabla^*_X(df(Y))^* - \nabla^*_Y(df(X))^* = (df([X, Y]))^*.
\]

\[\square\]

**Definition 2.4.** Let \((M, g)\) be a Riemannian manifold and \(f : \tilde{M} \to M\) be a smooth map from \(\tilde{M}\) to \(M\). Then the pull-back bundle \(f^*(TM)\) is defined by
\[
g^*(X, Y) = g(X, Y),
\]
where \(X, Y \in C^\infty(TM)\).

**Theorem 2.5.** Let \((M, g)\) be a Riemannian manifold and \(f : \tilde{M} \to M\) be a smooth map from \(\tilde{M}\) to \(M\). Then the pull-back bundle \((f^*(TM), \tilde{M}, \pi^*)\) equipped with the metric \(g^*\) and the connection \(\nabla^*\) is a Riemannian vector bundle.

**Proof.** Let \(X \in C^\infty(T\tilde{M})\) be a vector field on \(\tilde{M}\) and further let \(\overline{\varphi}, \overline{\psi} \in C^\infty(f^*(TM))\) be sections of \(f^*(TM)\). Let \(\{e_i\}\) be a local orthonormal frame for \(TM\) then \(\{e_i^*\}\) is a local orthonormal frame for \(f^*(TM)\). We have
\[
\overline{\varphi} = \sum \alpha_i e_i^* \text{ and } \overline{\psi} = \sum \beta_i e_i^*
\]
for some functions \(\alpha_i, \beta_i \in C^\infty(\tilde{M})\). By the definition of \(g^*\) and \(\nabla^*\) it is clear that
\[
g^*(\nabla^*_X e_i^*, e_j^*) + g^*(e_i^*, \nabla^*_X e_j^*) = \tilde{X}(g^*(e_i^*, e_j^*)) = 0.
\]
Then
\[
g^*(\nabla^*_X \overline{\varphi}, \overline{\psi}) + g^*(\overline{\varphi}, \nabla^*_X \overline{\psi})
= g^*(\nabla^*_X (\sum \alpha_i e_i^*), \sum \beta_j e_j^*) + g^*(\sum \alpha_i e_i^*, \nabla^*_X (\sum \beta_j e_j^*))
= \sum \tilde{X}(\alpha_i) \beta_j + \sum \alpha_i \beta_j \cdot g^*(\nabla^*_X e_i^*, e_j^*)
\]
\[ + \sum_{i} \alpha_i \tilde{X}(\beta_i) + \sum_{i,j} \alpha_i \beta_j \cdot g^*(e_i^* \cdot \nabla_{\tilde{X}} e_j^*) \]

\[ = \tilde{X}(\sum_{i} \alpha_i \beta_i) \]

\[ = \tilde{X}(g^*(\tilde{\varphi}, \tilde{\psi})). \]
CHAPTER 3

The Willmore Functional in $(M^3, g)$

In this chapter we will consider the special case of the pull-back bundle when $f$ is an immersion. Also we will show that the Willmore functional is conformal invariant.

1. The Tangent and Normal Bundles

From now on we will assume throughout this chapter that the map $f : \tilde{M} \to M$ is an immersion. Then $f$ is locally a diffeomorphism onto its image. This means that given a vector field $\bar{X} \in C^\infty(T\tilde{M})$ on $\tilde{M}$ and a section $\bar{\varphi} \in C^\infty(f^*(TM))$ of $f^*(TM)$ we can, at each point $p \in \tilde{M}$, find a neighbourhood $U$ of $p$, a neighbourhood $V$ in $M$ of $f(p)$ and local vector fields $X, \varphi \in C^\infty(TM)$ such that

$$df(\bar{X}_p) = X_{f(p)} \quad \text{and} \quad \bar{\varphi}_p = \varphi_{f(p)}^*,$$

for every $p \in U$. We will call $X$ and $\varphi$ local extensions of $\bar{X}$ and $\bar{\varphi}$, respectively.

The pull-back metric $g^*$ splits the vector space $f^*(TM)_p$ at each point $p \in \tilde{M}$ into the direct sum

$$(f^*(TM))_p = T_p\tilde{M} \oplus N_p\tilde{M},$$

where $T_p\tilde{M}$ is identified with the image $df(T_p\tilde{M})$ in $T_{f(p)}M$ and $N_p\tilde{M}$ with the normal space

$$(df(T_p\tilde{M}))^\perp$$

with respect to $g$ in $T_{f(p)}M$. This gives a splitting of the pull-back bundle $f^*(TM)$

$$f^*(TM) = T\tilde{M} \oplus N\tilde{M},$$

where $T\tilde{M}$ is the tangent bundle of $\tilde{M}$ and $N\tilde{M}$ the normal bundle of $f(\tilde{M})$ in $M$. We obtain a metric

$$\tilde{g} : C^\infty(T\tilde{M}) \otimes C^\infty(T\tilde{M}) \to C^\infty(\tilde{M})$$

on the tangent bundle and a metric

$$\hat{g} : C^\infty(N\tilde{M}) \otimes C^\infty(N\tilde{M}) \to C^\infty(\tilde{M})$$

on the normal bundle by restricting $g^*$ to $T\tilde{M}$ and $N\tilde{M}$, respectively.
Proposition 3.1. Let \((M, g)\) be a Riemannian manifold and \(f : \tilde{M} \to M\) be an immersion of \(\tilde{M}\) into \(M\). Then

\[
\tilde{\nabla} : C^\infty(T\tilde{M}) \times C^\infty(T\tilde{M}) \to C^\infty(T\tilde{M})
\]

is the Levi-Civita connection of \((\tilde{M}, \tilde{g})\) Here \(X, Y\) are some local extensions of \(\tilde{X}\) and \(\tilde{Y}\).

Proof. We show that \(\tilde{\nabla}\) preserves the metric \(\tilde{g}\). Let \(\tilde{X}, \tilde{Y}, \tilde{Z} \in C^\infty(T\tilde{M})\) and let \(X, Y, Z\) denote some local extensions. Then we have

\[
\tilde{X}(\tilde{g}(\tilde{Y}, \tilde{Z})) = X(g(Y, Z))
\]

\[
= g(\nabla_x Y, Z) + g(Y, \nabla_x Z)
\]

\[
= \tilde{g}(\nabla_x Y)^T \tilde{Z} + \tilde{g}(\tilde{Y}, (\nabla_x Z)^T)
\]

\[
= \tilde{g}(\tilde{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) + \tilde{g}(\tilde{Y}, \tilde{\nabla}_{\tilde{X}} \tilde{Z}).
\]

So the connection \(\tilde{\nabla}\) is metric. Furthermore

\[
\tilde{\nabla}_{\tilde{X}} \tilde{Y} - \tilde{\nabla}_{\tilde{Y}} \tilde{X} - [\tilde{X}, \tilde{Y}] = (\nabla_x Y)^T - (\nabla_y X)^T - [X, Y]^T
\]

\[
= (\nabla_x Y - \nabla_y X - [X, Y])^T = 0,
\]

which shows that \(\tilde{\nabla}\) is torsion-free.

Proposition 3.2. Let \(\tilde{N}\tilde{M}\) be the normal bundle of \(f(\tilde{M})\) in \(M\). Let \(X \in C^\infty(T\tilde{M})\) be a vector field on \(\tilde{M}\), \(\varphi \in C^\infty(\tilde{N}\tilde{M})\) be a section of \(\tilde{N}\tilde{M}\) and let \(X\) and \(\varphi\) be some local extensions. Then

\[
D : C^\infty(T\tilde{M}) \times C^\infty(\tilde{N}\tilde{M}) \to C^\infty(\tilde{N}\tilde{M})
\]

is defined by

\[
D_{\tilde{X}} \tilde{\varphi} = (\nabla_x \varphi)^\perp
\]

is a connection on \(\tilde{N}\tilde{M}\) which preserves the metric \(\tilde{g}\).

Proof. Let \(\tilde{X} \in C^\infty(T\tilde{M})\) be a vector field on \(\tilde{M}\) and \(\tilde{\varphi}, \tilde{\varphi} \in C^\infty(\tilde{N}\tilde{M})\) be two sections on \(\tilde{N}\tilde{M}\). Let \(X, \psi, \varphi\) be some local extensions. Then we have

\[
\tilde{X}(\tilde{g}(\tilde{\varphi}, \tilde{\varphi})) = X(g(\psi, \varphi))
\]

\[
= g(\nabla_x \psi, \varphi) + g(\psi, \nabla_x \varphi)
\]

\[
= \tilde{g}(\nabla_x \psi)^\perp, \tilde{\varphi} + \tilde{g}(\psi, (\nabla_x \varphi)^\perp)
\]

\[
= \tilde{g}(D_{\tilde{X}} \tilde{\psi}, \tilde{\varphi}) + \tilde{g}(\tilde{\psi}, D_{\tilde{X}} \tilde{\varphi}).
\]
Proposition 3.3. Let $(M, g)$ be a Riemannian manifold and $f : \tilde{M} \to M$ be an immersion of $\tilde{M}$ into $M$. Further let $\tilde{X}, \tilde{Y} \in C^\infty(T\tilde{M})$ be two vector fields on $\tilde{M}$ and denote by $X, Y$ some local extensions of $\tilde{X}, \tilde{Y}$. Then the second fundamental form of $f$

\[ B : C^\infty(T\tilde{M}) \otimes C^\infty(T\tilde{M}) \to C^\infty(N\tilde{M}) \]

defined by

\[ B(\tilde{X}, \tilde{Y}) = (\nabla_\tilde{X} Y)^\perp \]

is a symmetric tensor field.

Proof. We first show $B(\tilde{X}, \tilde{Y}) = B(\tilde{Y}, \tilde{X})$.

\begin{align*}
B(\tilde{X}, \tilde{Y}) - B(\tilde{Y}, \tilde{X}) & = (\nabla_\tilde{X} Y)^\perp - (\nabla_\tilde{Y} X)^\perp \\
& = (\nabla_\tilde{X} Y)^\perp - (\nabla_\tilde{X} Y)^\perp + [X, Y]^\perp \\
& = 0
\end{align*}

Since obviously $B(\mu \tilde{X}, \tilde{Y}) = \mu B(\tilde{X}, \tilde{Y})$ for any $\mu \in C^\infty(\tilde{M})$ we have

\[ B(\tilde{X}, \mu \tilde{Y}) = \mu B(\tilde{X}, \tilde{Y}) \]

by the symmetry of $B$ and thus $B$ is tensorial in both its arguments.

Definition 3.4. Let $(M, g)$ be a Riemannian manifold and $f : \tilde{M} \to M$ be an immersion of $\tilde{M}$ into $M$. Then the mean curvature vector field $\mathbb{H} \in C^\infty(N\tilde{M})$ of the immersion $f : \tilde{M} \to M$ is defined by

\[ \mathbb{H} = \frac{1}{\tilde{m}} \text{trace } B. \]

To the second fundamental form $B$ and a section $\tilde{\phi} \in C^\infty(N\tilde{M})$ we associate the corresponding shape-operator $A^\tilde{\phi} : T\tilde{M} \to T\tilde{M}$ given by

\[ \tilde{g}(A^\tilde{\phi}(\tilde{X}), \tilde{Y}) = \tilde{g}(B(\tilde{X}, \tilde{Y}), \tilde{\phi}) \]

for any vector fields $X, Y \in C^\infty(T\tilde{M})$.

Proposition 3.5. Let $\tilde{X} \in C^\infty(T\tilde{M})$ be a vector field on $\tilde{M}$, $\tilde{\phi} \in C^\infty(N\tilde{M})$ be a section of $N\tilde{M}$ and $X, \phi$ some local extension $\tilde{X}$ and $\tilde{\phi}$. Then

\[ A^\tilde{\phi}(\tilde{X}) = -(\nabla_\tilde{X} \phi)^\top. \]

Proof. Let $\tilde{Y} \in C^\infty(T\tilde{M})$ be a vector field on $\tilde{M}$ and $Y$ a local extension of $\tilde{Y}$. Then

\[ \tilde{g}(A^\tilde{\phi}(\tilde{X}), \tilde{Y}) = \tilde{g}(B(\tilde{X}, \tilde{Y}), \tilde{\phi}) = \tilde{g}((\nabla_\tilde{X} Y)^\perp, \tilde{\phi}) = g(\nabla_X \phi, \phi) \]
Levi-Civita connections on TM induced by the metrics $g$ and $g^\lambda$.

By the definition of the Levi-Civita connection we have

$$\nabla_Y g^\lambda = X(g(Y, \varphi)) - g(Y, \nabla_X \varphi)$$

$$= g(Y, -\nabla_X \varphi).$$

\[\square\]

2. The Conformal Invariance

In this section we will prove that the Willmore functional is invariant under conformal changes of the Riemannian metric $g$ on $M$.

**Proposition 3.6.** Let $M$ be a smooth manifold and $g$, $g^\lambda$ be two Riemannian metrics on $M$ such that

$$g^\lambda = \lambda^2 \cdot g$$

for some positive real valued function $\lambda : M \to \mathbb{R}^+$. Denote by $\nabla$ and $\nabla^\lambda$ the Levi-Civita connections on $TM$ induced by the metrics $g$ and $g^\lambda$. Then $\nabla$ and $\nabla^\lambda$ are related by

$$\nabla_Y^\lambda X - \nabla_X Y = X(\log \lambda) Y + Y(\log \lambda) X - g(X, Y) \grad(\log \lambda).$$

**Proof.** Let $X, Y \in C^\infty(TM)$ be two vector fields on $M$. Further let $\{e_i\}$ be a local frame on $M$ which is orthonormal with respect to $g^\lambda$. Then

$$\nabla^\lambda_X Y - \nabla_X Y = \sum_i (g^\lambda(\nabla^\lambda_X Y, e_i) - g^\lambda(\nabla_X Y, e_i) e_i)$$

$$= \sum_i (\lambda^2 g(\nabla^\lambda_X Y, e_i) - \lambda^2 g(\nabla_X Y, e_i) e_i).$$

By the definition of the Levi-Civita connection we have

$$\nabla^\lambda_X Y - \nabla_X Y = \sum_i \left\{ \lambda^2 g(\nabla^\lambda_X Y, e_i) e_i - \lambda^2 g(\nabla_X Y, e_i) e_i \right\}$$

$$= \sum_i \left\{ X(\lambda^2 g(Y, e_i)) e_i + Y(\lambda^2 g(e_i, X)) e_i - g(X, Y) e_i (\lambda^2) e_i \right\}$$

Thus we have

$$\nabla^\lambda_X Y - \nabla_X Y$$

$$= \sum_i \left\{ X(\lambda^2 g(Y, e_i)) e_i + Y(\lambda^2 g(e_i, X)) e_i - g(X, Y) e_i (\lambda^2) e_i \right\}$$

$$= \frac{1}{2} \sum \left\{ X(\lambda^2 g(Y, e_i)) e_i + Y(\lambda^2 g(e_i, X)) e_i - g(X, Y) e_i (\lambda^2) e_i \right\}.$$
\[ X(\log \lambda) Y + Y(\log \lambda) X - g(X, Y) \text{grad}(\log \lambda). \]

And so
\[ \nabla_X^\lambda Y - \nabla_X Y = X(\log \lambda) Y + Y(\log \lambda) X - g(X, Y) \text{grad}(\log \lambda) \]

\[ \square \]

After the necessary preparations we can now prove the following theorem.

**Theorem 3.7.** [14] Let \( \Sigma \) be a surface and \( f : \Sigma \to (M, g) \) be an immersion of \( \Sigma \) into a 3-dimensional Riemannian manifold. Then the Willmore functional
\[ \int_{\Sigma} (H^2 + K) dA \]
is conformal invariant. Here \( K \) is the sectional curvature in \( M \) evaluated at \( df(T\Sigma) \).

**Proof.** [2] As above let \( g^\lambda \) be a Riemannian metric on \( M \) such that
\[ g^\lambda = \lambda^2 \cdot g \]
for some positive real valued function \( \lambda : M \to \mathbb{R}^+ \). Denote by \( \nabla \) and \( \nabla^\lambda \) the Levi-Civita connection on \( TM \) induced by the metrics \( g \) and \( g^\lambda \). Let \( \bar{g} \) and \( \bar{g}^\lambda \) be the pull-back metrics on \( \Sigma \) induced by \( f \) and further let \( \bar{\nabla} \) and \( \bar{\nabla}^\lambda \) be the Levi-Civita connections on \( \Sigma \) induced by \( f \). Let \( \bar{X}, \bar{Y} \in C^\infty(T\Sigma) \) be two vector field on \( \Sigma \) and denote by \( B_0 \) and \( B_0^\lambda \) the second fundamental form on \( \Sigma \) with respect to the different metrics \( \bar{g} \) and \( \bar{g}^\lambda \). Then by the definition of second fundamental form and (4) we have
\[ B_0^\lambda(\bar{X}, \bar{Y}) - B_0(\bar{X}, \bar{Y}) = \bar{g}(\bar{X}, \bar{Y})(\text{grad}(\log \lambda))^\perp. \]

Let \( \bar{\varphi} \in C^\infty(N\Sigma) \) be a local section of the normal bundle of \( \Sigma \) of unit length with respect to \( \bar{g} \). Then from (6) and the duality between \( B_0 \) and \( A_0^\varphi \) and \( A_{0\varphi}^\lambda \), respectively, we obtain
\[ \bar{g}(A_{0\varphi}^\lambda(\bar{X}, \bar{Y})) = \bar{g}(A_0^\varphi(\bar{Y}), \bar{X}) + \bar{g}(\bar{X}, \bar{Y})\text{\hat{g}}((\text{grad}(\log \lambda))^\perp, \bar{\varphi}). \]

Since \( A_0^\varphi \) is a symmetric endomorphism we can find a local orthonormal frame \( \{E_1, E_2\} \) of \( \Sigma \) which are in the principal direction of \( A_0^\varphi \) with respect to \( \bar{g} \). Then \( \{\lambda^{-1} E_i\} \) is a local orthonormal frame in the principal directions of \( A_{0\varphi}^\lambda \) with respect to the metric \( \bar{g}^\lambda \) by (6). We denote by \( (k_1(\bar{\varphi}), k_2(\bar{\varphi})) \) and \( (k_1^\lambda(\bar{\varphi}), k_2^\lambda(\bar{\varphi})) \) the corresponding principal curvature of \( A_0^\varphi \) and \( A_{0\varphi}^\lambda \), respectively. We have, by (7)
\[ k_i^\lambda(\bar{\varphi}) = k_i(\bar{\varphi}) + \text{\hat{g}}((\text{grad}(\log \lambda))^\perp, \bar{\varphi}). \]
Since \( \tilde{\varphi} \in N \Sigma \) is a local normal section of unit length with respect to \( \hat{g} \) we know that \( \tilde{\varphi}^\lambda = \lambda^{-1} \tilde{\varphi} \) is a local normal section of unit length with respect to \( \hat{g}^\lambda \). From the fact that

\[
A^\varphi = \lambda A^\varphi^\lambda
\]

(8) and the Gauss equation we find

\[
\begin{align*}
\lambda(k_1^\varphi(\tilde{\varphi}^\lambda)) - k_2^\varphi(\tilde{\varphi}^\lambda) &= k_1(\tilde{\varphi}) - k_2(\tilde{\varphi}) \\
\lambda^2(k_1^\varphi(\tilde{\varphi}^\lambda) - k_2^\varphi(\tilde{\varphi}^\lambda))^2 &= (k_1(\tilde{\varphi}) - k_2(\tilde{\varphi}))^2 \\
\lambda^2(H_\lambda^2 - K_\lambda + \bar{K}_\lambda) &= H^2 - K + \bar{K}.
\end{align*}
\]

We notice

\[
dA_\lambda = \sqrt{E_\lambda G_\lambda - (F_\lambda)^2} \, dx \, dy = \lambda^3 \sqrt{EG - F^2} \, dx \, dy = \lambda^3 dA
\]

which implies

\[
(H_\lambda^2 - K_\lambda + \bar{K}_\lambda) dA_\lambda = (H^2 - K + \bar{K}) dA.
\]

If we now apply the Gauss-Bonnet theorem we get

\[
\int_\Sigma (H_\lambda^2 + \bar{K}_\lambda) dA_\lambda - 2\pi \chi(\Sigma) = \int_\Sigma (H^2 - K + \bar{K}) dA
\]

\[
= \int_\Sigma (H - K + \bar{K}) dA
\]

\[
= \int_\Sigma (H^2 + \bar{K}) dA - 2\pi \chi(\Sigma)
\]

which leads to

\[
\int_\Sigma (H^2 + \bar{K}) dA = \int_\Sigma (H_\lambda^2 + \bar{K}_\lambda) dA_\lambda.
\]

\[\square\]
Variational Methods

In this chapter we calculate the Euler-Lagrange equation for \( W \). For these calculation we need the first variational formula for the volume element. This also give some motivation for the notation of minimal surface for surfaces of constant zero mean-curvature.

1. Minimal Submanifolds

Let \( \tilde{M} \) be a smooth \( \tilde{m} \)-dimensional manifold and \((M, g)\) a smooth \( m \)-dimensional Riemannian manifold. Denote by \( I \) the interval \((-\varepsilon, \varepsilon)\). Let \( F : \tilde{M} \times I \rightarrow M \) be a differentiable mapping such that for each \( t \in I \) the map \( f_t : \tilde{M} \rightarrow M \) given by

\[
  f_t : x \mapsto F(x, t)
\]

is an immersion and \( F(x, 0) = f_0(x) = f(x) \). Then \( F \) is said to be a variation of \( f \). We will by \( V \) denote the section of the pull-back bundle \( f^*(TM) \) given by

\[
  V(p) = (dF(\frac{\partial}{\partial t}|_{(p,0)}))^*.
\]

For each \( t \in I \) denote by \( dA(t) \) the volume element induced by \( f_t \) and let \( dA(0) = dA \).

**Proposition 4.1.** Let \( \tilde{M} \) be a smooth \( \tilde{m} \)-dimensional manifold and let \( F : I \times \tilde{M} : M \) be a smooth variation of the immersion \( f : \tilde{M} \rightarrow M \) as above.

\[
\frac{d}{dt} |_{t=0} dA(t) = -\tilde{m} \cdot \hat{g}(H, V^+) dA + div(V^+) dA.
\]

**Proof.** For a point \( p \in \tilde{M} \) let \( \{\tilde{e}_1, \cdots, \tilde{e}_{\tilde{m}}\} \) be a local orthonormal frame for \( T\tilde{M} \) with respect to the metric \( \hat{g}_0 \) such that for at \( p \) we have

\[
(\nabla_{\tilde{e}_i} \tilde{e}_j)_p = 0.
\]

We have the corresponding 1-forms \( \omega_i \) given by \( \omega_i(\tilde{X}) = \hat{g}(\tilde{e}_i, \tilde{X}) \) constituting a local frame for the co-tangent bundle \( T\tilde{M}^* \). Thus the metric at a given time \( t \) is given by

\[
\tilde{g}_t = \sum_{i,j=1}^{\tilde{m}} \rho_{ij}(t) \omega_i \otimes \omega_j,
\]
where \( \rho_{i,j}(t) = g(d\tilde{f}_{i}(\tilde{x}), d\tilde{f}_{j}(\tilde{x})) \). By the definition of the volume form we have

\[
dA(t) = \sqrt{\rho(t)} \cdot \omega_1 \wedge \ldots \wedge \omega_m = \sqrt{\rho(t)} \cdot dA,
\]

where \( \rho(t) = \det(\rho_{i,j}(t)) \). So in order to determine \( \partial/\partial t(dA(t))|_{t=0} \) we must calculate \( \partial/\partial t(\sqrt{\rho(t)})|_{t=0} \). By lemma (4.2) we have

\[
\frac{d}{dt}dA(t)_{t=0} = \frac{1}{2} \sum_{k=1}^{m} \frac{d\rho_{kk}(0)}{dt} \cdot dA = \frac{1}{2} \text{trace}(\rho_{i,j}(0)) \cdot dA.
\]

We can locally extend \( \partial/\partial t, \tilde{e}_1, \ldots, \tilde{e}_m \) to a local frame in \( T(\tilde{M} \times I) \) and denote the images of this frame under \( dF \) by \( V, e_1, \ldots, e_m \). Hence locally at \( p \) we have

\[
\rho_{kk}(t) = g(e_k, e_k)
\]

and

\[
\frac{1}{2} \frac{d}{dt} \theta_{kk}(t) = \frac{1}{2} V(g(e_k, e_k)) = g(\nabla_V e_k, e_k) = e_k(g(V, e_k)) - g(V, \nabla e_k).
\]

Here we used the fact that \([e_k, V] = 0\). By construction we have \((\nabla e_k)^T = 0\) at \( p \) so

\[
\sum_{k=1}^{m} g(V, \nabla e_k) = \tilde{m} \cdot \hat{g}(\mathbb{H}, V^\perp)
\]

Hence at \( p \)

\[
\frac{1}{2} \sum_{k=1}^{m} \frac{d\rho_{kk}(0)}{dt} = -\tilde{m} \cdot \hat{g}(\mathbb{H}, V^\perp) + \sum_{k=1}^{m} \bar{e}_k(\bar{g}(V^T, \bar{e}_k)) = -\tilde{m} \cdot \hat{g}(\mathbb{H}, V^\perp) + \text{div}(V^T).
\]

\[\square\]

**Lemma 4.2.** Let \( B(t) = ((b_{ij}(t))) \) be a \( C^1 \) family of \( m \times m \) matrices with \( B(0) = I \). Then

\[
\frac{d}{dt}dB(t)_{t=0} = \text{trace}(B'(0)).
\]

**Proof.** For each fixed \( t \) we can consider \( B(t) \) to be a linear mapping \( B(t) : \mathbb{R}^m \to \mathbb{R}^m \) with respect to the canonical basis of \( \mathbb{R}^m \) which we will denote by \( \{e_1, \ldots, e_m\} \). From linear algebra we know that

\[
det(B(t)) = W(B(t)e_1, \ldots, B(t)e_m),
\]

where \( W(\cdot) \) is the determinant of the matrix. Therefore

\[
\frac{d}{dt}det(B(t))_{t=0} = \text{trace}(B'(0)).
\]
2. THE LAPLACE OPERATOR IN VECTOR BUNDLES

where $W : \mathbb{R}^m \to \mathbb{R}$ is the unique linear function given by the condition that $W$ is alternating and $W(e_1, \ldots, e_m) = 1$. We then have

$$
\frac{d}{dt}(\det(B(t)))|_{t=0} = \sum_{k=1}^{m} W(B(0)e_1, \ldots, B'_{k}(0)e_k, \ldots, B(0)e_m)
$$

$$
= \text{trace}(B'(0)).
$$

2. The Laplace Operator in Vector Bundles

Let $(E, M, \pi)$ be a vector bundle with connection $\nabla^E$. Then $(\text{Hom}(TM, E), M, \pi)$ is defined as the vector bundle which fiber at each point $p \in M$ consists of the space of homomorphisms from $T_pM$ to $E_p$. We state the following well know propositions and leave out the proofs which are straightforward calculations.

**Proposition 4.3.** Let $(E, M, \pi)$ be a vector bundle with connection $\nabla^E$. The connection $\nabla^E$ on $E$ induces a connection $\nabla^H : C^\infty(TM) \times C^\infty(\text{Hom}(TM, E)) \to C^\infty(\text{Hom}(TM, E))$ on $\text{Hom}(TM, E)$ defined by

$$
\nabla^H_X \varphi : Y \mapsto \nabla^E_X(\varphi(Y)) - \varphi(\nabla_X Y),
$$

where $\varphi \in \text{C}^\infty(\text{Hom}(TM, E))$ and $X, Y \in C^\infty(TM)$.

**Proposition 4.4.** Let $E$ be a Riemannian vector bundle with metric $h$ and $\text{Hom}(TM, E)$ the corresponding homomorphism bundle then $h$ induces a metric $H$ on $\text{Hom}(TM, E)$ given by

$$
H(\varphi, \varphi) = \sum_i h(\varphi(e_i), \varphi(e_i)),
$$

where $\varphi, \varphi \in \text{C}^\infty(\text{Hom}(TM, E))$ and $\{e_i\}$ is a local orthogonal frame for the tangent bundle $TM$ of $M$.

For a section $\psi \in \text{C}^\infty(E)$ of the vector bundle $E$ we have the section $(\nabla^E \psi) \in \text{C}^\infty(\text{Hom}(TM, E))$ given by

$$
\nabla^E \psi : X \mapsto \nabla^E_X \psi
$$

for $X \in \text{C}^\infty(TM)$.

With use of the connection $\nabla^H$ on the homeomorphism bundle $\text{Hom}(TM, E)$ we define the mapping

$$
\nabla^H_{X,Y} : \text{C}^\infty(TM) \times \text{C}^\infty(TM) \times \text{C}^\infty(E) \to \text{C}^\infty(E)
$$

by

$$
\nabla^H_{X,Y} \psi = (\nabla^H_X(\nabla^E \psi))(Y).
$$
Fixing a $\psi \in C^\infty(E)$, the mapping

$$(X, Y) \mapsto \nabla^H_{X,Y} \psi$$

is a $C^\infty(E)$-valued bilinear form.

**Definition 4.5.** Let $(M, g)$ be a Riemannian manifold and $(E, M, \pi)$ be a Riemannian vector bundle over $M$ with connection $\nabla^E$. We define the Laplacian of the section $\psi \in C^\infty(E)$, denoted by $\Delta^E \psi$, by

$$\Delta^E \psi = \text{trace} \left( (X, Y) \mapsto \nabla^H_{X,Y} \psi \right)$$

where $\nabla^H$ is the induced connection on the homeomorphism bundle $\text{Hom}(TM, E)$ and $\nabla^H_{X,Y} \psi = (\nabla^H_X(\nabla^E \psi))(Y)$.

**Proposition 4.6.** Let $(M, g)$ be a Riemannian manifold and $(E, M, \pi)$ be a Riemannian vector bundle over $M$ with connection $\nabla^E$. Then given a local orthogonal frame for the tangent bundle $\{e_i\}$ such that $(\nabla e_i e_j)_p = 0$ we have

$$(\Delta^E \psi)_p = \sum_i (\nabla^E_{e_i}(\nabla^E \psi))_p$$

for any $\psi \in C^\infty(E)$

**Proof.**

$$(\Delta^E \psi)_p = \sum_i (\nabla^H_{e_i}(\nabla^E \psi)(e_i))_p$$

$$= \sum_i (\nabla^E_{e_i}(\nabla^E \psi)(e_i))_p - \sum_i (\nabla^E \psi(e_i e_i))_p.$$ 

Here the second term vanishes since $(\nabla e_i e_j)_p = 0$ and we get

$$(\Delta^E \psi)_p = \sum_i (\nabla^E_{e_i}(\nabla^E \psi))_p.$$ 

**Proposition 4.7.** Let $(M, g)$ be a compact Riemannian manifold without boundary and $(E, M, \pi)$ be a Riemannian vector bundle with metric $h$ and connection $\nabla^E$. Then the Laplacian $\Delta^E$ satisfies

$$\int_M h(\Delta^E \psi, \varphi) dV_M = - \int_M H(\nabla^E \psi, \nabla^E \varphi) dV_M = \int_M b(\psi, \Delta^E \varphi) dV_M,$$

where $\psi, \varphi \in C^\infty(E)$ are sections of the vector bundle $E$.
3. The Euler-Lagrange Equation

PROOF. By Proposition (4.6) we have

\[ b(\Delta^E \psi, \varphi) = \sum_i b(\nabla^E_{e_i}(\nabla^E \psi), \varphi) \]
\[ = \sum_i e_i(b(\nabla^E_{e_i} \psi, \varphi)) - \sum_i b(\nabla_{e_i} \psi, \nabla_{e_i} \varphi) \]
\[ = \text{div}(X) - H(\nabla^E \psi, \nabla^E \varphi), \]

where \( X \in C^\infty(TM) \) is defined by

\[ X = \sum_i b(\nabla^E_{e_i} \psi, \varphi)e_i \]

and div(X) is the divergence of the \( X \) and \( H \) is the metric on \( \text{Hom}(TM, E) \). Since \( M \) is compact and without boundary we have by Green’s formula in [12]

\[ \int_M b(\Delta^E \psi, \varphi)dV_M = -\int_M H(\nabla^E \psi, \nabla^E \varphi)dV_M. \]

Let \((SM, M, \pi)\) be the sub-bundle of \( \text{Hom}(TM, TM) \) which fiber at each point \( p \in M \) consists of the vector space of symmetric endomorphisms. Let \( A : NM \to SM \) be the bundle homomorphism given by

\[ A : \varphi \mapsto A^p \]

for \( \varphi \in C^\infty(NM) \). Since we have metrics on \( SM \) and \( NM \) we can define the transpose ‘\( A \) of \( A \). Then ‘\( A : SM \to NM \) is given by

\[ \hat{g}(\text{‘}A(s), \varphi) = H(s, A(\varphi)) \]

for \( s \in C^\infty(SM) \) and \( \varphi \in C^\infty(NM) \). We define \( \tilde{A} : NM \to NM \) by

\[ \tilde{A} = \text{‘}A \circ A. \]

3. The Euler-Lagrange Equation

Let \( f : \tilde{M} \to M \) be an immersion and \( F : \tilde{M} \times I \to M \) be a variation of \( f \). It follows from Theorem (2.5) that the pull-back bundle

\((F^*(TM), \tilde{M} \times I, \pi)\)

over \( \tilde{M} \times I \) is a Riemannian vector bundle with the pull-back metric \( g^* \) and pull-back connection \( \nabla^* \) induced by \( F \). According to Proposition (2.3) we have

\[ \nabla^*_X(dF\tilde{Y})^* - \nabla^*_Y(dF\tilde{X})^* = (dF[X, \tilde{Y}])^* \]
where \( \bar{X}, \bar{Y} \in C^\infty(T(\bar{M} \times I)) \). If \( R \) is the curvature tensor on \( M \) we have

\[ (R(dF(\bar{X}), dF(\bar{Y})) \varphi^*) = \nabla^*_{\bar{X}} \nabla^*_{\bar{Y}} \varphi - \nabla^*_{\bar{Y}} \nabla^*_{\bar{X}} \varphi - \nabla^*_{[\bar{X}, \bar{Y}]} \varphi \]

where \( \bar{X}, \bar{Y} \in C^\infty(T(\bar{M} \times I)) \) and \( \varphi \in C^\infty F^*(M) \). The pull-back bundle \( F^*(TM) \) splits into the direct sum

\[ F^*(TM) = T \oplus N \]

where \( T_{\bar{p}, \bar{t}} = dF_{\bar{p}}(T_{\bar{p}}\bar{M}) \) and \( N_{\bar{p}, \bar{t}} \) is the orthogonal complement with respect to \( \bar{g}^* \). We have the connection \( \bar{\nabla} \) on \( T \) and \( D \) on \( N \) by taking the projection of \( \nabla^* \) onto \( T \) and \( N \), respectively. Also we have the metric \( \bar{g} \) on \( T \) and \( \hat{g} \) on \( N \) simply by restricting \( g^* \).

We want to characterize the critical point of the Willmore functional \( W \) and therefore we compute the Euler-Lagrange equation for \( W \). Following the method used by Weiner in [13] we consider

\[ \frac{\partial}{\partial t} W(F_t)|_{t=0} = \frac{\partial}{\partial t} \int_\Sigma (H^2 + \bar{K}) dA|_{t=0} \]

where \( \Sigma = \bar{M} \) is a surface and \( M \) is m-dimensional Riemannian manifold. First we compute \( \partial / \partial t (H^2) \). Let \( \nu = \partial / \partial t \) then we have

\[ \nu(W(f_t))|_{t=0} = \int_\Sigma \nu((H^2 + \bar{K}) dA)|_{t=0} \]

\[ = \int_\Sigma (\nu(H^2 + \bar{K}) dA|_{t=0} + (H^2 + \bar{K}) \nu(dA))|_{t=0} \]

Let \( V \in C^\infty(f^*(TM)) \) be let be given by

\[ V(p) = df(\nu)(p). \]

and let \( H \in C^\infty(N) \) be the mean curvature of \( \bar{M} \) at \( p \). We have

\[ \nu(H^2) = 2\bar{g}(D_\nu H, H). \]

Let \( E_i \) where \( i \in (1, 2) \) and \( E_j \) where \( j \in (3, \ldots, m) \) be frames at \( (p, 0) \) in \( T \) and \( N \) respectively. Parallel transport \( E_i \) and \( E_j \) along \( \{(p, t) - \varepsilon < t < \varepsilon \} \) in \( T \) and \( N \). That is \( \bar{\nabla}_t E_i = 0 \) and \( D_\nu E_j = 0 \) at \( (p, t) \). Further parallel transport \( E_i \) in \( T \) and \( E_j \) in \( N \) along all geodesics through \( p \) in \( M \). Since \( E_i \) is tangential for each \( i \in (1, 2) \), we can find local vector fields \( \{e_1, e_2\} \) on \( \bar{M} \times I \) such that \( dF_{\bar{p}}(e_i) = E_i \). By the definition of the mean curvature \( H \) we have

\[ H = \frac{1}{2} \sum_{i=1}^2 D_\nu E_i \]

thus

\[ D_\nu H = \frac{1}{2} \sum_i D_\nu D_\nu E_i = \frac{1}{2} \sum_i D_\nu \nabla^*_\nu E_i. \]
We have from (10)

\[
\frac{1}{2} \sum_i (R(V, E_i)E_i)^* = \frac{1}{2} \left( \sum_i \nabla_{\varepsilon_i}^* \nabla_{\varepsilon_i}^* E_i - \nabla_{\varepsilon_i}^* \nabla_{\varepsilon_i}^* E_i + \nabla_{[v, \varepsilon_i]}^* E_i \right)
\]

and hence

(11) \quad \frac{1}{2} \left( \sum_i (R(V, E_i)E_i)^* \right) = \frac{1}{2} \sum_i D_\varepsilon \nabla_{\varepsilon_i}^* E_i - D_\varepsilon \nabla_{\varepsilon_i}^* E_i + D_{[v, \varepsilon_i]} E_i.

So from (9) and (11) we have

\[
D_\varepsilon \mathcal{H} = \frac{1}{2} \left\{ \left( \sum_i (R(V, E_i)E_i)^* \right) + D_\varepsilon \nabla_{\varepsilon_i}^* V \right\}
\]

Since \([v, \varepsilon_i]\) is tangential we have by definition of the second fundamental form \(B\)

\(D_{[v, \varepsilon_i]} E_i = B([v, \varepsilon_i], \varepsilon_i) = D_\varepsilon dF[v, \varepsilon_i] = -D_\varepsilon \nabla_{\varepsilon_i} V\)

where the last equality come from (9) and the fact that \(\nabla_{\partial_i \partial_j} E_i = 0\) at \((p, t)\).

We obtain

\[
D_\varepsilon \mathcal{H} = \frac{1}{2} \left\{ \left( \sum_i (R(V^\perp, E_i)E_i)^* \right) + D_\varepsilon \nabla_{\varepsilon_i}^* V - 2D_\varepsilon \nabla_{\varepsilon_i} V \right\}
\]

Then

\[
D_\varepsilon \mathcal{H} = \frac{1}{2} \left\{ \left( \sum_i (R(V^\perp, E_i)E_i)^* \right) + D_\varepsilon dF[V, \varepsilon_i] - D_\varepsilon \nabla_{\varepsilon_i} V \right\}
\]

\[
= \frac{1}{2} \left\{ \left( \sum_i (R(V^\perp, E_i)E_i)^* \right) + \Delta^N V - \sum_i B(\varepsilon_i, \nabla_{\varepsilon_i} V) \right\}
\]

\[
+ \sum_i (D_\varepsilon B(\varepsilon_i, V^\perp) - D_\varepsilon \nabla_{\varepsilon_i} V^\perp) \}
\]
\(\frac{1}{2}\{ (\sum_i (R(V^\perp, E_i)E_i)^\perp + \Delta^N V^\perp - \sum_i B(e_i, A^V(e_i)) + (\sum_i (R(V^T, E_i)E_i)^\perp + \sum_i (D_{\alpha_i} B(e_i, V^T) - D_{\alpha_i} \nabla_{e_i} V^T)\}.\)

We notice
\[
\sum_{i=1}^2 B(e_i, A^{V^\perp}(e_i)) = \sum_{j=3}^m \sum_{i=1}^2 \hat{g}(B(e_i, A^{V^\perp}(e_i)), E_j) E_j
\]
\[
= \sum_{j=3}^m \sum_{i} \hat{g}(A^{E_i}(e_i), A^{V^\perp}(e_i)) E_j
\]
\[
= \sum_{j=3}^m H(A^{E_i}, A^{V^\perp}) E_j
\]
\[
= \sum_{j=3}^m \hat{g}(A(A^{V^\perp}), E_j) E_j
\]
\[
= ^t A \circ A(V^\perp) = \tilde{A}(V^\perp)
\]
and at \((p, 0)\)
\[
(R(V^T, E_i)E_i)^\perp + D_{\alpha_i} B(e_i, V^T) - D_{\alpha_i} \nabla_{e_i} V^T
\]
\[
= (-R(E_i, V^T)E_i)^\perp + D_{\alpha_i} D_{V^T} E_i - D_{\alpha_i} \nabla_{V^T} E_i - D_{[\alpha_i, V^T]} E_i
\]
\[
= -D_{\alpha_i} \nabla_{V^T} E_i + D_{V^T} \nabla^*_{e_i} E_i + D_{[\alpha_i, V^T]} E_i
\]
\[
+ D_{\alpha_i} D_{V^T} E_i - D_{\alpha_i} \tilde{\nabla}_{V^T} E_i - D_{[\alpha_i, V^T]} E_i
\]
\[
= D_{V^T} D_{\alpha_i} E_i.
\]
So we find
\[
D_{\alpha_i} = \frac{1}{2}\{ (\sum_{i=1}^2 R(V^\perp, E_i)E_i)^\perp + \Delta^N V^\perp + \tilde{A}(V^\perp)\} + D_{V^T} \mathbb{H}.
\]

Restricted to the case when \(\Sigma\) is a surface without boundary and \(M\) has constant constant sectional curvature \(\tilde{K}\) we have at \((p, 0)\)
\[
v((H^2 + \tilde{K})dA)
\]
\[
v(H^2) \cdot dA + (H^2 + \tilde{K}) \cdot v(dA)
\]
\[
= 2\hat{g}(D_{\partial_i}[\partial, \mathbb{H}, \mathbb{H}) \cdot dA + (H^2 + \tilde{K}) \cdot \frac{\partial}{\partial t} (dA)
\]
\[
= \hat{g}(2\tilde{K} V^\perp + \Delta^N V^\perp + \tilde{A}(V^\perp) + D_{V^T} \mathbb{H}, \mathbb{H}) \cdot dA
\]
\[
+ (H^2 + \tilde{K})(-2\hat{g}(V^\perp, \mathbb{H})) \cdot dA + (H^2 + \tilde{K}) \cdot \text{div}(V^T) \cdot dA
\]
\[
\begin{align*}
\tilde{A} &= \hat{g}(\Delta^N H \nu - 2H^2 \mathbb{H} + \tilde{A}(\nu^\perp), \mathbb{H}) \cdot dA \\
& \quad + V^\top H^2 \cdot dA + (H^2 + \tilde{K}) \cdot \text{div}(V^\top) \cdot dA.
\end{align*}
\]

Since \(\tilde{A}\) is symmetric with respect to \(\hat{g}\), \(\Sigma\) has no boundary and
\[
V^\top H^2 \cdot dA + (H^2 + \tilde{K}) \cdot \text{div}(V^\top) \cdot dA = \text{div}((H^2 + \tilde{K}) V^\top) \cdot dA
\]
we have by Proposition (4.7) and Green’s theorem
\[
v\{ \int_{\Sigma} (H^2 + \tilde{K}) dA(t) \}_{t=0} = \int_{\Sigma} \hat{g}(\Delta^N H - 2H^2 \mathbb{H} + \tilde{A}(\mathbb{H}), \nu^\perp) dA.
\]

Hence we can conclude:

**Theorem 4.8.** [13] Let \(f : \Sigma \to (M, g)\) be a smooth immersion from a compact orientable surface \(\Sigma\) without boundary to a Riemannian manifold \((M, g)\) of constant sectional curvature \(\tilde{K}\). Then \(f\) is a stationary point of the Willmore functional \(W\) if and only if

\[
\Delta^N H - 2H^2 \mathbb{H} + \tilde{A}(\mathbb{H}) = 0.
\]

Sometimes the condition that a surface satisfies (12) is taken as the definition of a Willmore surface.

![Figure 1. Example of a Willmore surface in \(\mathbb{R}^3\) obtained at [5].](image)

In the case when \(M\) is a 3-dimensional manifold we have the following corollary.

**Corollary 4.9.** Let \((M^3, g)\) be a 3-dimensional Riemannian manifold of constant sectional curvature \(\tilde{K}\) and \(\Sigma\) be a compact orientable surface without boundary. Then a smooth immersion \(f : \Sigma \to M\) is a stationary point of the Willmore functional \(W(f)\) if and only if

\[
\Delta H + 2H(H^2 - G) = 0,
\]

where \(G\) is the determinant of the second fundamental form \(B\).
PROOF. The normal space is spanned by a vector \( \gamma \) and we have \( \mathbb{H} = H \cdot \gamma \) and \( \Delta^N \mathbb{H} = \Delta H \cdot \gamma \). Let \( \{e_1, e_2\} \) be a local orthonormal frame for the tangent bundle \( T\Sigma \). We then have

\[
\widetilde{A}(\mathbb{H}) = \sum_i B(e_i, A^{\gamma^+}(e_i))
\]

\[
= \sum_i B(e_i, \sum_j \tilde{g}(A^{\mathbb{H}}(e_i), e_j), e_j)
\]

\[
= \sum_{ij} \tilde{g}(A^{\mathbb{H}}(e_i), e_j) B(e_i, e_j)
\]

\[
= \sum_{ij} \tilde{g}(B(e_i, e_j), \mathbb{H}) B(e_i, e_j)
\]

\[
= \sum_{ij} H(B(e_i, e_j)^2) \cdot \gamma
\]

\[
= H(B(e_1, e_1)^2 + 2B(e_1, e_2)^2 + B(e_2, e_2)^2)
\]

\[
= H((2H)^2 + 2(|B(e_1, e_2)|^2 - |B(e_1, e_1)| B(e_2, e_2))
\]

\[
= (4H^3 - 2GH) \cdot \gamma
\]

and the theorem follows. \( \square \)

**Theorem 4.10.** Let \( \Sigma \) be a minimal surface in \( S^3 \) and further let \( \sigma : S^3 \setminus \{n\} \to \mathbb{R}^3 \) be the stereo graphic projection of \( S^3 \) into \( \mathbb{R}^3 \) from the north pole. Then the image \( \sigma(\Sigma) \) is a Willmore surface in \( E^3 \).

**Proof.** By Corollary (4.9) a minimal surface in \( S^3 \) is also a Willmore surface \( S^3 \). By Theorem (3.7) we know that \( W \) is invariant under conformal transformation of the metric in \( E^3 \). Hence taking the stereo graphic projection from \( S^3 \) to \( E^3 \), which is a conformal transformation, yields a Willmore surface in \( E^3 \). \( \square \)

**Corollary 4.11.** Let \( g \) be a natural number, then there exist an embedded compact Willmore surface \( \Sigma \) in \( E^3 \) of genus \( g \).

**Proof.** By a result of Lawson in [7] we know there exists compact embedded minimal surfaces in \( S^3 \) of arbitrary genus. Hence by Theorem (4.10) taking the stereo graphic projection of these surfaces gives the desired surfaces in \( E^3 \). \( \square \)

In the article [7] from 1985 U.Pinkall has shown that there exists Willmore surfaces in \( E^3 \) not obtainable by stereo graphic projection of minimal surfaces in \( S^3 \).
Figure 2. A Willmore surface of positive genus obtained at [5].
Bibliography
