SURFACES OF CONSTANT MEAN CURVATURE

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Abstract

The aim of this Master's dissertation is to give a survey of some basic results regarding surfaces Σ of constant mean curvature (CMC) in $\mathbb{R}^3$. Such surfaces are often called soap bubbles since a soap film in equilibrium between two regions is characterized by having constant mean curvature. The surface area of these surfaces is critical under volume-preserving deformations.

CMC surfaces may also be characterized by the fact that their Gauss map $N\colon \Sigma \to S^2$ is harmonic i.e. it satisfies

$$\tau(N) = 0,$$

where $\tau(N)$ is the tension field of N, generalizing the classical Laplacian. This is a non-linear system of partial differential equations. It was proved in the 1990s that this system has global solutions on compact surfaces of any genus $g \geq 0$.

In this dissertation we study necessary and sufficient conditions for a surface to have CMC. We study the minimal case (characterized by mean curvature $H \equiv 0$) and the well-known Weierstrass representation for such surfaces. Also CMC surfaces with rotational symmetry are considered and a generalization of the Weierstrass representation to surfaces of non-zero constant mean curvature is presented. Finally we show that the only compact embedded CMC surfaces in $\mathbb{R}^3$ are spheres.

It has been my intention throughout this work to give references to the stated results and credit to the work of others. The only part of this Master's dissertation which I claim is my own is the elementary proof of a special case of Ruh-Vilms theorem for surfaces in $\mathbb{R}^3$ given in Theorem 4.1.

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Contents

Short History

In 1841 Delaunay characterized in [**1**] a class of surfaces in Euclidean space which he described explicitly as surfaces of revolution of roulettes of the conics. These surfaces are the *catenoids*, *unduloids*, *nodoids* and *right circular cylinders*. Today they are known as the *surfaces of Delaunay* and are the first non-trivial examples of surfaces having constant mean curvature, the sphere being the trivial case.

In an appended note to Delaunay's paper M. Sturm characterized these surfaces variationally as the extremals of surfaces of rotation having fixed volume while maximizing lateral area. Using this characterization the following theorem was obtained:

THEOREM (Delaunay's theorem). *The complete immersed surfaces of revolution in $\mathbb{R}^3$ having constant mean curvature are exactly those obtained by rotating about their axis the roulettes of the conics.*

These surfaces where also recognized by Plateau using soap film experiments. In 1853 J. H. Jellet showed in [**2**] that if Σ is a compact star-shaped surface in $\mathbb{R}^3$ having constant mean curvature then it is the standard sphere. A hundred years later, in 1956, H. Hopf conjectured that this, in fact, holds for all compact immersions:

CONJECTURE (Hopf's conjecture). *Let Σ be an immersion of an oriented, compact hypersurface with constant mean curvature $H \neq 0$ in $\mathbb{R}^n$. Then Σ must be the standard embedded $(n-1)$-sphere.*

Hopf proved the conjecture in [**3**] for the case of immersions of S^2 into $\mathbb{R}^3$ having constant mean curvature and a few years later A. D. Alexandrov showed the conjecture to hold for any *embedded* hypersurface in $\mathbb{R}^n$, see [**4**]. It was widely believed that this conjecture was true until 1982 when Wu-Yi Hsiang constructed a counterexample in $\mathbb{R}^4$. Two years later Wente constructed in [**5**] an immersion of the torus T^2 in $\mathbb{R}^3$ having constant mean curvature.

Wente's construction has been thoroughly studied but has only been able to create surfaces having genus $g = 1$. A different method for constructing surfaces in $\mathbb{R}^3$ having constant mean curvature of any genus $g \geq 3$ was presented in 1987 by N. Kapouleas [**6**]. A proof of the fact that there exist CMC-immersions of compact surfaces of any genus was published in [**8**] in 1995 by the same author.

CHAPTER 1

Some basic surface theory

In this chapter we introduce the notation to be used in this text. We also introduce some basic results concerning isothermal coordinates and the tension field of the Gauss map of a surface in $\mathbb{R}^3$.

1.1. Notation

DEFINITION 1.1. A non-empty subset Σ on $\mathbb{R}^3$ is said to be a *regular surface* if for each point $p \in \Sigma$ there exists an open neighborhood U in Σ around p and a bijective map $\varphi = (x, y) \colon U \subset \Sigma \to \mathbb{R}^2$ such that its inverse $X \colon \varphi(U) \to U$

i. is a homeomorphism,
ii. is a differentiable map,
iii. $(X_x \times X_y)(q) \neq 0$ for all $q \in \varphi(U)$.

The functions x, y are called *local coordinates* around p. The map X is called a *local parametrization* of Σ around p.

Let Σ be a regular surface in $\mathbb{R}^3$ and $p \in \Sigma$ be an arbitrary point. By a *tangent vector* to Σ , at the point p, we mean the tangent vector $\alpha'(0)$ of a differentiable parametrized curve $\alpha\colon (-\varepsilon, \varepsilon) \to \Sigma$ with $\alpha(0) = p$. The set of tangent vectors of Σ at a point $p \in \Sigma$ is called the *tangent space* of Σ at $p \in \Sigma$ and is denoted by $T_p\Sigma$. A local parametrization X determines a basis

$$\left\{X_x, X_y\right\}$$

of $T_p\Sigma$, called the basis associated with X.

On the tangent plane we have the usual induced metric from the ambient space $\mathbb{R}^3$ with the associated quadratic form $I_p \colon T_p\Sigma \to \mathbb{R}$ called the *first fundamental form* of Σ at $p \in \Sigma$. Given a local parametrization X of Σ and a parametrized curve $\alpha(t) = X(x(t), y(t))$ for $t \in (-\varepsilon, \varepsilon)$ with $p = \alpha(0)$ we have the following form

$$\begin{aligned} I_p(\alpha'(0)) &= \left\langle X_x x' + X_y y', X_x x' + X_y y' \right\rangle \\ &= \left\langle X_x, X_x \right\rangle_p (x')^2 + 2\left\langle X_x, X_y \right\rangle_p x'y' + \left\langle X_y, X_y \right\rangle_p (y')^2 \\ &= E(x')^2 + 2Fx'y' + G(y')^2, \end{aligned} \tag{1.1}$$

where the values of E, F and G are computed for $t = 0$.

By condition (iii) of the definition of a regular surface (1.1) we have, given a local parametrization X of a surface Σ in $\mathbb{R}^3$ at a point $p \in \Sigma$ that the map

$N\colon \Sigma \to S^2$ defined by

$$N(p) = \frac{X_x \times X_y}{\left|X_x \times X_y\right|}(p) \tag{1.2}$$

is well defined. This map is known as the *Gauss map* of Σ. The quadratic form II_p defined in $T_p\Sigma$ by

$$II_p(v) = -\left\langle dN_p(v), v\right\rangle$$

is called the *second fundamental form* of Σ at p. Given a local parametrization X on Σ at a point $p \in \Sigma$ and, as above, letting α be a parametrized curve such that $\alpha(t) = X(x(t), y(t))$ for $t \in (-\varepsilon, \varepsilon)$ with $p = \alpha(0) = X(x(0), y(0))$, we get

$$dN_p(\alpha') = N'(x(t), y(t)) = N_x x' + N_y y'$$

Since $\langle N, N\rangle = 1$ we must have that $N_x, N_y \in T_p\Sigma$ and hence

$$\begin{aligned} N_x &= a_{11}X_x + a_{21}X_y, \\ N_y &= a_{12}X_x + a_{22}X_y, \end{aligned} \tag{1.3}$$

for some functions a_{ij}. We find that

$$\begin{aligned} II_p(\alpha') &= -\left\langle dN_p(\alpha'), \alpha'\right\rangle = -\left\langle N_x x' + N_y y', X_x x' + X_v x'\right\rangle \\ &= e(x')^2 + 2fx'y' + g(y')^2, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} e &= -\left\langle N_x, X_x\right\rangle = \left\langle N, X_{xx}\right\rangle, \\ f &= -\left\langle N_y, X_x\right\rangle = \left\langle N, X_{xy}\right\rangle = \left\langle N, X_{yx}\right\rangle = -\left\langle N_x, X_y\right\rangle, \\ g &= -\left\langle N_y, X_y\right\rangle = \left\langle N, X_{yy}\right\rangle. \end{aligned}$$

Using the terms from equations (1.1), (1.3) and (1.4) we arrive at

$$\begin{aligned} a_{11} &= \frac{fF - eG}{EG - F^2}, & a_{12} &= \frac{gF - fG}{EG - F^2}, \\ a_{21} &= \frac{eF - fE}{EG - F^2}, & a_{22} &= \frac{fF - gE}{EG - F^2}, \end{aligned}$$

known as the *Weingarten equations*.

Continuing by using that $\{X_x, X_y\}$ is a basis for $T_p\Sigma$ and that N is orthogonal to both X_x and X_y we have that $\{X_x, X_y, N\}$ is a basis for $\mathbb{R}^3$. We find that

$$\begin{aligned} X_{xx} &= \Gamma_{11}^1 X_x + \Gamma_{11}^2 X_y + eN, \\ X_{yx} &= \Gamma_{12}^1 X_x + \Gamma_{12}^2 X_y + fN, \\ X_{xy} &= \Gamma_{21}^1 X_x + \Gamma_{21}^2 X_y + fN, \\ X_{yy} &= \Gamma_{22}^1 X_x + \Gamma_{22}^2 X_y + gN. \end{aligned} \tag{1.5}$$

The Γ_{ij}^k are known as the *Christoffel symbols* and are invariant under isometries (i.e. can be computed from the first fundamental form alone). Using that

$$(X_{xx})_y - (X_{xy})_x = 0,$$

$$(X_{yy})_x - (X_{xy})_y = 0,$$
$$N_{xy} - N_{yx} = 0,$$

we find that

$$\begin{aligned} e_y - f_x &= e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2, \\ f_y - g_x &= e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2. \end{aligned} \tag{1.6}$$

These equalities are known as the *Mainardi-Codazzi equations*.

DEFINITION 1.2. Let Σ be a surface in $\mathbb{R}^3$ and $p \in \Sigma$ an arbitrary point. Let $dN_p\colon T_p\Sigma \to T_p\Sigma$ be the differential of the Gauss map. Then the determinant of dN_p is called the *Gaussian curvature* K of Σ at p. The negative half of the trace of dN_p is called the *mean curvature* H of Σ at p.

In terms of the first and second fundamental forms K and H are given by

$$K = \frac{eg - f^2}{EG - F^2}, \tag{1.7}$$

$$H = \frac{eG - 2fF + gE}{2(EG - F^2)}. \tag{1.8}$$

1.2. Isothermal coordinates

In this section we introduce the notion of isothermal coordinates which is a useful tool in differential geometry.

DEFINITION 1.3. Let Σ be a surface in $\mathbb{R}^3$. Then local coordinates $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ on Σ are said to be *isothermal* if there exists a strictly positive function, called the *dilation*, $\lambda\colon U \subset \Sigma \to \mathbb{R}$ such that

$$E = \langle X_x, X_x \rangle = \lambda^2 = \langle X_y, X_y \rangle = G, \quad F = 0.$$

We have the following result regarding existence of isothermal coordinates on an arbitrary surface in $\mathbb{R}^2$.

THEOREM 1.4. *Let Σ be a differentiable surface in $\mathbb{R}^3$ and $p \in \Sigma$ be a point on Σ. Then there exists an open neighborhood U of p and isothermal coordinates $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ around p.*

This was proved in the analytic case by Gauss. For a complete proof in the general case please see [**9**]. Having chosen isothermal coordinates the mean curvature simplifies

$$H = \frac{eG + gE - 2fF}{2(EG - F^2)} = \frac{e + g}{2\lambda^2}.$$

The Christoffel symbols similarly simplify

$$\begin{aligned} \Gamma_{11}^1 = \Gamma_{12}^2 = -\Gamma_{22}^1 &= \frac{1}{2\lambda^2}\frac{\partial \lambda^2}{\partial x}, \\ -\Gamma_{11}^2 = \Gamma_{12}^1 = \Gamma_{22}^2 &= \frac{1}{2\lambda^2}\frac{\partial \lambda^2}{\partial y}. \end{aligned} \tag{1.9}$$

Using this we get the following form of the Mainardi-Codazzi equations (1.6):

$$\begin{aligned} e_y - f_x &= (e+g)\Gamma^2_{22}, \\ g_x - f_y &= (e+g)\Gamma^1_{11}. \end{aligned} \tag{1.10}$$

The Weingarten relations (eq. 1.3) reduce to

$$a_{11} = -\frac{e}{\lambda^2}, \quad a_{21} = a_{12} = -\frac{f}{\lambda^2}, \quad a_{22} = -\frac{g}{\lambda^2}. \tag{1.11}$$

Suppose Σ is a surface in $\mathbb{R}^3$ and $\varphi = (x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ are local isothermal coordinates on Σ. We may then consider the local coordinates (x, y) as a complex-valued map $z = x + iy\colon U \subset \Sigma \to \mathbb{C}$. The inverse $X\colon z(U) \subset \mathbb{C} \to \Sigma$ can then be considered as map from an open subset $z(U)$ in $\mathbb{C}$ into Σ i.e. a local parametrization of Σ. We then have

$$X_z = \frac{1}{2}\left(X_x - iX_y\right).$$

The complex notation for surfaces in $\mathbb{R}^3$ has many advantages which we will be useful in chapters 2 and 3.

Letting $\langle\cdot,\cdot\rangle$ be the usual inner product in $\mathbb{R}^3$ and let $(\cdot,\cdot)$ be the complex bilinear extension of $\langle\cdot,\cdot\rangle$ in $\mathbb{C}^3$ we have the following result.

PROPOSITION 1.5. *Let Σ be a surface in $\mathbb{R}^3$ and let $z = x+iy\colon U \subset \Sigma \to \mathbb{C}$ be local isothermal coordinates on Σ. Then the inverse $X\colon z(U) \subset \mathbb{C} \to \Sigma$ of z is conformal i.e. satisfies*

$$\begin{aligned} 4\left(X_z, X_z\right) &= \left|X_x\right|^2 - \left|X_y\right|^2 - 2i\left\langle X_x, X_y\right\rangle = 0, \\ 4\left(X_{\bar z}, X_{\bar z}\right) &= \left|X_x\right|^2 - \left|X_y\right|^2 + 2i\left\langle X_x, X_y\right\rangle = 0. \end{aligned} \tag{1.12}$$

Conversely, if $z = x+iy$ are local coordinates on Σ satisfying equation (1.12) then they are isothermal.

PROOF. The first statement follows by a direct computation. The reverse implication follows by considering real and imaginary parts of equation (1.12). □

PROPOSITION 1.6. *Let Σ be a surface in $\mathbb{R}^3$ and $z = x+iy\colon U \subset \Sigma \to \mathbb{C}$ be local isothermal coordinates on Σ with dilation λ. Then the inverse $X\colon \varphi(U) \subset \mathbb{C} \to \Sigma$ of z satisfies*

$$4X_{\bar z z} = X_{xx} + X_{yy} = 2\lambda^2 HN,$$

where $N\colon \Sigma \to S^2$ is the Gauss map of Σ.

PROOF. By a direct computation using the differentiated form of equation (1.12) we have

$$\begin{aligned} 4X_{\bar z z} &= \frac{2}{\lambda^2}\left[\left(X_{\bar z z}, X_z\right)X_z + \left(X_{\bar z z}, X_{\bar z}\right)X_{\bar z}\right] + 4\left(X_{\bar z z}, N\right)N \\ &= \left\langle X_{xx} + X_{yy}, N\right\rangle N = 4(e+g)N \\ &= 2\lambda^2 HN \end{aligned}$$

This immediately gives our sought result. □

1.3. The tension field

In this section we give an explicit formula for the tension field (see appendix A) for maps from a surface in $\mathbb{R}^3$ into S^2 in terms of local isothermal coordinates.

PROPOSITION 1.7. *Let Σ be a surface in $\mathbb{R}^3$ and $\varphi\colon \Sigma \to S^2$ be a map into the unit sphere S^2 in $\mathbb{R}^3$. If $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ are local isothermal coordinates on Σ then the tension field $\tau(\varphi)$ of φ is locally given by*

$$\tau(\varphi) = \frac{1}{\lambda^2}\left(\Delta\varphi\right)^{\mathsf{T}}$$

i.e. as the tangential part of the classical Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in $\mathbb{R}^2$.

PROOF. By the definition of the tension field of a smooth map $\varphi\colon \Sigma \to S^2$ we have

$$\tau(\varphi) = \sum_{k=1}^{2}\left(\nabla^*_{e_k} d\varphi(e_k) - d\varphi(\nabla_{e_k} e_k)\right),$$

where ∇^* is the pull-back connection on the pull-back bundle $\varphi^{-1}TS^2$ over Σ via φ.

Let $p \in \Sigma$ be an arbitrary point and $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ be isothermal coordinates around p. We then have

$$\begin{aligned}\nabla_{e_k} e_k &= \partial_{e_k} e_k + \frac{\lambda^2}{2}\left(2e_k(\frac{1}{\lambda^2})e_k - g(e_k, e_k)\,\mathrm{grad}\,\frac{1}{\lambda^2}\right)\\ &= \partial_{e_k} e_k + \frac{\lambda^2}{2}\left(2e_k(\frac{1}{\lambda^2})e_k - \mathrm{grad}\,\frac{1}{\lambda^2}\right)\end{aligned}$$

for $k = 1, 2$. Then using the definition of the gradient grad we obtain

$$\begin{aligned}\nabla_{e_1} e_1 + \nabla_{e_2} e_2 &= \partial_{e_1} e_1 + \partial_{e_2} e_2 + \lambda^2\left(e_1(\frac{1}{\lambda^2})e_1 + e_2(\frac{1}{\lambda^2})e_2 - \mathrm{grad}\,\frac{1}{\lambda^2}\right)\\ &= \nabla_{\frac{1}{\lambda}\frac{\partial}{\partial x}}\left(\frac{1}{\lambda}\frac{\partial}{\partial x}\right) + \nabla_{\frac{1}{\lambda}\frac{\partial}{\partial y}}\left(\frac{1}{\lambda}\frac{\partial}{\partial y}\right)\\ &= \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\right)\frac{\partial}{\partial x} + \frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\right)\frac{\partial}{\partial y}\end{aligned}$$

This implies that

$$d\varphi(\nabla_{e_1} e_1 + \nabla_{e_2} e_2) = \frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\right)\frac{\partial\varphi}{\partial x} + \frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\right)\frac{\partial\varphi}{\partial y} \tag{1.13}$$

The other term is given by

$$\nabla^*_{e_1} d\varphi(e_1) = \left[\frac{1}{\lambda}\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\frac{\partial \varphi}{\partial x}\right)\right]^{\mathsf{T}} = \frac{1}{\lambda}\left[\frac{\partial}{\partial x}\left(\frac{1}{\lambda}\right)\frac{\partial \varphi}{\partial x} + \frac{1}{\lambda}\frac{\partial^2 \varphi}{\partial x^2}\right]^{\mathsf{T}} \tag{1.14}$$

$$\nabla^*_{e_2} d\varphi(e_2) = \left[\frac{1}{\lambda}\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\frac{\partial \varphi}{\partial y}\right)\right]^{\mathsf{T}} = \frac{1}{\lambda}\left[\frac{\partial}{\partial y}\left(\frac{1}{\lambda}\right)\frac{\partial \varphi}{\partial y} + \frac{1}{\lambda}\frac{\partial^2 \varphi}{\partial y^2}\right]^{\mathsf{T}} \tag{1.15}$$

It follows by equations (1.13), (1.14) and (1.15) that

$$\begin{aligned}\tau(\varphi) &= \nabla^*_{e_1} d\varphi(e_1) + \nabla^*_{e_2} d\varphi(e_2) - d\varphi(\nabla_{e_1} e_1) - d\varphi(\nabla_{e_2} e_2)\\ &= \frac{1}{\lambda^2}\left[\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}\right]^{\mathsf{T}} = \frac{1}{\lambda^2}\left[\Delta\varphi\right]^{\mathsf{T}}\end{aligned}$$

□

Harmonic maps generalize the concept of harmonic functions well known from complex analysis. A harmonic map is one for which the tension field vanishes everywhere and, as stated in Appendix A, arises as a critical point of a certain variational problem.

THEOREM 1.8. *Let Σ be a surface in $\mathbb{R}^3$ and $\varphi\colon \Sigma \to S^2$ be a map into the unit sphere S^2 in $\mathbb{R}^3$. If φ is conformal then it is harmonic.*

PROOF. Let $p \in \Sigma$ be an arbitrary point and $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ be local isothermal coordinates around p. Then the conformality of φ means that

$$\langle \varphi_x, \varphi_y \rangle = 0 \quad \text{and} \quad \langle \varphi_x, \varphi_x \rangle = \langle \varphi_y, \varphi_y \rangle .$$

By differentiating we then obtain

$$\begin{aligned}\langle \varphi_{xx}, \varphi_y \rangle &= -\langle \varphi_{yx}, \varphi_x \rangle , & \langle \varphi_{yy}, \varphi_x \rangle &= -\langle \varphi_{xy}, \varphi_y \rangle ,\\ \langle \varphi_{xx}, \varphi_x \rangle &= \langle \varphi_{yx}, \varphi_y \rangle , & \langle \varphi_{yy}, \varphi_y \rangle &= \langle \varphi_{xy}, \varphi_x \rangle ,\end{aligned}$$

and therefore

$$\begin{aligned}\langle \varphi_{xx} + \varphi_{yy}, \varphi_x \rangle &= \langle \varphi_{yx}, \varphi_{yy} \rangle - \langle \varphi_{yx}, \varphi_{yy} \rangle = 0,\\ \langle \varphi_{xx} + \varphi_{yy}, \varphi_y \rangle &= -\langle \varphi_{yx}, \varphi_{xx} \rangle + \langle \varphi_{xy}, \varphi_{xx} \rangle = 0.\end{aligned}$$

These relations imply that

$$\tau(\varphi) = \frac{1}{\lambda^2}\left[\Delta\varphi\right]^{\mathsf{T}} = 0.$$

□

CHAPTER 2

Minimal surfaces

In this chapter we introduce some results concerning minimal surfaces. We also prove the famous Weierstrass representation for minimal surfaces.

DEFINITION 2.1. A surface Σ in $\mathbb{R}^3$ is said to be *minimal* if its mean curvature H satisfies $H \equiv 0$.

2.1. Conformality of the Gauss map

PROPOSITION 2.2. *Let Σ be a minimal surface in $\mathbb{R}^3$. Then the Gauss map $N\colon \Sigma \to S^2$ of Σ is conformal.*

PROOF. Let $p \in \Sigma$ be an arbitrary point on Σ and (x, y) be local isothermal coordinates around p. Then it follows by

$$H = \frac{e+g}{2\lambda^2} = 0$$

and equation (1.11) that

$$N_x = \frac{1}{\lambda^2}\left(eX_x + fX_y\right), \qquad N_y = \frac{1}{\lambda^2}\left(fX_x - eX_y\right).$$

This implies that

$$\left\langle N_x, N_y \right\rangle = 0 \quad \text{and} \quad \left\langle N_x, N_x \right\rangle = \left\langle N_y, N_y \right\rangle = \frac{e^2+f^2}{\lambda^2} \geq 0.$$

Hence N is conformal. □

A partial reverse implication of the previous theorem is obtained via the following.

PROPOSITION 2.3. *Let Σ be a real analytic surface in $\mathbb{R}^3$ and $N\colon \Sigma \to S^2$ be a Gauss map of Σ. If N is conformal then Σ is either minimal or part of a sphere.*

PROOF. For local isothermal coordinates (x, y) on Σ we have

$$\begin{aligned} 0 = \left\langle N_x, N_y \right\rangle &= \left\langle \frac{1}{\lambda^2}\left(eX_x + fX_y\right), \frac{1}{\lambda^2}\left(fX_x + gX_y\right) \right\rangle \\ &= \frac{1}{\lambda^4}\left(ef\left\langle X_x, X_x\right\rangle + fg\left\langle X_y, X_y\right\rangle\right) \\ &= \frac{f}{\lambda^2}\left(e+g\right) = 2fH. \end{aligned}$$

Let $p \in \Sigma$ be a point. Suppose $H(p) \neq 0$ then there exists an open neighborhood $V \subset \Sigma$ around p such that $f|_V = 0$. For every point in $q \in V$ we have, by equation (1.11), that N_x and N_y are parallel with X_x and X_y, respectively. By conformality we have that $|N_x| = |N_y|$ and, since (x, y) are isothermal, that $|X_x| = |X_y|$. Hence q is umbilical i.e. the principal curvatures coincide. Let $k = k_1 = k_2$, where k_1 and k_2 are the principal curvatures. Differentiating $N_x = -kX_x$ and $N_y = -kX_y$ gives

$$(kX_x)_y = (N_x)_y = (N_y)_x = (kX_y)_x$$

and since X_x and X_y are linearly independent we must have $k_x = k_y = 0$ so k is constant. If $k = 0$ then $H = 0$ which contradicts the assumption. Hence $k \neq 0$ and

$$N = -kX + a$$

where a is a constant vector. Then X is a local parametrization for a sphere having radius $1/k$ centered at a/k since

$$\|X - \frac{1}{k}a\|^2 = \|\frac{1}{k}N\|^2 = \frac{1}{k^2}.$$

Thus by real analyticity Σ is either minimal or part of a sphere. □

2.2. The Weierstrass representation formula

The Weierstrass representation formula was first presented by Karl Weierstrass in [**10**]. It states that given two holomorphic functions defined on some simply connected subset of $\mathbb{C}$ there exists an associated minimal surface. This surface is unique up to motions.

THEOREM 2.4. *Let Σ be a surface in $\mathbb{R}^3$ and $\varphi = x + iy\colon U \subset \Sigma \to \mathbb{C}$ be local isothermal coordinates on Σ. Suppose U is an open simply connected subset of Σ. If $X\colon z(U) \subset \mathbb{C} \to \Sigma$ is the inverse of z then Σ is minimal if and only if the derivative $X_z\colon z(U) \to \mathbb{C}^3$ is holomorphic.*

PROOF. This is a direct consequence of Proposition 1.6 and the fact that a map $f\colon U \subseteq \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if $f_{\bar{z}} = 0$. □

Integration gives us the following corollary.

COROLLARY 2.5. *Let Σ be a surface in $\mathbb{R}^3$ and $\varphi = x + iy\colon U \subset \Sigma \to \mathbb{C}$ be local isothermal coordinates on Σ. Suppose U is an open simply connected subset of Σ. Then the inverse $X\colon z(U) \subset \mathbb{C} \to \Sigma$ of z is given by*

$$X(z) = 2\,\mathrm{Re} \int_{z_0}^{z} X_z(z)dz + C, \tag{2.1}$$

where C is some constant vector in $\mathbb{R}^3$.

PROOF. We have

$$\begin{aligned} X_z dz &= \frac{1}{2}\left((X_x - iX_y)(dx + idy)\right) \\ &= \frac{1}{2}\left(X_x dx + X_y dy + i(X_x dy - X_y dx)\right), \\ X_{\bar{z}} d\bar{z} &= \frac{1}{2}\left(X_x dx + X_y dy - i(X_x dy - X_y dx)\right). \end{aligned}$$

Integrating $dX = X_z dz + X_{\bar{z}} d\bar{z} = 2\,\mathrm{Re}\, X_z dz$ gives us our sought relation. □

This corollary gives us the famous Weierstrass representation for minimal surfaces.

THEOREM 2.6 (Weierstrass Representation). *Let V be an open simply connected subset of $\mathbb{C}$. Suppose $f\colon V \to \mathbb{C}$ is holomorphic on V, $g\colon V \to \mathbb{C}$ is meromorphic on V and the product fg^2 is holomorphic on V. Then $X\colon V \to \mathbb{R}^3$ defined by*

$$X(z) = \mathrm{Re} \int_{z_0}^{z} X_z(z) dz, \tag{2.2}$$

where

$$X_z(z) = f(z)(1 - g(z)^2, i(1 + g(z)^2), 2g(z))$$

is a minimal surface.

PROOF. Using the above results the only thing we need to show is that equation (2.2) define isothermal coordinates. This, however, follows by direct computation using Proposition 1.5. □

Examples of minimal surfaces are the surfaces of Sherk (Fig. 2.1) and Catalan (Fig. 2.2).

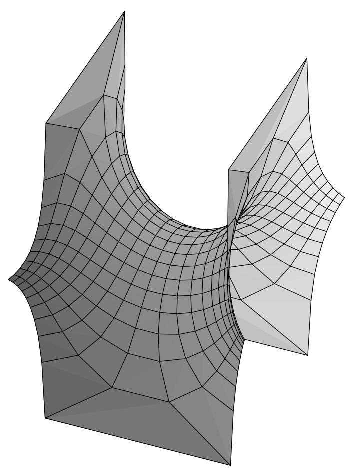

FIGURE 2.1. Sherk's minimal surface. $(f, g) = (\frac{2}{1-z^4}, z)$

An interesting observation is the fact that the Gauss map of minimal surfaces generated using Theorem 2.6 can be identified with the complex valued function g.

PROPOSITION 2.7. *Let Σ be a minimal surface in $\mathbb{R}^3$ given by the Weierstrass representation*

$$X(z) = \mathrm{Re} \int_{z_0}^{z} f(z)(1 - g(z)^2, i(1 + g(z)^2), 2g(z)) dz.$$

Then the Gauss map $N\colon \Sigma \to S^2$ of Σ may be identified, via stereographic projection σ, with g.

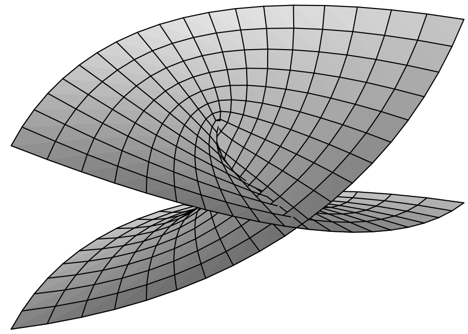

FIGURE 2.2. Catalan's minimal surface. $(f, g) = \left(i\left(\frac{1}{z} - \frac{1}{z^3}\right), z\right)$

PROOF. Let $G = \sigma^{-1} \circ g$ then

$$G = \frac{(2\operatorname{Re} g, 2\operatorname{Im} g, g^2 - 1)}{1 + g^2}.$$

Now the real and imaginary parts of

$$\big(f(1 - g^2), if(1 + g^2), 2fg\big)$$

represent two orthogonal tangent vectors on Σ in $\mathbb{R}^3$. Then

$$\left\langle \frac{fg}{2}\left(\frac{1}{g} - g, \frac{i}{2}(\frac{1}{g} + g), 1\right), G\right\rangle = \frac{fg}{2}\left(\frac{g}{g} - g\bar{g} + |g|^2 - 1\right) = 0$$

and since $|G| = 1$ it is clear that G is a Gauss map for Σ. □

We can conclude that the above examples (Figures 2.1, 2.2) have bijective Gauss map i.e. for every point $p \in S^2$ there exists only one point on Σ having that point as a normal. Amongst the Enneper surfaces, defined by $(f, g) = (1, z^n)$ for every $n \in \mathbb{N}$, only the case of $n = 1$ satisfies this property (see Figures 2.3, 2.4 and 2.5).

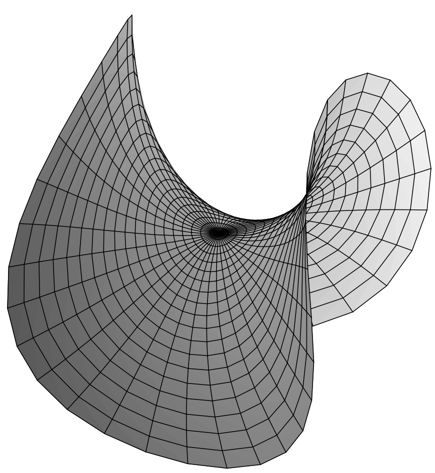

FIGURE 2.3. First order Enneper surface. $(f, g) = (1, z)$

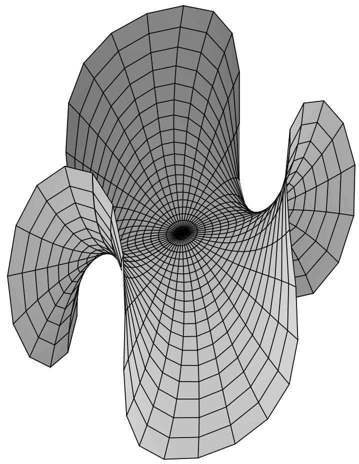

FIGURE 2.4. Second order Enneper surface. $(f, g) = (1, z^2)$

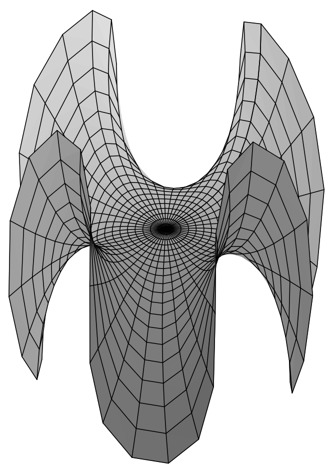

FIGURE 2.5. Third order Enneper surface. $(f, g) = (1, z^3)$

CHAPTER 3

CMC surfaces of revolution

In this chapter we study complete CMC surfaces with rotational symmetry. We present Kenmotsu's modern solution given in [**11**] to the problem of finding all such surfaces. Furthermore we describe the classical construction of the same due to Delaunay, see [**1**].

DEFINITION 3.1. A surface Σ in $\mathbb{R}^3$ is said to have constant mean curvature (CMC) if and only if there exists a $c \in \mathbb{R}$ such that $H \equiv c$.

3.1. Kenmotsu's solution

Let $\varphi\colon I \subset \mathbb{R} \to \mathbb{R}^2$ with $\varphi(s) = \big(x(s), y(s)\big)$ be a parametrization of some regular planar C^2 curve. Assume that φ is an arclength parametrization and that 0 is contained in the open interval I. Let Σ be the surface of revolution in $\mathbb{R}^3$ defined by

$$(s,\theta) \mapsto (x(s), y(s)\cos\theta, y(s)\sin\theta), \qquad s \in I, 0 \le \theta \le 2\pi.$$

Then the first and second fundamental forms are given by

$$I_p = ds^2 + y^2 d\theta^2,$$
$$II_p = (x''y' - x'y'')ds^2 + x'yd\theta^2.$$

Assuming $y(s) > 0$ for $s \in I$ we have, by definition of H, that

$$2Hy - x' - x''yy' + x'yy'' = 0, \qquad s \in I. \tag{3.1}$$

Multiplying by x' and y', respectively, and simplifying using the fact that

$$(x')^2 + (y')^2 = 1 \quad \text{and} \quad x'x'' + y'y'' = 0, \qquad s \in I,$$

we obtain

$$2Hyx' + (yy')' - 1 = 0 \quad \text{and} \quad 2Hyy' - (yx')' = 0.$$

Setting $Z(s) = y(s)y'(s) + iy(s)x'(s)$ and combining these equations the following first order complex linear differential equation is obtained

$$Z' - 2iHZ - 1 = 0, \qquad s \in I. \tag{3.2}$$

Restricting our attention to the case of H being constant we have:

If $H = 0$ then the solution is given by

$$Z(s) = s + C = s + c_1 + ic_2$$

for some $C = c_1 + ic_2 \in \mathbb{C}$. This gives us

$$\begin{aligned} y(s) &= |Z(s)| = \sqrt{(s+c_1)^2 + c_2^2}, \\ x'(s) &= \frac{\operatorname{Im} Z}{y} = \frac{c_2}{\sqrt{(s+c_1)^2 + c_2^2}}. \end{aligned} \tag{3.3}$$

By integrating x we obtain

$$x = c_2 \operatorname{arcsinh}\left(\frac{s+c_1}{c_2}\right) \quad \text{hence} \quad s + c_1 = \sinh\left(\frac{x}{c_2}\right) c_2.$$

Substituting into equation (3.3) we obtain

$$y = \sqrt{(s+c_1)^2 + c_2^2} = \sqrt{\sinh^2\left(\frac{x}{c_2}\right) c_2^2 + c_2^2} = c_2 \cosh\left(\frac{x}{c_2}\right).$$

It is clear that this is a parametrization of a catenary.

If $H \neq 0$ then

$$\begin{aligned} Z(s) &= \left(\frac{1}{2iH}\left(1 - e^{-2iHs}\right) + C\right) e^{2iHs} \\ &= \frac{1}{2iH}\left((1 + 2iHC) - e^{-2iHs}\right) e^{2iHs} \\ &= \frac{Be^{i(2Hs+\theta)} - 1}{2iH}, \end{aligned} \tag{3.4}$$

where $Be^{i\theta} = 1 + 2iHC$ for some $B, \theta \in \mathbb{R}$ and $C \in \mathbb{C}$ is an arbitrary constant. Using the fact that $y(s) > 0$ we have by translation of the arclength and by restricting our attention to $H > 0$

$$\begin{aligned} y(s) &= |Z| = \frac{1}{2H}\sqrt{1 + B^2 + 2B\sin 2Hs}, \\ x'(s) &= \frac{\operatorname{Im} Z}{y} = \frac{1 + B\sin 2Hs}{\sqrt{1 + B^2 + 2B\sin 2Hs}}. \end{aligned}$$

Hence the solution to equation (3.4) is the one-parameter family of surfaces of revolution having constant mean curvature H given by

$$\begin{aligned} \varphi(s; H, B) = \Bigg(\int_0^s \frac{1 + B\sin 2Ht}{\sqrt{1 + B^2 + 2B\sin 2Ht}} dt, \\ \frac{1}{2H}\sqrt{1 + B^2 + 2B\sin 2Hs}\Bigg) \end{aligned} \tag{3.5}$$

for any $B \in \mathbb{R}$ and $H > 0$.

Studying φ for varying B (see Fig. 3.1) we find that $\varphi(s; H, 0)$ is a generating curve for a right circular cylinder and $\varphi(s; H, 1)$ is a generating curve for a sequence of continuous half-circles centered on the x-axis. For $0 < B < 1$ the function $x(s)$ increases monotonously whereas in the case of $B > 1$ it does not.

THEOREM 3.2 (Delaunay's theorem). *Any complete surface of revolution with constant mean curvature is either a sphere, a catenoid or a surface whose generating curve is given by* $\varphi(s; H, B)$ *for some* $B \in \mathbb{R}$.

PROOF. Let $H \in \mathbb{R}$ be given and let $\varphi_H(s)$ be a generating curve parametrized by arclength for a complete surface of revolution having constant mean curvature H. By uniqueness of solution of (3.2) we have $\varphi_H(s) = \varphi(s; H, B)$ for some $B \in \mathbb{R}$. □

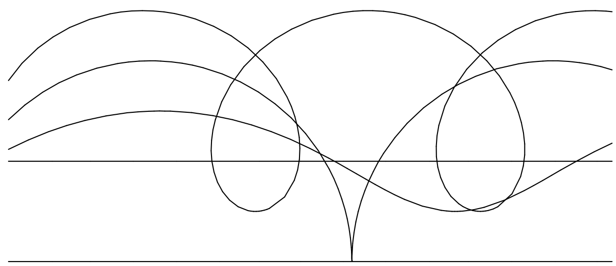

FIGURE 3.1. Solutions for $H = 0.5$ and $B = 0, 0.5, 1, 1.5$

3.2. Delaunay's construction

The surfaces of Delaunay are constructed by rolling a conic ℓ along a straight line in the plane and taking the trace of the focus F. This is called a *roulette* of the conic ℓ. This trace then describes a planar curve which is rotated about the axis along which it was rolled. This gives a surface of revolution having constant mean curvature. The construction presented here is based on the article [**12**] by J. Eells.

3.2.1. ℓ is a parabola. Let ℓ be a parabola given by $\ell\colon t \mapsto (t, at^2)$ for some a which we take to be strictly positive. Let F be the focus and A be the vertex of ℓ (see Figure 3.2). Let K be a point on ℓ and denote by P the intersection of the tangent line of ℓ at K with the horizontal axis. By solving the line equation for the tangent at K we find that for $K = \ell(t) = (t, at^2)$ then $P = (t/2, 0)$. This implies that $\overline{PK} = \overline{OP}$. And since $\angle FOP = \angle PKF$ we

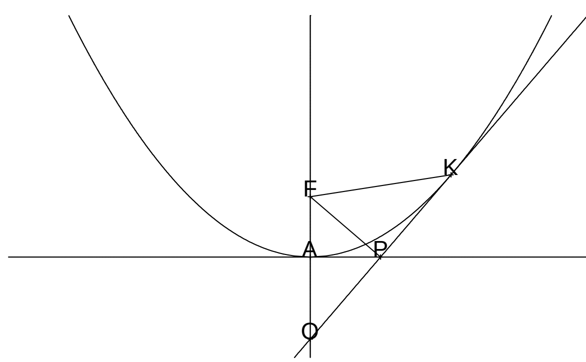

FIGURE 3.2. ℓ is parabola

FIGURE 3.3. Catenary

also have $\angle OPF = \angle KPF = \frac{\pi}{2}$. By definition of the trigonometric functions we have

$$\overline{FA} = \overline{FP} \cos \angle AFP = \overline{FP} \cos \angle PFK.$$

Now let FP denote the x-axis along which our parabola rolls. Then the ordinate of F in this system of coordinates is given by $\overline{PF}$. Denote this by y. We have

$$\cos \angle PFK = \frac{dx}{ds}.$$

where s is the arclength of the locus of F. This is equivalent to

$$\frac{dx}{ds} = \cos \alpha, \tag{3.6}$$

where α denotes the angle made by the tangent of F with the x-axis. We then arrive at

$$c = y\frac{dx}{ds} = y\frac{\frac{dx}{ds}}{\sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2}} = \frac{y}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

or, equivalently,

$$\frac{dy}{dx} = \sqrt{\frac{y^2 - c^2}{c^2}}. \tag{3.7}$$

The solution to this differential equation is given by

$$y = \frac{c}{2}\left(e^{x/c} + e^{-x/c}\right) = c \cosh \frac{x}{c}$$

which is a *catenary* (Fig. 3.3). The corresponding surface of revolution is the *catenoid* (Fig. 3.4). The Gauss map of the locus of F into S^1 is given by $x \mapsto \alpha_x$ where

$$\cos \alpha_x = \frac{dx}{ds} = \frac{c}{y}$$

showing that the Gauss map is injective onto an open semicircle.

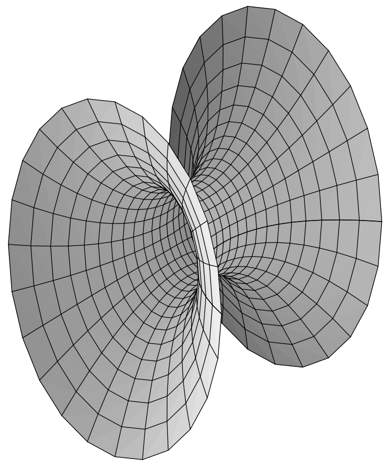

FIGURE 3.4. Catenoid

3.2.2. ℓ is an ellipse. Let F and F' be foci of ℓ and O its center. Take a point K on ℓ and let P and P' be the points on the tangent at K closest to F and F', respectively (Fig. 3.5). As above, letting PK be the x-axis and PF ($P'F'$) the ordinate y (y'). Let T and T' denote the intersection with the x-axis of the tangent of the locus of F and F', respectively.

We have $\angle FKP = \angle F'KP'$. Also the tangent of the locus of F (F') is orthogonal to FK (FL') and hence $\angle KFT = \angle KF'T' = \frac{\pi}{2}$. This gives us

$$\frac{y}{\overline{FK}} = \sin\angle FKP = \cos\angle FTP = \frac{dx}{ds},$$
$$\frac{y'}{\overline{F'K}} = \sin\angle F'KP' = \cos\angle F'T'P' = \frac{dx}{ds}.$$

From the characterization of the ellipse, $\overline{FK} + \overline{F'K} = 2a$ for some $a > 0$, and the pedal equation, $\overline{PF} \cdot \overline{P'F'} = b^2$ for some $b > 0$, we find

$$y + y' = 2a\frac{dx}{ds} \quad \text{and} \quad yy' = b^2 \quad \text{so}$$

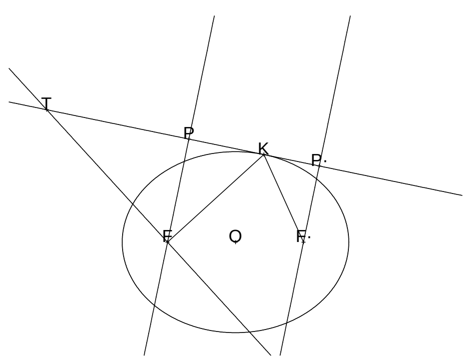

FIGURE 3.5. ℓ is ellipse

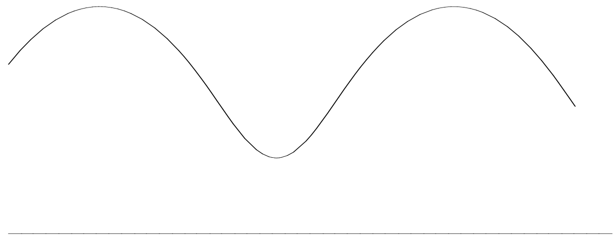

FIGURE 3.6. Undulary, $H = 0.5$, $B = 0.5$.

$$y^2 - 2ay\frac{dx}{ds} + b^2 = 0.$$

Taking $a \leq b$ we get the following cases (when the angle is obtuse and acute, respectively)

$$y^2 \pm 2ay\frac{dx}{ds} + b^2 = 0. \tag{3.8}$$

A solution to this problem is given in Section 3.1

$$\begin{aligned} x(s) &= \int_0^s \frac{1 + B\sin 2Ht}{\sqrt{1 + B^2 + 2B\sin 2Ht}}dt, \\ y(s) &= \frac{1}{2H}\sqrt{1 + B^2 + 2B\sin 2Hs}, \end{aligned} \tag{3.9}$$

where $H = \frac{1}{2a}$ and $B = \pm\sqrt{1 - \frac{b^2}{4H^2}}$. The locus of either foci is called the *undulary* (Fig. 3.6). The corresponding surfaces is called the *unduloid* (Fig. 3.7). The Gauss map of the undulary is given by $x \mapsto \alpha_x$ where

$$\cos\alpha_x = \frac{dx}{ds} = \mp\frac{y^2 + b^2}{2ay}.$$

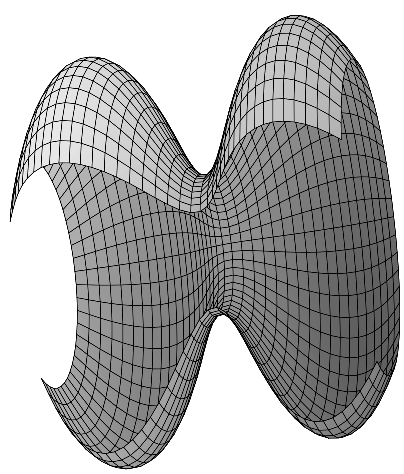

FIGURE 3.7. Unduloid, $H = 0.5$, $B = 0.5$.

3.2.3. ℓ is a hyperbola. We proceed as in the case of the ellipse but instead use the characterization $\overline{FK} - \overline{F'K} = 2a > 0$ of the hyperbola and the pedal equation $\overline{PF} \cdot \overline{P'F'} = -b^2$ (Fig. 3.8). We arrive at the differential equation

$$y^2 \pm 2ay\frac{dx}{ds} - b^2 = 0.$$

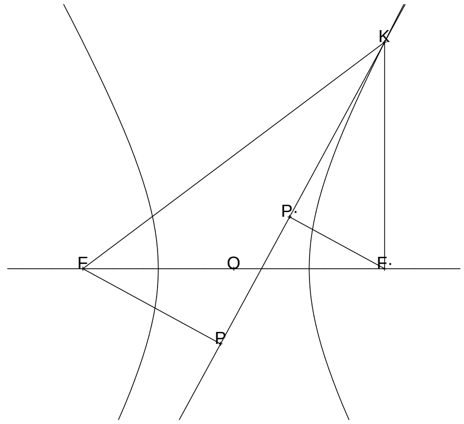

FIGURE 3.8. Hyperbola

This differential equation can be solved in the same manner as for the ellipse with the exception that B in equation (3.9) is given by

$$B = \pm\sqrt{1 + \frac{b^2}{4H^2}}.$$

Here the two loci fit together to form the curve known as the *nodary* (Fig. 3.9) and the corresponding surface is called the *nodoid* (Fig. 3.10) The Gauss map of the nodary is given by $x \mapsto \alpha_x$ where

$$\cos\alpha_x = \mp\frac{y^2 - b^2}{2ay}.$$

This map has no extreme points and is clearly surjective.

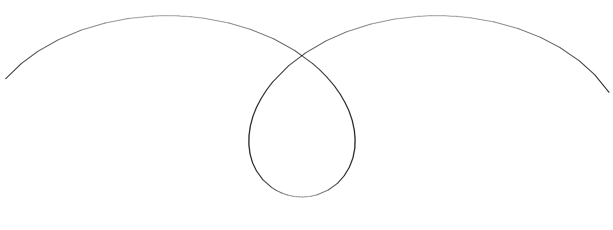

FIGURE 3.9. Nodary, $H = 0.5$, $B = 1.5$.

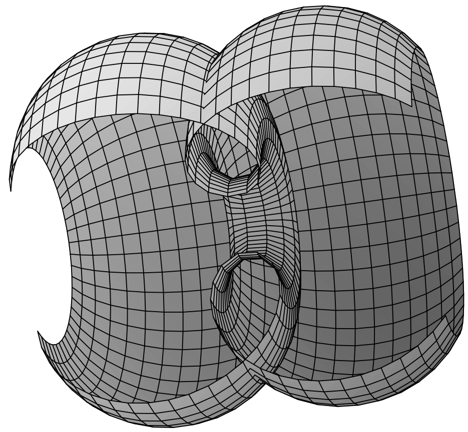

FIGURE 3.10. Nodoid, $H = 0.5$, $B = 1.5$.

CHAPTER 4

CMC surfaces

The main aim of this chapter is to give a new elementary proof of a special case of the Ruh-Vilms' theorem. We also present the Kenmotsu representation formula for CMC surfaces with $H \neq 0$.

4.1. Harmonicity of the Gauss map

Let (M, g) be an orientable m-dimensional Riemannian manifold, $i\colon M \to \mathbb{R}^{m+p}$ be an isometric immersion and $N\colon M \to G_p^o(\mathbb{R}^{m+p})$ be the associated Gauss map, mapping $x \in M$ to the oriented normal space of $i(M)$ at $i(x)$. Then Ruh-Vilms' theorem presented in [**13**] states that the tension field $\tau(N)$ of N satisfies

$$\tau(N) = -m\nabla H,$$

where ∇H is the covariant derivative of the mean curvature vector field H. This implies that the Gauss map N is harmonic if and only if the mean curvature vector field H is parallel. For surfaces in $\mathbb{R}^3$ this is equivalent to the surface having constant mean curvature.

THEOREM 4.1. *Let Σ be an oriented surface in $\mathbb{R}^3$. Then Σ has constant mean curvature if and only if its Gauss map $N\colon \Sigma \to S^2$ is harmonic.*

PROOF. We prove that the following equation

$$\tau(N) = -2\,\mathrm{grad}\, H$$

holds. Our sought result then follows trivially.

Let $p \in \Sigma$ be an arbitrary point and $(x, y)\colon U \subset \Sigma \to \mathbb{R}^2$ be isothermal coordinates around p. Then we have

$$\begin{aligned}
(N_{xx})^{\mathsf{T}} &= \frac{\langle N_{xx}, X_x\rangle X_x + \langle N_{xx}, X_y\rangle X_y}{\lambda^2} \\
&= \frac{1}{\lambda^2}\left[(-e_x - \langle N_x, X_{xx}\rangle)X_x + (-f_x - \langle N_x, X_{yx}\rangle)X_y\right] \\
&= \frac{1}{\lambda^2}\left[(-e_x - \langle N_x, X_{xx}\rangle)X_x + (-e_y + (e+g)\Gamma_{22}^2 - \langle N_x, X_{yx}\rangle)X_y\right].
\end{aligned}$$

This follows by using

$$\begin{aligned}
-e_x &= \frac{\partial}{\partial x}\langle N_x, X_x\rangle = \langle N_{xx}, X_x\rangle + \langle N_x, X_{xx}\rangle, \\
-f_x &= \frac{\partial}{\partial x}\langle N_x, X_y\rangle = \langle N_{xx}, X_y\rangle + \langle N_x, X_{yx}\rangle,
\end{aligned}$$

and the Mainardi-Codazzi equations (1.10). Similarly we have

$$(N_{yy})^{\mathsf{T}} = \frac{1}{\lambda^2}\left[(-g_x + (e+g)\Gamma^1_{11} - \langle N_y, X_{xy}\rangle)X_y + (-g_y - \langle N_y, X_{yy}\rangle)X_y\right].$$

By using the Christoffel relations (1.9) we find

$$\begin{aligned}
\langle N_x, X_{xx}\rangle &= \langle a_{11}X_x + a_{21}X_y, \Gamma^1_{11}X_x + \Gamma^2_{11}X_y\rangle \\
&= \lambda^2\left(a_{11}\Gamma^1_{11} + a_{21}\Gamma^2_{11}\right) = -\left(e\Gamma^1_{11} - f\Gamma^2_{22}\right) \\
\langle N_x, X_{yx}\rangle &= \lambda^2\left(a_{11}\Gamma^1_{12} + a_{21}\Gamma^2_{12}\right) = -\left(f\Gamma^1_{11} + e\Gamma^2_{22}\right) \\
\langle N_y, X_{yx}\rangle &= \lambda^2\left(a_{12}\Gamma^1_{21} + a_{22}\Gamma^2_{21}\right) = -\left(g\Gamma^1_{11} + f\Gamma^2_{22}\right) \\
\langle N_y, X_{yy}\rangle &= \lambda^2\left(a_{12}\Gamma^1_{22} + a_{22}\Gamma^2_{22}\right) = -\left(-f\Gamma^1_{11} + g\Gamma^2_{22}\right)
\end{aligned}$$

Adding and using the above equations we get

$$\begin{aligned}
\langle \Delta N, X_x\rangle &= \langle N_{xx} + N_{yy}, X_x\rangle \\
&= \frac{1}{\lambda^2}\left(-(e_x + g_x) + (e+g)\Gamma^1_{11} - \langle N_x, X_{xx}\rangle - \langle N_y, X_{xy}\rangle\right) \\
&= \frac{1}{\lambda^2}\left(-(e_x + g_x) + 2(e+g)\Gamma^1_{11}\right), \\
\langle \Delta N, X_y\rangle &= \langle N_{xx} + N_{yy}, X_y\rangle \\
&= \frac{1}{\lambda^2}\left(-(e_y + g_y) + 2(e+g)\Gamma^2_{22}\right).
\end{aligned}$$

Hence we have

$$\begin{aligned}
-\tau(N) = -\frac{1}{\lambda^2}(\Delta N)^{\mathsf{T}} = &\quad \frac{1}{\lambda^2}\left[\frac{e_x + g_x}{\lambda^2} - \frac{e+g}{\lambda^4}\frac{\partial\lambda^2}{\partial x}\right]X_x \\
&+ \frac{1}{\lambda^2}\left[\frac{e_y + g_y}{\lambda^2} - \frac{e+g}{\lambda^4}\frac{\partial\lambda^2}{\partial y}\right]X_y \\
= &\quad \frac{1}{\lambda^2}\left[\frac{e_x + g_x}{\lambda^2} + (e+g)\frac{\partial}{\partial x}(\frac{1}{\lambda^2})\right]X_x \\
&+ \frac{1}{\lambda^2}\left[\frac{e_y + g_y}{\lambda^2} + (e+g)\frac{\partial}{\partial y}(\frac{1}{\lambda^2})\right]X_y \\
= &\quad \left[\frac{1}{\lambda}\frac{\partial}{\partial x}(e+g)\frac{1}{\lambda^2} + (e+g)\frac{1}{\lambda}\frac{\partial}{\partial x}(\frac{1}{\lambda^2})\right]\frac{1}{\lambda}X_x \\
&+ \left[\frac{1}{\lambda}\frac{\partial}{\partial y}(e+g)\frac{1}{\lambda^2} + (e+g)\frac{1}{\lambda}\frac{\partial}{\partial y}(\frac{1}{\lambda^2})\right]\frac{1}{\lambda}X_y \\
= &\ 2\,\mathrm{grad}\,H.
\end{aligned}$$

This proves our theorem. □

4.2. Kenmotsu's representation formula

In this section we show a corresponding result to Weierstrass' representation formula for surfaces having non-zero constant mean curvature.

Let S^2 be the unit sphere in $\mathbb{R}^3$. Cover S^2 by open sets U_i, $i = 1, 2$ where $U_1 = S^2 - \{n\}$ and $U_2 = S^2 - \{s\}$ and n and s are the north and south pole, respectively. Let σ be the stereographic projection with respect to the north pole n:

$$\sigma(x) = \frac{x_1 + ix_2}{1 -{}_3} \qquad \text{for} \quad x = (x_1, x_2, x_3) \in U_1. \tag{4.1}$$

For a surface Σ in $\mathbb{R}^3$ having Gauss map $N\colon \Sigma \to S^2$ consider the following composition

$$\varphi\colon \Sigma \xrightarrow{N} S^2 \xrightarrow{\sigma} \mathbb{C}$$

which we also call the Gauss map of Σ. This map is considered as a complex mapping from a 1-dimensional complex manifold Σ in $\mathbb{R}^3$ into the Riemann sphere. Using this notation we have the following theorem presented in [**14**] due to K. Kenmotsu.

THEOREM 4.2 (Kenmotsu's representation formula). *Let V be an open simply connected subset of $\mathbb{C}$ and H be an arbitrary non-zero real constant. Suppose $\varphi\colon V \to \mathbb{C}$ is a harmonic function into the Riemann sphere. If $\varphi_{\bar{z}} \neq 0$ then $X\colon V \to \mathbb{R}^3$ defined by*

$$X(z) = \operatorname{Re} \int_{z_0}^{z} X_z(z)\,dz, \tag{4.2}$$

with

$$X_z(z) = \frac{(-1)}{H(1 + \varphi(z)\bar{\varphi}(z))^2} \left(1 - \varphi(z)^2, i(1 + \varphi(z)^2), 2\varphi(z)\right) \overline{\frac{\partial \varphi}{\partial \bar{z}}}(z),$$

for $z \in V$, is a regular surface having φ as a Gauss map and mean curvature H.

First we derive an explicit formula of the tension field of the Gauss map $\varphi\colon \Sigma \to \mathbb{C}$ of an arbitrary surface Σ.

PROPOSITION 4.3. *Let Σ be a surface in $\mathbb{R}^3$ and $\varphi\colon \Sigma \to \mathbb{C}$ be a Gauss map on Σ. If $z = x + iy$ are local isothermal coordinates with dilation λ then*

$$\tau(\varphi) = \frac{4}{\lambda^2} \left[\frac{\partial^2 \varphi}{\partial z \partial \bar{z}} - \frac{2\bar{\varphi}}{1 + \varphi\bar{\varphi}} \frac{\partial \varphi}{\partial z} \frac{\partial \varphi}{\partial \bar{z}} \right]. \tag{4.3}$$

PROOF. Let $z = x + iy$ be local isothermal coordinates and denote $\varphi(z) = u + iv$. Then

$$g = \lambda^2(dx^2 + dy^2) \quad \text{and} \quad h = \frac{4}{(1 + u^2 + v^2)^2}(du^2 + dv^2)$$

which gives us Christoffel symbols on the Riemann sphere

$$\begin{aligned} \Gamma_{11}^1 = \quad \Gamma_{12}^2 = -\Gamma_{22}^1 &= \frac{-2u}{1 + u^2 + v^2}, \\ -\Gamma_{11}^2 = \quad \Gamma_{12}^1 = \quad \Gamma_{22}^2 &= \frac{-2v}{1 + u^2 + v^2}. \end{aligned}$$

By the explicit formula for the tension field (eq. A.2) we get

$$\tau(u) = \frac{1}{\lambda^2}\Bigg\{ \Delta u + \Gamma^1_{11}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] + \Gamma^1_{22}\left[\left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right] + 2\Gamma^1_{12}\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\right]\Bigg\},$$

where Δ is the classical Laplacian. Adding this and the similar formula for $\tau(v)$ we arrive at

$$\begin{aligned}
\lambda^2\tau(\varphi) &= \lambda^2\big(\tau(u) + i\tau(v)\big) \\
&= \Delta(u+iv) - \\
&\quad - 2\frac{u-iv}{1+u^2+v^2}\Bigg\{\left(\frac{\partial u}{\partial x}\right)^2 - 2i\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} - \left(\frac{\partial v}{\partial x}\right)^2 + \\
&\qquad \left(\frac{\partial u}{\partial y}\right)^2 - 2i\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} - \left(\frac{\partial v}{\partial y}\right)^2\Bigg\} \\
&= \Delta\varphi - 8\frac{\bar\varphi}{1+\varphi\bar\varphi}\frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial\bar z} \\
&= 4\frac{\partial^2\varphi}{\partial z\partial\bar z} - 8\frac{\bar\varphi}{1+\varphi\bar\varphi}\frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial\bar z}
\end{aligned}$$

This proves our sought formula. □

Next we show that a surface having prescribed mean curvature H satisfies the following equation.

LEMMA 4.4. *Let Σ be a surface in $\mathbb{R}^3$ having mean curvature $H\colon \Sigma \to \mathbb{R}$ and let $\varphi\colon \Sigma \to \mathbb{C}$ be a Gauss map of Σ. Then*

$$H\frac{\partial\varphi}{\partial z} = \psi\frac{\partial\varphi}{\partial\bar z},$$

where $\psi = \frac{1}{\lambda^2}\big((e-g)/2 - if\big)$.

PROOF. We show that the following equation holds

$$\frac{\partial\varphi}{\partial\bar z} = -\frac{H}{2}\left(1+\varphi\bar\varphi\right)^2\frac{\partial(X^1+iX^2)}{\partial\bar z}. \tag{4.4}$$

By direct computation using the Weingarten equations (1.11) together with equation (4.1) we have

$$\begin{aligned}
\frac{\partial\varphi}{\partial\bar z} = \frac{\partial}{\partial\bar z}\left(\frac{N^1+iN^2}{1-N^3}\right) = \frac{1}{2\lambda^2(1-N^3)^2}\Bigg[\\
\Big\{\left(-X^1_x + N^3X^1_x - N^1X^3_x\right) - i\left(X^2_x - N^3X^2_x + N^2X^3_x\right)\Big\}e \\
-\Big\{(1-N^3)\left(X^2_y - X^2_x + iX^2_y + iX^1_x\right) + (N^1+iN^2)\left(X^3_y + iX^3_x\right)\Big\}f
\end{aligned}$$

$$+\left\{\left(X_y^2 - N^3X_y^2 + N^2X_y^3\right) - \left(X_y^1 - N^3X_y^1 + N^1X_y^3\right)\right\}g\Big].$$

Using the definition

$$N = \frac{1}{\lambda^2}\left(X_x^2X_y^3 - X_x^3X_y^2, X_x^3X_y^1 - X_x^1X_y^3, X_x^1X_y^2 - X_x^2X_y^1\right) \tag{4.5}$$

and the fact that $z = x + iy$ are isothermal together with equation

$$(1 + \varphi\bar{\varphi})(1 - N^3) = 2 \qquad \text{on} \qquad U_1, \tag{4.6}$$

our sought relation is obtained via

$$\begin{aligned}\frac{\partial\varphi}{\partial\bar{z}} &= \frac{1}{2\lambda^2(1-N^3)^2}\left[(-2)(e+g)\frac{\partial(X^1 + iX^2)}{\partial\bar{z}}\right] \\ &= \frac{-2H}{(1-N^3)^2}\frac{\partial(X^1 + iX^2)}{\partial\bar{z}} \\ &= -\frac{H}{2}(1 + \varphi\bar{\varphi})^2\frac{\partial(X^1 + iX^2)}{\partial\bar{z}}.\end{aligned}$$

By similar calculations we have

$$\frac{\partial\varphi}{\partial z} = -\frac{\psi}{2}(1 + \varphi\bar{\varphi})^2\frac{\partial(X^1 + iX^2)}{\partial\bar{z}}. \tag{4.7}$$

From equations (4.4) and (4.7) we may conclude that

$$H\frac{\partial\varphi}{\partial z} = \psi\frac{\partial\varphi}{\partial\bar{z}}.$$

□

The computations carried out in the proof of the previous lemma give us a way of describing the dilation λ in terms of the Gauss map φ.

COROLLARY 4.5. *Let Σ be an orientable surface in $\mathbb{R}^3$ with mean curvature $H\colon \Sigma \to \mathbb{R}$ and $\varphi\colon \Sigma \to \mathbb{C}$ be the Gauss map of Σ. Let $z = x + iy$ be local isothermal coordinates on Σ with dilation λ. Then*

$$\left|\frac{\partial\varphi}{\partial\bar{z}}\right| = \frac{\lambda}{2}(1 + \varphi\bar{\varphi})\left|H\right|, \tag{4.8}$$

$$\left|\frac{\partial\varphi}{\partial z}\right| = \frac{\lambda}{2}(1 + \varphi\bar{\varphi})\left|\psi\right|. \tag{4.9}$$

PROOF. By equations (4.5) we have

$$\begin{aligned}4\left|X_{\bar{z}}^1 + iX_{\bar{z}}^2\right|^2 &= 2\lambda^2(1 - N^3) - 4\left|X_{\bar{z}}^3\right|^2, \\ 4\left|X_{\bar{z}}^3\right|^2 &= \lambda^2((N^1)^2 + (N^2)^2)^2.\end{aligned}$$

The above formulas gives us

$$4\left|X_{\bar{z}}^1 + iX_{\bar{z}}^2\right|^2 = \lambda^2(1 - N^3)^2. \tag{4.10}$$

From equations (4.4) and (4.6) we get our first sought result. Equation (4.9) follows from equations (4.10) and (4.7). □

Next we show how the derivatives of the components of X are related to the Gauss map.

PROPOSITION 4.6. *Let Σ be an orientable surface in $\mathbb{R}^3$ with mean curvature $H\colon \Sigma \to \mathbb{R}$ and $\varphi\colon \Sigma \to \mathbb{C}$ be the Gauss map of Σ. Let $X\colon V \subset \mathbb{C} \to \Sigma$ be a local conformal immersion of Σ, then the following equations hold on U_1:*

$$\begin{aligned} H\frac{\partial X^1}{\partial \bar{z}} &= -\frac{1-(\bar{\varphi})^2}{(1+\varphi\bar{\varphi})^2}\frac{\partial \varphi}{\partial \bar{z}}, \\ H\frac{\partial X^2}{\partial \bar{z}} &= i\frac{1+(\bar{\varphi})^2}{(1+\varphi\bar{\varphi})^2}\frac{\partial \varphi}{\partial \bar{z}}, \\ H\frac{\partial X^3}{\partial \bar{z}} &= -2\frac{\bar{\varphi}}{(1+\varphi\bar{\varphi})^2}\frac{\partial \varphi}{\partial \bar{z}}. \end{aligned} \tag{4.11}$$

PROOF. Let $\tilde{\sigma}\colon U_2 \subset S^2 \to \mathbb{C}$ denote the stereographic projection with respect to the south pole s of S^2 and put $\rho = \tilde{\sigma} \circ N$. Then by similar calculations as in the proof of Lemma 4.4 we have

$$\frac{\partial \rho}{\partial \bar{z}} = -\frac{H}{2}\left(1+\rho\bar{\rho}\right)^2 \frac{\partial (X^1 - iX^2)}{\partial \bar{z}}. \tag{4.12}$$

Using that $\varphi\rho = 1$ we may conclude that the following equations hold on $U_1 \cap U_2$:

$$0 = \frac{\partial(\varphi\rho)}{\partial \bar{z}} = -\frac{H}{2}\left[\left(\varphi+\bar{\rho}\right)\frac{\partial (X^1 - iX^2)}{\partial \bar{z}} + \left(\bar{\varphi}+\rho\right)\frac{\partial (X^1 + iX^2)}{\partial \bar{z}}\right]$$

By $(\bar{\varphi}+\rho)/(\varphi+\bar{\rho}) = \bar{\varphi}^2$ we have

$$H\left[\frac{\partial (X^1 - iX^2)}{\partial \bar{z}} + \bar{\varphi}^2\frac{\partial (X^1 + iX^2)}{\partial \bar{z}}\right] = 0.$$

Rewriting as

$$H(1+\bar{\varphi}^2)\frac{\partial X^1}{\partial \bar{z}} = iH(1-\bar{\varphi}^2)\frac{\partial X^2}{\partial \bar{z}}$$

and substituting into equation (4.4) we arrive at

$$H\frac{\partial X^2}{\partial \bar{z}} = i\frac{1+\bar{\varphi}^2}{(1+\varphi\bar{\varphi})^2}\frac{\partial \varphi}{\partial \bar{z}}. \tag{4.13}$$

Similarly, we obtain the first formula of equation (4.11). The third equation follows from

$$\frac{\partial X^3}{\partial z}\left(\frac{\partial X^1}{\partial \bar{z}} + i\frac{\partial X^2}{\partial \bar{z}}\right) = \lambda^2\frac{\varphi}{(1+\varphi\bar{\varphi})^2}.$$

This last equation follows by rewriting equation (4.6) to

$$\begin{aligned} &4\left\{\frac{\partial X^3}{\partial z}\left(\frac{\partial X^1 + iX^2}{\partial \bar{z}}\right) - \lambda^2\frac{\varphi}{(1+\varphi\bar{\varphi})^2}\right\} \\ &\quad = 4\frac{\partial X^3}{\partial z}\left(\frac{\partial X^1 + iX^2}{\partial \bar{z}}\right) - \lambda^2(1-N^3)(N^1+iN^2). \end{aligned}$$

Expanding using equation (4.5) and separating out real and imaginary parts we find that both vanish and the equation follows. If $\frac{\partial\varphi}{\partial\bar z} \neq 0$ then by equations (4.10) and (4.6) we have

$$H\frac{\partial X^3}{\partial z} = \lambda^2\frac{\varphi}{(1+\varphi\bar\varphi)^2}\frac{1}{X^1_{\bar z}+iX^2_{\bar z}} = \lambda^2\frac{\varphi}{(1+\varphi\bar\varphi)^2}\frac{\overline{X^1_{\bar z}+iX^2_{\bar z}}}{|X^1_{\bar z}+iX^2_{\bar z}|^2}$$
$$= \frac{4\varphi}{(1+\varphi\bar\varphi)^2}\frac{\overline{X^1_{\bar z}+iX^2_{\bar z}}}{(1-N^3)^2} = \varphi\overline{\left(X^1_{\bar z}+iX^2_{\bar z}\right)}.$$

This implies the last equation of (4.11) by conjugating and substituting using equation (4.13) and the similar formula for $\frac{\partial X^1}{\partial\bar z}$. If $\frac{\partial\varphi}{\partial\bar z} = 0$ then H must be 0. $\square$

PROOF OF KENMOTSU'S REPR. FORMULA. We have shown that for an arbitrary surface Σ having constant mean curvature $H \neq 0$ and complex Gauss map $\varphi\colon \Sigma \to \mathbb{C}$ the following equations hold

$$ds^2 = \left(\frac{2}{H(1+\varphi\bar\varphi)}\left|\frac{\partial\varphi}{\partial\bar z}\right|\right)^2|dz|^2, \tag{4.14}$$

$$\frac{\partial^2\varphi}{\partial z\partial\bar z} - \frac{2\bar\varphi}{1+\varphi\bar\varphi}\frac{\partial\varphi}{\partial z}\frac{\partial\varphi}{\partial\bar z} = 0. \tag{4.15}$$

Similarly, given φ and H, we may construct a surface Σ which is locally parametrized by X using equation (4.2). $\square$

The last thing we need to show is that this representation is unique up to conformal transformations.

THEOREM 4.7. *Let Σ be a simply connected surface in $\mathbb{R}^3$ and $\varphi\colon U \to \mathbb{C}$ and $\widetilde\varphi\colon U \to \mathbb{C}$ be smooth mappings satisfying equation (4.15) for some constant $H \neq 0$. We define a surface X and $\widetilde X$ by Theorem 4.2 i.e.*

$$X = \operatorname{Re}\int_{z_0}^{z} X_z dz \quad \textit{where}$$

$$X_z = \frac{(-1)}{H}\left(\frac{1}{2}\frac{1-\varphi^2}{(1+\varphi\bar\varphi)^2}, i\frac{1+\varphi^2}{(1+\varphi\bar\varphi)^2}, 2\frac{\varphi}{(1+\varphi\bar\varphi)^2}\right)\overline{\frac{\partial\varphi}{\partial\bar z}}$$

Then the following conditions are equivalent:

i. *There exists a holomorphic mapping $\omega = f(z)$ with $f'(z) \neq 0$ on Σ and a motion θ of $\mathbb{R}^3$ such that $\widetilde X \circ f(z) = \theta \circ x(z)$ for $z \in \Sigma$.*

ii. *There exists a holomorphic mapping $\omega = f(z)$ with $f'(z) \neq 0$ on Σ satisfying*

$$\varphi(z) = \widetilde\varphi \circ f(z) \qquad \textit{for} \qquad z \in \Sigma.$$

PROOF. Assume condition i. hold. We may assume $\theta = \mathrm{id}$. Then $X_z = \widetilde X_\omega \circ f'$ and $X_{\bar z} = \widetilde X_{\bar\omega} \circ \overline{f'}$. We have

$$\frac{1}{\lambda}\left(X_x + iX_y\right) = \frac{f'}{|f'|}\frac{1}{\tilde\lambda}\left(\widetilde X_{\tilde x} + i\widetilde X_{\tilde y}\right)$$

and therefore

$$2N(z) = \frac{i}{\lambda^2}\left(X_x + iX_y\right) \times \left(X_x - iX_y\right) = 2\widetilde{N} \circ f(z).$$

Hence $\varphi(z) = \widetilde{\varphi} \circ f(z)$ and ii. follows from equation (4.4).

Conversely suppose condition ii. hold. Then it follows from Theorem 4.2 $\frac{\partial X}{\partial z} = \frac{\partial \widetilde{X}}{\partial \omega} \circ f'(z)$ that

$$X(z) = \widetilde{X}(f(z)) + C,$$

where $C \in \mathbb{R}^3$ is some constant. □

Kenmotsu's representation formula allows us to, given a harmonic mapping $\varphi\colon U \subset \mathbb{C} \to \mathbb{C}$, construct a surface having specified mean curvature and φ as a Gauss map.

EXAMPLE 4.8. Let $\varphi\colon \mathbb{C} - \{0\} \to \mathbb{C}$ be given by $\varphi(z) = -1/\bar{z}$ and $H = 1$. The immersion X obtained via Theorem 4.2 is the standard immersion of the unit sphere (see fig. 4.1) in $\mathbb{R}^3$:

$$X(z) = \frac{1}{1 + z\bar{z}}\left(z + \bar{z}, i(z - \bar{z}), z\bar{z} - 1\right), \qquad z \in \mathbb{C} - \{0\}.$$

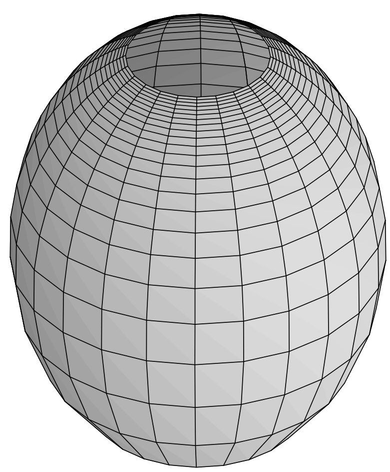

FIGURE 4.1. Unit sphere in $\mathbb{R}^3$.

EXAMPLE 4.9. Let $\varphi\colon \mathbb{C} - \{0\} \to \mathbb{C}$ be given by $\varphi(z) = 1/\bar{z}^2$. The by Theorem 4.2 we have

$$X(z) = \operatorname{Re} \frac{1}{H} \int_{z_0}^{z} \left(\frac{1}{2} \frac{1 - \bar{z}^{-4}}{1 + (z\bar{z})^{-2}}, i\frac{1 + \bar{z}^{-4}}{1 + (z\bar{z})^{-2}}, 2\frac{\bar{z}^{-2}}{1 + (z\bar{z})^{-2}} \right) dz.$$

The corresponding surface of mean curvature $H = 1$ is given in figure 4.2.

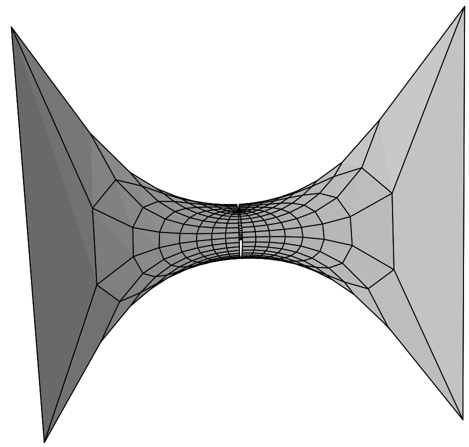

FIGURE 4.2. Surface with $\varphi = 1/\bar{z}^2$ with $H = 1$.

CHAPTER 5

Compact CMC surfaces

In this chapter we prove a result due to Alexandrov [**4**] stating that every compact embedded CMC surface in $\mathbb{R}^3$ is a sphere.

THEOREM 5.1 (Alexandrov's theorem). *Let Σ be a compact embedded surface in $\mathbb{R}^3$ having constant mean curvature $H \neq 0$. Then Σ is a sphere.*

PROPOSITION 5.2 (Willmore's theorem). [**15**] *Let Σ be a compact surface embedded in $\mathbb{R}^3$. Then*

$$\int_\Sigma H^2 \, dA \geq 4\pi$$

with equality if and only if Σ is a round sphere.

PROOF. Let $K^+(p) = \max\{K, 0\}$ then we will show that

$$\int_\Sigma K^+ \, dA \geq 4\pi \tag{5.1}$$

holds for any compact surface in $\mathbb{R}^3$. This follows from the observation that the left hand side of the above equation represents the area of the image under the Gauss map N of the part of Σ having $K \geq 0$. So the only thing we need to prove is that the image of N covers all of S^2. Assume that N is oriented outwards, then the point of Σ having minimal y-component has a normal $(0, -1, 0)$ and $K \geq 0$ (since otherwise it would be a saddle point). But we may orient our surface however we want (while maintaining K) and hence we have shown that there exists a point for every normal direction having $K \geq 0$. The surface area of the unit round sphere is 4π implying equation (5.1).

This gives us

$$\int_\Sigma |K| \, dA \geq \int_\Sigma K^+ \, dA \geq 4\pi.$$

If $K < 0$ for any point then $K < 0$ in a neighborhood of that point and the first inequality is strict. Hence equality implies $K \geq 0$ everywhere and by the previous equation and

$$H^2 = \left(\frac{k_1 + k_2}{2}\right)^2 = k_1 k_2 + \left(\frac{k_1 - k_2}{2}\right)^2 \geq K$$

we have proved the first part of our result. If $H^2 = K$ then $k_1 = k_2$ and we know by Proposition 2.3 that Σ is a round sphere. If Σ is a round sphere then $H^2 = K$ proving our second statement. □

LEMMA 5.3. [**16**] *Let Σ be a compact embedded surface in $\mathbb{R}^3$ bounding a domain D of volume V. If the mean curvature H of Σ is positive everywhere, then*

$$\int_\Sigma \frac{1}{H} dA \geq 3V \tag{5.2}$$

Equality holds if and only if Σ is a round sphere.

PROOF. Let N be the normal of Σ and for every $\varepsilon \in [0,1]$ let Σ_ε be the shell of Σ defined by

$$\Sigma_\varepsilon = \{p + \varepsilon h(p) N \mid p \in \Sigma\},$$

where

$$h(p) = \sup\{r \mid \text{the point } p \text{ is the unique nearest point on } \Sigma \text{ to the point } q \text{ at distance } r \text{ from } p \text{ along the normal } S \text{ at } p\}.$$

The volume of Σ_ε is then given by

$$V = \int_\Sigma F\, dA, \tag{5.3}$$

where

$$F = \int_0^{h(p)} |(1-k_1 t)(1-k_2 t)|\ dt = \int_0^{h(p)} \left|1 - 2Ht + Kt^2\right|\ dt$$

Note that the definition of h we prevents overlap i.e. every point in the interior of Σ_ε lies on a unique normal to Σ.

For every point $q \in D$ it lies in the closed shell $\overline{\Sigma_\varepsilon}$. Let $d = \operatorname{dist}(q, \Sigma)$ then the open ball $B_q(d)$ centered at p with radius d satisfies $B_q(d) \cap \Sigma = \emptyset$. But there exists at least one point $p \in \Sigma$ contained in the boundary of $B_q(d)$. For any q' on the radius from q to p, if r is the distance from q' to p then the closed ball $\overline{B}_{q'}(d)$ is contained in $B_q(d)$ except for the point p. Hence p is the unique point of Σ having distance r from q'. By definition of $h(p)$ we must have $d \leq h(p)$ and hence q lie on the closed shell.

Since all points of D are covered and the only points covered twice are in the image of the boundary we have that the volume V of D is exactly equal to the volume of the shell given by equation (5.3)

The equation

$$\frac{1}{h(p)} \geq \max\{k_1(p), k_2(p)\} \tag{5.4}$$

hold for every $p \in \Sigma$ and we may conclude that

$$(1 - k_1 t), \quad (1 - k_2 t)$$

are non-negative for $0 \leq t \leq h(p)$ and hence

$$F = \int_0^{h(p)} |(1-k_1 t)(1-k_2 t)|\ dt = \int_0^{h(p)} (1-k_1 t)(1-k_2 t)\ dt. \tag{5.5}$$

By the inequalities of geometric and arithmetic mean

$$(1-k_1 t)(1-k_2 t) \leq (1 - Ht)^2 \tag{5.6}$$

and by equation (5.4) we have

$$\frac{1}{h(p)} \geq H.$$

Then

$$F \leq \int_p^{1/H} (1 - Ht)^2 \, dt = \frac{1}{3H}$$

and by equation (5.3) we arrive at the sought equation (5.2).

Equality gives us, by Willmore's theorem (5.2), that Σ is in fact a round sphere. □

PROOF OF ALEXANDROV'S THEOREM. By the divergence formula

$$\iint_\Sigma \langle X, N \rangle \; dA = \iiint_D \operatorname{div} X \; dx\, dy\, dz$$

we have

$$A = H \iint \langle N, X \rangle \; dA = H \iiint \operatorname{div} X \; dx\, dy\, dz = 3HV.$$

Then by Lemma 5.3 the surface Σ must be a sphere since

$$\iint \frac{1}{H} \, dA = \frac{1}{H} \iint dA = \frac{A}{H} = 3V.$$

□

Recent developments

Wente was the first to show that there exists a compact CMC immmersion in $\mathbb{R}^3$ which is not the standard sphere. He provided this by explicitly constructing an immersion of the torus T^2 in $\mathbb{R}^3$ having constant mean curvature.

THEOREM 5.4 (Wente's counterexample). [**5**] *There exists a countable number of isometrically distinct conformal immersion of T^2 into $\mathbb{R}^3$ with constant mean curvature $H \neq 0$.*

The problem whether there exists immersions of genus $g \geq 2$ was addressed by N. Kapouleas in [**7, 8**]. He successfully showed that there exists CMC immersions of every genus $g \geq 2$ by fusing Delaunay surfaces and Wente tori.

THEOREM 5.5. *For any $g \geq 2$ there exists infinitely many smooth CMC-surfaces immersed in $\mathbb{R}^3$ having constant mean curvature $H = 1$ and genus g.*

A recent development in the field is the DPW method [**17, 18**] named after its creators J. Dorfmeister, F. Pedit and H. Wu. This method gives a description of all immersed CMC surfaces in $\mathbb{R}^3$ with or without umbilics. It is often characterized as a Weierstrass type method as it allows the construction of CMC-immersions from a meromorphic and a holomorphic function.

Harmonic maps

Harmonic mappings arise as critical points of a certain variational problem. Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map between two Riemannian manifolds of dimension m and n, respectively. We then define the *energy density function*, $e(\varphi)$, of φ at $p \in M$ by

$$e(\varphi) = \frac{1}{2}\,\mathrm{trace}_g(\varphi^* h) = \frac{1}{2}\sum_{i=1}^{m} h(d\varphi(e_i), d\varphi(e_i)) = \frac{1}{2}\sum_{i=1}^{m} \|d\varphi(e_i)\|_h^2,$$

where $\varphi^* h$ is the pull back of h via φ and $\{e_i\}_{i=1}^m$ is a local orthonormal frame of T_pM with respect to g_p. Let $p \in M$,

$$(x_1, x_2, \ldots, x_m) \quad \text{and} \quad (y_1, y_2, \ldots, y_n)$$

be local coordinates around p and $\varphi(p) \in N$ and for $\alpha = 1, 2, \ldots, n$ put $\varphi^\alpha = y_\alpha \circ \varphi$. Then at each point q in a neighborhood of p

$$e(\varphi)(q) = \frac{1}{2}\sum_{i,j=1}^{m}\left\{ g^{ij}(q) \sum_{\alpha,\beta=1}^{n} h_{\alpha\beta}(\varphi(q)) \frac{\partial \varphi^\alpha}{\partial x_i}(q) \frac{\partial \varphi^\beta}{\partial x_j}(q) \right\},$$

where (g^{ij}) is the inverse of the matrix (g_{ij}) with

$$g_{ij} = g(e_i, e_j)$$

We define the *energy* or *action integral* of φ on M by

$$E(\varphi) = \int_M e(\varphi)\, d\mathrm{vol}_g,$$

where $\mathrm{vol}_g = \det(g_{ij}) dx_1 \cdots dx_n$ is the volume form on M.

A map $F\colon (-\varepsilon, \varepsilon) \times M \to N$ is said to be a *smooth variation* of φ if it satisfies

$$\begin{cases} F(0,p) = \varphi(p) & p \in M, \\ F\colon (-\varepsilon, \varepsilon) \times M \to N & \text{of class } C^\infty. \end{cases}$$

For every $t \in (-\varepsilon, \varepsilon)$ we define the smooth variation φ_t of φ by

$$\varphi_t(p) = F(t,p), \qquad p \in M.$$

DEFINITION A.1. A map $\varphi\colon (M,g) \to (N,h)$ is said to be a *harmonic mapping* if φ is a critical point to E at $C^\infty(M,N)$ i.e. for any smooth variation $\varphi_t\colon M \to N$ with $-\varepsilon < t < \varepsilon$ and $\varphi_0 = \varphi$ we have

$$\frac{d}{dt}E(\varphi_t)\bigg|_{t=0} = 0$$

Let $\varphi^{-1}TN$ denote the *pull-back* of the tangent bundle TN of N via φ,

$$\varphi^{-1}TN = \left\{(p,u) \in M \times TN \mid p \in M,\ u \in T_{\varphi(p)}M\right\}.$$

The *variation vector field* V along φ defined by

$$V(p) = \frac{d}{dt}\varphi_t(p)\bigg|_{t=0} \quad \text{for all} \quad p \in M,$$

is a C^∞ mapping from M into TN satisfying

$$V(p) \in T_{\varphi(p)}N \quad \text{for all} \quad p \in M.$$

We may then define a connection ∇^* called the *pull-back connection*, on the set of smooth sections of $X \in \varphi^{-1}TN$ by

$$(\nabla^*_X V)(p) = {}^N\nabla_{d\varphi(X)}V = \frac{d}{dt}{}^N P^{-1}_{\varphi\circ\sigma_t} V(\sigma(t))\bigg|_{t=0}, \quad p \in M,$$

where $t \mapsto \sigma(t) \in M$ is a C^1 curve in M satisfying $\sigma(0) = p$, $\sigma'(0) = X_p \in T_pM$, and σ_t is a curve given by $\sigma_t(s) = \sigma(s)$, $0 \le s \le t$ that is, the restriction of σ to the part between p and $\sigma(t)$. The map ${}^N P_{\varphi\circ\sigma_t}\colon T_{\varphi(p)}N \to T_{\varphi(\sigma(t))}N$ is the parallel transport along a C^1 curve $\varphi\circ\sigma_t$ with respect to the Levi-Civita connection ${}^N\nabla$ on N.

FACT A.2 (First variational formula). *Let $\varphi\colon (M,g) \to (N,h)$ be a smooth map. For a smooth variation φ_t of φ put*

$$V(p) = \frac{d}{dt}\varphi_t(p)\bigg|_{t=0}, \qquad p \in M$$

then

$$\frac{d}{dt}E(\varphi_t)\bigg|_{t=0} = -\int_M h(V, \tau(\varphi))d\mathrm{vol}_M$$

Thus φ is a harmonic mapping if and only if

$$\tau(\varphi)(p) = 0 \quad \textit{for all} \quad p \in M.$$

This is known as the *Euler-Lagrange equation* for harmonic maps. The section $\tau(\varphi)$ of the pull-back bundle $\varphi^{-1}TN$ is called the *tension field* of φ and is given by

$$\tau(\varphi)(p) = \sum_{i=1}^{m}\left(\nabla^*_{e_k}d\varphi(e_k) - d\varphi(\nabla_{e_k}e_k)\right). \tag{A.1}$$

The tension field can be given in local coordinates by the following: let $(x_1, x_2, \ldots, x_m)$ and $(y_1, y_2, \ldots, y_n)$ be local coordinates around p and $\varphi(p)$ in M and N, respectively. Then $\tau(\varphi)$ is given by

$$\tau(\varphi) = \sum_{\gamma=1}^{n}\tau(\varphi)^\gamma\frac{\partial}{\partial y_\gamma}, \quad \text{where} \tag{A.2}$$

$$\tau(\varphi)^\gamma(p) = \sum_{i,j=1}^{m} g^{ij}\left\{\frac{\partial^2\varphi^\gamma}{\partial x_i\partial x_j} - \sum_{k=1}^{m}\Gamma^k_{ij}(p)\frac{\partial\varphi^\gamma}{\partial x_k} + \sum_{\alpha,\beta=1}^{n}{}^N\Gamma^\gamma_{\alpha\beta}(\varphi(p))\frac{\partial\varphi^\alpha}{\partial x_i}\frac{\partial\varphi^\beta}{\partial x_j}\right\}.$$

Here Γ_{ij}^{k}, ${}^{N}\Gamma_{\alpha\beta}^{\gamma}$ are the Christoffel symbols on (M^m, g) and (N^n, h), respectively.

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