Polynomial Harmonic Morphisms

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Contents

Abstract

The aim of this Master's thesis is to be the first survey of known results on polynomial harmonic morphims between Euclidean spaces. These were first studied by Baird in [**4**] in the early 1980s. He obtained several results on the subject but left open the, still unsolved, classification of such maps. In the article [**25**] from 1995, Eells and Yiu classified the homogeneous polynomial harmonic morphisms whose restrictions to spheres are again harmonic morphisms to spheres. These are the well known Hopf polynomials of degree 2. This result revitalized the subject and soon thereafter, Ou obtained a complete classification of the homogeneous polynomial harmonic morphisms of degree 2. During the preparation of this thesis, a very interesting development has taken place with Ababou, Baird and Brossard writing the article [**1**], proving that this is still a very active area of research.

In Chapter 1 we discuss the Weierstrass representation of minimal surfaces as a motivating example and the history of general harmonic morphisms, beginning with Jacobi some 150 years ago.

Chapters 2 and 3 are devoted to the introduction of harmonic maps and harmonic morphisms where we also derive some of their basic properties.

In Chapter 4 we then study polynomial harmonic morphisms. We show that every globally defined harmonic morphism between Euclidean spaces of sufficiently high dimensions is necessarily polynomial. We give the complete classification due to Ou of those homogeneous of degree 2 and discuss some examples of higher degree. A general method for constructing non-trivial examples is provided and we make a conjecture on the structure of polynomial harmonic morphisms based on known results on those of degree 2.

In Chapter 5 we use the results of Chapter 4 to give a new proof of the above mentioned result by Eells and Yiu regarding homogeneous polynomial harmonic morphisms between Euclidean spheres. Finally, we show how the results derived so far can be used to give information concerning the singularities of general harmonic morphisms.

It has been my firm intention throughout this work to give references to the stated results and credit to the work of others. The only results I claim are mine will appear in chapter 4 and 5 and have been marked with an asterix [$*$]. Any statement, example or proof left unmarked, is considered to be too well known for a reference to be given.

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CHAPTER 1

Introduction

1. Motivation

The study of harmonic morphisms involves to a large extent the study of harmonicity and minimal submanifolds, two concepts which themselves are strongly related. In this section we illustrate this relationship with an example from classical differential geometry relating the mean curvature of a surface in $\mathbb{R}^3$ to the Laplacian of the coordinate functions. For details see [**19**].

A *regular parametrized surface* is a C^2-map

$$X : U \subseteq \mathbb{C} \to \mathbb{R}^3$$

where U is open and connected. The map X is assumed to have injective differential, so that along the image $X(U)$ in $\mathbb{R}^3$ we have a normal vector field

$$N = \frac{\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}}{\left|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right|}.$$

It is customary to use the following notation:

$$E = \left|\frac{\partial X}{\partial u}\right|^2, \quad F = \left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \right\rangle, \quad G = \left|\frac{\partial X}{\partial v}\right|^2,$$

and

$$e = \left\langle N, \frac{\partial^2 X}{\partial u^2} \right\rangle, \quad f = \left\langle N, \frac{\partial^2 X}{\partial u \partial v} \right\rangle, \quad g = \left\langle N, \frac{\partial^2 X}{\partial v^2} \right\rangle.$$

Then the *mean curvature* of the surface is defined as

$$H = -\frac{1}{2}\mathrm{trace}(dN) = \frac{1}{2}\frac{eG + gE - 2fF}{EG - F^2}$$

and X is said to be a parametrized *minimal* surface if the mean curvature vanishes everywhere.

Now let V be a relatively compact subset of the domain U and h be a C^1-function on $\bar{V}$. Then

$$X_t(u,v) = X(u,v) + th(u,v)N(u,v)$$

is called *the normal variation of $X(\bar{V})$ determined by h.* The following very geometric result motivates the name minimal (see do Carmo [**17**] for a proof):

Theorem 1.1. *Let $X : U \to \mathbb{R}^3$ be a regular parametrized C^2-surface. Then X is minimal if and only if for every bounded $V \subset \bar{V} \subset U$ and every normal variation X_t of $X(\bar{V})$ we have*

$$\frac{d}{dt}\text{Area}(X_t(\bar{V}))\Big|_{t=0} = 0.$$

It is well known that every regular C^2-surface in $\mathbb{R}^3$ may locally be parametrized by *isothermal* coordinates i.e. coordinates for which $E = G$ and $F = 0$. Let us therefore assume that X is isothermal. Then we have for the Laplacian $\triangle(X)$ of X:

$$\begin{aligned}\Big\langle \frac{\partial X}{\partial u}, \triangle(X)\Big\rangle &= \Big\langle \frac{\partial X}{\partial u}, \frac{\partial^2 X}{\partial u^2}\Big\rangle + \Big\langle \frac{\partial X}{\partial u}, \frac{\partial^2 X}{\partial v^2}\Big\rangle \\ &= \Big\langle \frac{\partial X}{\partial u}, \frac{\partial^2 X}{\partial u^2}\Big\rangle - \Big\langle \frac{\partial^2 X}{\partial v \partial u}, \frac{\partial X}{\partial v}\Big\rangle \\ &= \frac{1}{2}\frac{\partial}{\partial u}\Big|\frac{\partial X}{\partial u}\Big|^2 - \frac{1}{2}\frac{\partial}{\partial u}\Big|\frac{\partial X}{\partial v}\Big|^2 \\ &= 0\end{aligned}$$

and similarly $\Big\langle \frac{\partial X}{\partial v}, \triangle(X)\Big\rangle = 0$. Hence $\triangle(X)$ is normal to the surface and

$$H = \frac{e+g}{2E} = \frac{\langle N, \triangle(X)\rangle}{2E}.$$

This implies the following.

Theorem 1.2. *If $X : U \to \mathbb{R}^3$ is a parametrized C^2-surface with $|\frac{\partial X}{\partial u}|^2 = |\frac{\partial X}{\partial v}|^2$ and $\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\rangle = 0$, then X is minimal if and only if X is harmonic.*

In the spirit of Theorem 1.2 a minimal parametrized surface can be *defined* as a map

$$X : U \subseteq \mathbb{C} \to \mathbb{R}^3$$

satisfying

$$\Big|\frac{\partial X}{\partial u}\Big|^2 = \Big|\frac{\partial X}{\partial v}\Big|^2, \quad \Big\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}\Big\rangle = 0$$

and

$$\triangle(X) = 0.$$

If we in addition to this assume that U is simply connected, it follows from elementary complex analysis that there exists a holomorphic map $\Psi : U \to \mathbb{C}^3$ such that

$$X = \text{Re } \Psi.$$

That X is isothermal is then equivalent to

$$(\frac{\partial \Psi_1}{\partial z})^2 + (\frac{\partial \Psi_2}{\partial z})^2 + (\frac{\partial \Psi_3}{\partial z})^2 = 0.$$

Choosing Ψ suitably leads us to the famous representation by Weierstrass:

Theorem 1.3 (The Weierstrass Representation)**.** [**57**] *Let U be an open, simply connected subset of $\mathbb{C}$ and $X : U \to \mathbb{R}^3$ a parametrized surface satisfying*

$$\left|\frac{\partial X}{\partial u}\right|^2 = \left|\frac{\partial X}{\partial v}\right|^2, \quad \left\langle \frac{\partial X}{\partial u}, \frac{\partial X}{\partial v} \right\rangle = 0$$

and

$$\triangle(X) = 0.$$

Then there is exists pair of meromorphic functions f, g in U such that f and fg^2 are holomorphic, $f, g \not\equiv 0$ and

$$X(z) = X(z_0) + Re \int_{z_0}^{z} f(w)((1 - g(w)^2), i(1 + g(w)^2), 2g(w))dw$$

for all $z_0 \in U$. Conversely, every pair f, g of meromorphic functions as above define a minimal parametrized surface in this way.

In the next section we shall see how the results presented here demonstrate a certain *duality* between minimal conformal immersions and harmonic morphisms.

2. History

The history of harmonic morphisms is generally thought to have begun with the article [**44**] of Jacobi from 1847 on the solutions of Laplace's equation in three dimensions. Here Jacobi investigated necessary conditions for a complex valued function ϕ, defined on an open subset of $\mathbb{R}^3$ such that for any holomorphic function f, the composition $f \circ \phi$ is harmonic i.e.

$$\triangle(f \circ \phi) = 0.$$

A *harmonic morphism* though ought to be a map that in some sense preserves a *harmonic structure.* It was for that purpose, more than a century after Jacobi, that harmonic morphisms were formally introduced by Constantinescu and Cornea in [**18**] in the context of *harmonic spaces* in abstract potential theory.

In general, a harmonic space (in the sense of Brelot, see [**15**]) is a locally compact Hausdorff space $\mathcal{X}$ endowed with a sheaf $\mathcal{H}$, assigning to each open subset U of $\mathcal{X}$ a real subspace $\mathcal{H}(U)$ of the continuous functions on U such that the following conditions are satisfied:

1. $\mathcal{X}$ has an open base for its topology consisting of *regular* sets. A regular set is an open, relatively compact subset V of $\mathcal{X}$ with non-empty boundary ∂V, such that for every continuous function f on ∂V, there is a unique element $H_f^V \in \mathcal{H}(V)$ which can be extended to $\bar{V}$ and equals f on ∂V. Furthermore, if $f \geq 0$ then $H_f^V \geq 0$.
2. If $U \subseteq \mathcal{X}$ is open and connected and $\{u_\alpha\}_{\alpha \in A}$ is an up-directed family in $\mathcal{H}(U)$, then either $\sup_{\alpha \in A} u_\alpha$ is in $\mathcal{H}(U)$ or $\sup_{\alpha \in A} u_\alpha \equiv +\infty$.

For an open subset U of $\mathcal{X}$ we call $\mathcal{H}(U)$ the *harmonic functions on* U. It is well known that $\mathbb{R}^n$ is a harmonic space with the harmonic functions as solutions to Laplace's equation. More generally, every Riemannian manifold is a harmonic space with the harmonic functions as zeros to the Laplace-Beltrami operator. These results are essentially due to R. M. Hervé who showed (see [**40**], Chapter 7) that the solutions to a uniformly elliptic equation

$$\sum_{i,k=1}^{m} a_{ik} \frac{\partial^2 f}{\partial x_i \partial x_k} + \sum_{i=1}^{m} b_i \frac{\partial f}{\partial x_i} + cf = 0$$

with coefficients a_{ik}, b_i and c locally Lipschitz in a domain $\Omega \subseteq \mathbb{R}^m$ defines a system satisfying the axioms of a harmonic space.

As defined by Constantinescu and Cornea in [**18**], a *harmonic morphism* is a continuous map

$$\phi : \mathcal{X} \to \mathcal{X}'$$

between harmonic spaces $\mathcal{X}$ and $\mathcal{X}'$ such that for every open $U \subseteq \mathcal{X}'$ and harmonic function f on U, the composition

$$f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$$

is harmonic. Since every harmonic function on $\mathbb{C}$ is locally the real part of a holomorphic function, we see that this is exactly what Jacobi was investigating. The aim of Constantinescu and Cornea was to generalize results from the theory of Riemann surfaces to harmonic spaces, with harmonic morphisms replacing the holomorphic maps.

Some decade after Constantinescu and Cornea's article, Fuglede and Ishihara published, independently, their investigations on harmonic morphisms in Riemannian geometry (see [**28**] and [**43**]). Their results showed that in the special case when the harmonic spaces are Riemannian manifolds, the harmonic morphisms are rich in geometric features, with several interesting applications and problems.

If we return to Jacobi for a while, assume that $\phi : \Omega \to \mathbb{C}$ is a harmonic morphism, where $\Omega \subseteq \mathbb{R}^3$ is open. By choosing $f(w) = w$ for $w \in \mathbb{C}$ we see that ϕ is in fact smooth. Furthermore for a holomorphic function f, whenever the composition is defined, we have:

$$\begin{aligned} 0 &= \triangle(f \circ \phi) \\ &= \frac{\partial^2 f}{\partial w^2}\Big((\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2\Big) + \frac{\partial f}{\partial w}\triangle(\phi). \end{aligned}$$

Since f may be chosen arbitrary we see, in this case, that the following two conditions are necessary and sufficient for ϕ to be a harmonic morphism:

$i)$ The map ϕ is harmonic, that is

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0,$$

$ii)$ the map ϕ satisfies

$$(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 + (\frac{\partial \phi}{\partial z})^2 = 0.$$

These conditions were obtained by Jacobi and both Fuglede and Ishihara noticed that they have natural generalizations to the case when $\phi : (M, g) \to (N, h)$ is a map between arbitrary Riemannian manifolds. These generalized conditions together remain necessary and sufficient for ϕ to be a harmonic morphism.

The condition $i)$ says that ϕ must be a *harmonic map*. Such maps were introduced as maps $\phi : (M, g) \to (N, h)$ which are, in the sense of the calculus of variations, the critical points of the *energy functional*

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 \nu.$$

The manifolds M and N are here assumed to be compact and oriented, but the Euler-Lagrange equation of this variational problem makes it possible to define harmonic maps between arbitrary Riemannian manifolds. The first formal definition of a harmonic map was given by Fuller in the year 1954 in [**30**] after some preliminary work by Bochner and Morrey. A thorough investigation of harmonic maps was also conducted by Eells and Sampson some decade later in their celebrated article [**24**].

If we write $\phi = \phi_1 + i\phi_2$, in the special case of $N = \mathbb{C}$, then the second condition $ii)$ obtained by Jacobi is equivalent to

$$|\mathrm{grad}(\phi_1)|^2 = |\mathrm{grad}(\phi_2)|^2, \quad \langle \mathrm{grad}(\phi_1), \mathrm{grad}(\phi_2) \rangle = 0.$$

This means that for $x \in \Omega$, either $d\phi_x = 0$ or $\mathrm{grad}(\phi_1)$ and $\mathrm{grad}(\phi_2)$ span a 2-dimensional subspace of $T_x\mathbb{R}^3 \cong \mathbb{R}^3$ which is mapped conformally onto $T_{\phi(x)}\mathbb{C} \cong \mathbb{C}$ by $d\phi_x$. This is expressed by saying that ϕ is *horizontally weakly conformal.*

If we compare the equations $i)$ and $ii)$ with those obtained in the previous section for minimal isothermal parametrized surfaces, we see the duality mentioned earlier: the concept of a harmonic morphism is in a sense dual to that of a harmonic conformal map.

The harmonicity and the weak horizontal conformality give the theory of harmonic morphisms both analytic and geometric dimensions but also make the question of existence very difficult, since we are dealing with an over-determined non-linear system of partial differential equations.

Soon after Fuglede and Ishihara, several mathematicians followed in the study of harmonic morphisms. To solve the question of existence, attempts were made to classify harmonic morphisms in different contexts. Jacobi had himself investigated the conditions for a function $F = F(x, y, z, w)$ where $(x, y, z, w) \in \mathbb{R}^3 \times \mathbb{C}$, such that every local solution $w = \phi(x, y, z)$ to the equation

$$F(x, y, z, w) = 0$$

is a harmonic morphism. He proved that this is true if F is holomorphic in w and a harmonic morphism in the first three variables. In particular he studied the case when the equation is given by

$$A(w)x + B(w)y + C(w)z = 1,$$

where A, B and C are holomorphic functions satisfying

$$A^2 + B^2 + C^2 = 0.$$

In the late 1980's, Baird and Wood gave in [**8**] a complete classification of harmonic morphisms from domains of $\mathbb{R}^3$ to any Riemann surface N^2. This was one of the first classification result for harmonic morphisms and could be seen as a complete solution to the problem posed by Jacobi. Their result is essentially that every harmonic morphism ϕ :

$\Omega \subseteq \mathbb{R}^3 \to N^2$ arises as a local solution $\sigma = \phi(x, y, z)$ of an equation of the kind:

$$\langle f(w)(1 - g(w)^2), i(1 + g(w)^2), 2g(w)), (x, y, z)\rangle = 1,$$

for two meromorphic functions $f, g : N^2 \to \mathbb{C} \cup \{\infty\}$. With the duality mentioned earlier in mind, this should be compared with Theorem 1.3. After this result was published, the theory of harmonic morphisms has grown rapidly as can be seen on the Bibliography of Harmonic Morphisms [**38**]. At present, Baird and Wood are writing the first book [**11**] on the subject.

CHAPTER 2

Harmonic Maps

Throughout this work we shall by a Riemannian manifold (M^m, g) mean a smooth (i.e. C^∞), real *connected* manifold of dimension m together with a smooth Riemannian metric g. The dimension m is always meant to be finite. We denote by ∇^M the Levi-Civita connection of M associated with the metric g and by ${}^M\Gamma_{ij}^k$ its coefficients. The letters M, N and P are reserved to mean Riemannian manifolds. All maps $M \to N$ and functions $M \to \mathbb{R}$ are unless otherwise stated understood to be smooth and so is any section of a smooth vector bundle $\xi : V \to M$ over M. We denote by $C^\infty(V)$ the totality of such sections.

In this chapter we present the basic notion of a harmonic map between Riemannian manifolds. To do this in an invariant way, we first introduce the second fundamental form of a map in terms of vector bundles and sections, thus relating it to the common notion of the second fundamental form of an immersion and its mean curvature vector. For a deeper exposition of harmonic maps we refer the reader to the reports [**20**], [**21**], [**22**] and the books [**56**] and [**60**].

1. The Second Fundamental Form

In this section we shall define the pull-back bundle of a map between Riemannian manifolds and equip it with a suitable Riemannian metric together with a compatible connection.

Definition 2.1. Let $\phi : M \to N$ be a map. The *pull-back bundle of* ϕ is the bundle $\eta : \phi^{-1}(TN) \to M$ over M with

$$\phi^{-1}(TN) = \{(x, v) \mid x \in M,\ v \in T_{\phi(x)}N\}$$

and

$$\eta(x, v) = x \text{ for } x \in M,\ v \in T_{\phi(x)}N.$$

Thus $\phi^{-1}(TN)$ is the *induced vector bundle of* TN *by* ϕ. Obviously, if n is the dimension of N, this is an n-dimensional vector bundle over M: for $x \in M$ choose a neighbourhood U of $\phi(x)$ in N and a smooth trivialization $\psi : U \times \mathbb{R}^n \to \pi^{-1}(U)$, where $\pi : TN \to N$ is the canonical projection. Then

$$\phi^{-1}(U) \times \mathbb{R}^n \ni (y, v) \mapsto (y, \psi(\phi(y), v)) \in \eta^{-1}(\phi^{-1}(U))$$

is a smooth trivialization of $\eta^{-1}(\phi^{-1}(U))$. A section $V \in C^\infty(\phi^{-1}(TN))$ of the pull-back bundle is by definition a map

$$V : M \to \phi^{-1}(TN)$$

such that

$$V_x \in T_{\phi(x)}N$$

for every x in M. Thus for $Z \in C^\infty(TN)$, $x \mapsto Z_{\phi(x)}$ is an element of $C^\infty(\phi^{-1}(TN))$ denoted by $\phi^*(Z)$ or simply Z. Another important example of a section of $\phi^{-1}(TN)$ is the map

$$M \ni x \mapsto d\phi_x(X_x)$$

for a section $X \in C^\infty(TM)$ of the tangent bundle of M.

Definition 2.2. By a *smooth variation of* $\phi : M \to N$ we mean a family ϕ_t of maps $\phi_t : M \times (-\epsilon, \epsilon) \to N$, $\epsilon > 0$, such that $\phi_0 = \phi$.

If ϕ_t is a smooth variation of ϕ then

$$M \ni x \mapsto \frac{\partial \phi_t(x)}{\partial t}\Big|_{t=0} \in T_{\phi_0(x)}N = T_{\phi(x)}N$$

is a section of $\phi^{-1}(TN)$. Conversely, for a section $V \in C^\infty(\phi^{-1}(TN))$ define a family $\phi_t(x) = \exp_{\phi(x)}(tV_x)$. If N is complete this will be defined throughout $M \times \mathbb{R}$ and

$$\frac{\partial \phi_t(x)}{\partial t}\Big|_{t=0} = d(\exp_{\phi(x)})_0(V_x) = V_x.$$

Thus we have:

Proposition 2.3. *If N is complete then for a map $\phi : M \to N$, every section in $C^\infty(\phi^{-1}(TN))$ is of the form*

$$M \ni x \mapsto \frac{\partial \phi_t(x)}{\partial t}\Big|_{t=0} \in T_{\phi(x)}N$$

for some smooth variation ϕ_t of ϕ.

Since we have $V_x \in T_{\phi(x)}N$ for a section $V \in C^\infty(\phi^{-1}(TN))$ and $x \in M$, we may define a Riemannian metric, also denoted by h, on $\phi^{-1}(TN)$ by

$$h(V, W)(x) = h_{\phi(x)}(V_x, W_x)$$

for $x \in M$ and sections $V, W \in C^\infty(\phi^{-1}(TN))$ of the pull-back bundle. Thus we have made $\phi^{-1}(TN)$ into a smooth Riemannian vector bundle over M. Our next step is to define a connection on $\phi^{-1}(TN)$ compatible with h. For $X, Y \in C^\infty(TM)$ and $x \in M$ choose a curve $\gamma : (-\epsilon, \epsilon) \to M$ with $\gamma(0) = x$ and $\gamma'(0) = X_x$. Let $P_{\gamma,t} : T_xM \to T_{\gamma(t)}M$ denote parallel transport along γ. Then from the compatibility of the Levi-Civita connection on M we have

$$\nabla^M_X Y(x) = \frac{d}{dt}\Big|_{t=0} P^{-1}_{\gamma,t}(Y_{\gamma(t)}). \tag{2.1}$$

It is therefore natural to make the following definition:

Definition 2.4. The *pull-back connection of* $\phi : M \to N$ is the connection

$$\nabla^\phi : C^\infty(TM) \times C^\infty(\phi^{-1}(TN)) \to C^\infty(\phi^{-1}(TN))$$

on $\phi^{-1}(TN)$ defined by

$$\nabla^\phi_X V(x) = \frac{d}{dt}\Big|_{t=0} P^{-1}_{\phi\circ\gamma,t}(V_{\gamma(t)}),$$

for $x \in M$, $X \in C^\infty(TM)$, $V \in C^\infty(\phi^{-1}(TN))$ and $\gamma : (-\epsilon, \epsilon) \to M$ a curve with $\gamma(0) = x$, $\gamma'(0) = X_x$. Here $P_{\phi\circ\gamma,t} : T_{\phi(x)}N \to T_{\phi(\gamma(t))}N$ is the parallel transport along $\phi \circ \gamma$.

It is a direct consequence of Definition 2.4 and equation (2.1) that for $Z \in C^\infty(TN)$ we have:

$$\nabla^\phi_X \phi^*(Z)(x) = \frac{d}{dt}\Big|_{t=0} P^{-1}_{\phi\circ\gamma,t}(Z_{\phi\circ\gamma(t)}) = \phi^*(\nabla^N_{d\phi_x(X)} Z)(x). \tag{2.2}$$

It is easy to see that ∇^ϕ is a well defined connection on $\phi^{-1}(TN)$ and uniquely determined by equation (2.2). Furthermore it is an easy consequence of the fact that parallel transport is an isometry that ∇^ϕ is compatible with the metric h on $\phi^{-1}(TN)$ (see [**21**], page 4 and [**56**], page 126).

Proposition 2.5. *If $\phi : M \to N$ is a map and X, $Y \in C^\infty(TM)$, then*

$$\nabla^\phi_X d\phi(Y) - \nabla^\phi_Y d\phi(X) - d\phi([X, Y]) = 0.$$

PROOF. Since the left hand side is tensorial in X and Y, it is enough to prove the statement for $X = \frac{\partial}{\partial x^i}$ and $Y = \frac{\partial}{\partial x^j}$ for local coordinates (x^k) around $x \in M$. For that purpose we choose local coordinates (y^α)

on N around $y = \phi(x)$. Then

$$\begin{aligned}
\nabla^{\phi}_{\frac{\partial}{\partial x^i}} d\phi(\frac{\partial}{\partial x^j}) - \nabla^{\phi}_{\frac{\partial}{\partial x^j}} d\phi(\frac{\partial}{\partial x^i}) &= \sum_\alpha \nabla^{\phi}_{\frac{\partial}{\partial x^i}} \frac{\partial \phi^\alpha}{\partial x^j}\frac{\partial}{\partial y^\alpha} - \sum_\alpha \nabla^{\phi}_{\frac{\partial}{\partial x^j}} \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial}{\partial y^\alpha} \\
&= \sum_\alpha (\frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j} - \frac{\partial^2 \phi^\alpha}{\partial x^j \partial x^i})\frac{\partial}{\partial y^\alpha} \\
&\quad + \sum_\alpha (\frac{\partial \phi^\alpha}{\partial x^j} \nabla^{\phi}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} - \frac{\partial \phi^\alpha}{\partial x^i} \nabla^{\phi}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial y^\alpha}).
\end{aligned}$$

The symmetry of the second derivatives implies that the first sum in the last expression vanishes. The second is also zero since by equation (2.2):

$$\begin{aligned}
\sum_\alpha \frac{\partial \phi^\alpha}{\partial x^j} \nabla^{\phi}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial y^\alpha} &= \sum_{\alpha,\beta} \frac{\partial \phi^\alpha}{\partial x^j}\frac{\partial \phi^\beta}{\partial x^i} \nabla^{N}_{\frac{\partial}{\partial y^\beta}} \frac{\partial}{\partial y^\alpha} \\
&= \sum_{\alpha,\beta} \frac{\partial \phi^\alpha}{\partial x^j}\frac{\partial \phi^\beta}{\partial x^i} \nabla^{N}_{\frac{\partial}{\partial y^\alpha}} \frac{\partial}{\partial y^\beta} \\
&= \sum_{\beta} \frac{\partial \phi^\beta}{\partial x^i} \nabla^{\phi}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial y^\beta}.
\end{aligned}$$

□

On the cotangent bundle T^*M of M we have a metric g^* obtained by identifying $\lambda, \sigma \in T^*_x M$ with their inverse images in $T_x M$ under the isomorphism

$$T_x M \ni Z \mapsto g_x(Z, \ \cdot\ ) \in T^*_x M.$$

Thus if $\lambda = g_x(X, \ \cdot\ )$ and $\sigma = g_x(Y, \ \cdot\ )$ for some $X, Y \in T_x M$, then

$$g^*_x(\lambda, \sigma) = g_x(X, Y).$$

On the tensor product $T^*M \otimes \phi^{-1}(TN)$ we may then define a metric $\langle\ \cdot\ ,\ \cdot\ \rangle$ by

$$\langle \lambda \otimes V, \sigma \otimes W\rangle(x) = g^*(\lambda, \sigma) h_{\phi(x)}(V, W)$$

for $\lambda, \sigma \in C^\infty(T^*M)$ and $V, W \in C^\infty(\phi^{-1}(TN))$. Since the differential $d\phi$ of ϕ is a section of $T^*M \otimes \phi^{-1}(TN)$ we get by definition

$$\begin{aligned}
\langle d\phi, d\phi\rangle(x) &= \sum_i h_{\phi(x)}(d\phi_x(e_i), d\phi_x(e_i)) \\
&= \text{trace}_g \phi^* h(x),
\end{aligned}$$

where (e_i) is any orthonormal base of the tangent space T_xM.

Recall that on T^*M we have a connection ∇^* dual to ∇^M given by

$$(\nabla^*_X\sigma)(Y) = X(\sigma(Y)) - \sigma(\nabla^M_X Y)$$

for $X, Y \in C^\infty(TM)$ and $\sigma \in C^\infty(T^*M)$, i.e. ∇^* is the ordinary *covariant differential* of 1-forms on M. Thus we may define a connection on $C^\infty(T^*M \otimes \phi^{-1}(TN))$ by the following:

Definition 2.6. For a map $\phi : M \to N$, $\hat{\nabla}$ is the connection on $C^\infty(T^*M \otimes \phi^{-1}(TN))$ given by

$$\hat{\nabla}_X(\lambda \otimes V) = (\nabla^*_X\lambda) \otimes V + \lambda \otimes (\nabla^\phi_X V)$$

for $X \in C^\infty(TM)$, $\lambda \in C^\infty(T^*M)$ and $V \in C^\infty(\phi^{-1}(TN))$.

That $\hat{\nabla}$ is a well defined connection on $T^*M \otimes \phi^{-1}(TN)$ is clear and it is easy to verify that it will be compatible with the metric $\langle \cdot\, , \,\cdot \rangle$.

Definition 2.7. For $\phi : M \to N$ the *second fundamental form of* ϕ is the covariant derivative $\hat{\nabla} d\phi$ of $d\phi$ by $\hat{\nabla}$.

By definition we have:

$$\hat{\nabla} d\phi(X, Y) = (\hat{\nabla}_X d\phi)(Y) = \nabla^\phi_X d\phi(Y) - d\phi(\nabla^M_X Y) \tag{2.3}$$

for $X, Y \in C^\infty(TM)$. Using Proposition 2.5 it is easy to see that the second fundamental form

$$\hat{\nabla} d\phi : C^\infty(TM) \times C^\infty(TM) \to C^\infty(\phi^{-1}(TN))$$

is symmetric and *tensorial* i.e. bi-linear over the ring of smooth functions $M \to \mathbb{R}$.

Definition 2.8. For a map $\phi : M \to N$ the *tension field of* ϕ is the trace of the second fundamental form of ϕ:

$$\tau(\phi) = \operatorname{trace}(\hat{\nabla} d\phi).$$

For maps $\phi : M \to N$ and $\psi : N \to P$ we write $d\psi(\hat{\nabla} d\phi)$ for the section $d\psi(\hat{\nabla} d\phi(\cdot))$ and $\hat{\nabla} d\psi(d\phi, d\phi)$ for $\hat{\nabla} d\psi(d\phi(\cdot), d\phi(\cdot))$. From the chain rule we now deduce the following:

Proposition 2.9. *If $\phi : M \to N$ and $\psi : N \to P$ are maps between Riemannian manifolds, then*

$$\hat{\nabla} d(\psi \circ \phi) = d\psi(\hat{\nabla} d\phi) + \hat{\nabla} d\psi(d\phi, d\phi)$$

and

$$\tau(\psi \circ \phi) = d\psi(\tau(\phi)) + \operatorname{trace}(\hat{\nabla} d\psi(d\phi, d\phi)).$$

2. Harmonic Maps

We now have the proper tools for defining the concept of a harmonic map between Riemannian manifolds and to derive its fundamental properties.

Definition 2.10. Let M and N be Riemannian manifolds. A map $\phi : M \to N$ is said to be *harmonic* if its tension field vanishes everywhere:

$$\tau(\phi) = 0.$$

We also define a stronger related concept:

Definition 2.11. Let M and N be Riemannian manifolds. A map $\phi : M \to N$ is said to be *totally geodesic* if its second fundamental form vanishes everywhere:

$$\hat{\nabla} d\phi = 0.$$

We see from Proposition 2.9 that the composition of two totally geodesic maps is totally geodesic but that this need not be true for two harmonic maps. As indicated in the previous section the theory has a close connection with the calculus of variations. We therefore proceed to give a variational characterization of harmonic maps.

Definition 2.12. Let M and N be Riemannian manifolds and assume that M is compact and oriented. For a map $\phi : M \to N$ the *energy functional* is the integral

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 \nu,$$

where ν is the volume form of M and $|d\phi|^2 = \langle d\phi, d\phi \rangle$ is the squared norm of $d\phi$ as defined in the previous section. The map ϕ is said to be a *critical point of the energy functional* if

$$\frac{d}{dt} E(\phi_t) \Big|_{t=0} = 0.$$

for any smooth variation ϕ_t of ϕ.

Theorem 2.13. [**24**] *Let M and N be compact Riemannian manifolds. If M is oriented then a map $\phi : M \to N$ is harmonic if and only if it is a critical point of the energy functional.*

Proof. We mainly follow Urakawa in [**56**]. Let ϕ_t be a smooth variation of ϕ and write $\Phi(t, x) = \phi_t(x) : (-\epsilon, \epsilon) \times M \to N$. Choose a local orthonormal frame (e_i) of the tangent bundle TM and write

e_i for $(0, e_i)$ as vector fields of the product manifold $(-\epsilon, \epsilon) \times M$. By Proposition 2.5 we have

$$\nabla^{\Phi}_{\frac{\partial}{\partial t}} d\Phi(e_i) = \nabla^{\Phi}_{e_i} d\Phi(\frac{\partial}{\partial t}).$$

For $|t| < \epsilon$, define $X_t \in C^\infty(TM)$ by

$$g(X_t, Y) = h(d\Phi(\frac{\partial}{\partial t}), d\Phi(Y))$$

for an arbitrary $Y \in C^\infty(TM)$. Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\sum_i h(d\Phi(e_i), d\Phi(e_i)) &= \sum_i h(\nabla^{\Phi}_{\frac{\partial}{\partial t}} d\Phi(e_i), d\Phi(e_i)) \\
&= \sum_i h(\nabla^{\Phi}_{e_i} d\Phi(\frac{\partial}{\partial t}), d\Phi(e_i)) \\
&= \sum_i \Big(e_i(g(X_t, e_i)) - h(d\Phi(\frac{\partial}{\partial t}), d\Phi(\nabla^M_{e_i} e_i))\Big) \\
&\quad - \sum_i h(d\Phi(\frac{\partial}{\partial t}), \hat{\nabla} d\Phi(e_i, e_i)) \\
&= \operatorname{div}(X_t) - h(d\Phi(\frac{\partial}{\partial t}), \sum_i d\Phi(e_i, e_i)).
\end{aligned}$$

The integral of the first term vanishes by Stokes' theorem. Thus we obtain

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_M h(\frac{\partial \phi_t}{\partial t}\Big|_{t=0}, \tau(\phi))\nu. \tag{2.4}$$

Since this holds for any smooth variation ϕ_t of ϕ the statement follows.

□

Remark 2.14. Equation (2.4) is generally refered to as *the first variation*. As mentioned in the previous chapter, harmonic maps were originally introduced as solutions to this variational problem. Intuitively, deforming ϕ in a manner that increases the "topological irregularity" (Fuller [**30**], page 987) of ϕ, will also increase the energy $E(\phi)$. Thus the idea was to find harmonic representatives in each homotopy class of maps $M \to N$ to be used as homotopic normalizers of given maps. This was the main theme of the article [**24**] of Eells and Sampson where the following result was achieved:

Theorem 2.15 (The Eells-Sampson Existence Theorem)**.** [**24**] *Let* (M, g) *and* (N, h) *be compact oriented Riemannian manifolds where*

(N,h) has non-positive sectional curvature. Then any homotopy class of continuous maps $(M,g) \to (N,h)$ has an energy minimizing harmonic representative.

For more details on the Eells-Sampson Existence Theorem and its history we refer to the book [**56**] of Urakawa.

For a function $f : M \to \mathbb{R}$ and a local orthonormal frame (e_i) for the tangent bundle TM of M we have

$$\begin{aligned}\tau(f) &= \mathrm{trace}(\hat{\nabla} df)\\ &= \sum_i \hat{\nabla} df(e_i, e_i)\\ &= \sum_i (\nabla^f_{e_i} df(e_i) - df(\nabla^M_{e_i} e_i))\\ &= \sum_i (e_i(e_i(f)) - \nabla^M_{e_i} e_i(f))\\ &= \sum_i g(\nabla^M_{e_i} \mathrm{grad}\ f, e_i)\\ &= \mathrm{div}(\mathrm{grad}(f))\end{aligned}$$

and we recover the familiar Laplace-Beltrami operator $\triangle^M$=div(grad) on M. For this reason we shall in the case of a function henceforth write $\triangle^M$ instead of τ.

Choosing local coordinates (x^k) and (y^α) around points $x \in M^m$ and $y \in N^n$ with $y = \phi(x)$ for $\phi : M \to N$, a straightforward calculation gives

(2.5)

$$\begin{aligned}\tau(\phi)^\alpha &= \sum_{i,j} g^{ij} (\hat{\nabla} d\phi)^\alpha_{ij}\\ &= \sum_{i,j} g^{ij} \frac{\partial^2 \phi^\alpha}{\partial x^i \partial x^j} - \sum_{i,j,k} g^{ij}\, {}^M\Gamma^k_{ij} \frac{\partial \phi^\alpha}{\partial x^k} + \sum_{i,j,\beta,\gamma} g^{ij}\, {}^N\Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j}\\ &= \triangle^M(\phi^\alpha) + \sum_{i,j,\beta,\gamma} g^{ij}\, {}^N\Gamma^\alpha_{\beta\gamma} \frac{\partial \phi^\beta}{\partial x^i} \frac{\partial \phi^\gamma}{\partial x^j}\end{aligned}$$

for $\alpha = 1, \dots, n$. This implies that if $N = \mathbb{R}^n$ then ϕ is harmonic if and only if each of its components are harmonic functions and

$$\tau(\phi) = (\triangle^M(\phi^1), \dots, \triangle^M(\phi^n)).$$

Example 2.16. Let I be an open interval of $\mathbb{R}$, $\gamma : I \to M$ be a regular curve on M and (x^k) local coordinates on M. Then from equation (2.5) we get:

$$\tau(\gamma)^k = \frac{d^2\gamma^k}{dt^2} + \sum_{i,j} {}^M\Gamma^k_{ij} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$$

Thus γ is harmonic if and only if it is a geodesic.

Example 2.17. If $\phi : M^m \to N^n$ is an isometric immersion we may identify $X \in C^\infty(TM)$ with $d\phi(X) \in C^\infty(\phi^{-1}(TN))$ and consider T_xM to be a subspace of $T_{\phi(x)}N$ for $x \in M$. Since for $X, Y \in T_xM$:

$$\nabla^N_{d\phi(X)} d\phi(Y) = \hat{\nabla} d\phi(X,Y) + d\phi(\nabla^M_X Y)$$

and we see that $\hat{\nabla} d\phi$ is the second fundamental form of M in the classical sense i.e. the orthogonal projection of $\nabla^N_X Y$ onto the normal space $(T_xM)^-$ with the identification mentioned above. Recall that the mean curvature vector of M in N is the trace of the second fundamental form divided by m and that ϕ is said to be a minimal immersion of M into N if the mean curvature vector vanishes. Thus we have:

Theorem 2.18. [**24**] *An isometric immersion is minimal if and only if it is harmonic.*

The name minimality is motivated by the fact that if M is compact and orientable, then ϕ is minimal if and only if ϕ is a critical point of the volume functional

$$V(\phi) = \int_M \nu_\phi$$

where ν_ϕ is the volume form of M associated with the induced metric $g = \phi^* h$. Actually, for every smooth variation by immersions (ϕ_t) of ϕ one may prove (see [**23**], page 21) that

$$\frac{d}{dt} V(\phi_t)\Big|_{t=0} = -\int_M \langle \frac{\partial \phi_t}{\partial t}\Big|_{t=0}, \tau(\phi)\rangle \nu_\phi.$$

There is a natural generalization of isometric immersions to *(weakly) conformal maps* i.e. maps $\phi : (M,g) \to (N,h)$ with $\phi^* h = \mu^2 g$ for some function

$$\mu : M \to \mathbb{R}_+ \cup \{0\}$$

called the *conformal factor of* ϕ. The adjective weak indicates that μ may take the value 0 in which case $d\phi = 0$. Theorem 2.18 can now be generalized to the following:

Theorem 2.19. *Let $m \geq 2$ and $\phi : (M^m, g) \to (N^n, h)$ be a conformal immersion. Then*

a) if $m = 2$ then ϕ is harmonic if and only if M is minimal in N.
b) if $m > 2$ then two of the following conditions implies the other:

1) *ϕ is harmonic,*
2) *M is minimal in N,*
3) *ϕ is a homothety i.e. its conformal factor is constant.*

PROOF. Denote by μ the conformal factor of ϕ so that $\phi^* h = \mu^2 g$. We define a new Riemannian metric $\tilde{g}$ on M as the pull-back of h by ϕ: $\tilde{g} = \phi^* h$. If ∇^M and $\tilde{\nabla}^M$ are the Levi-Civita connections of (M, g) and $(M, \tilde{g})$, respectively, then since $\tilde{g} = \mu^2 g$ one may easily deduce that (see page 90 of [**34**])

$$\tilde{\nabla}^M_X Y = \nabla^M_X Y + \frac{\mu^{-2}}{2} X(\mu^2) Y + \frac{\mu^{-2}}{2} Y(\mu^2) X - \frac{\mu^{-2}}{2} g(X,Y) \mathrm{grad}_g(\mu^2)$$

for $X, Y \in C^\infty(TM)$. Here grad_g is the gradient in (M, g). Thus if $\{X_1, \dots, X_m\}$ is a local orthonormal frame of (M, g) we then have for $i = 1, \dots, m$:

$$\tilde{\nabla}^M_{X_i} X_i = \nabla^M_{X_i} X_i + \mu^{-2} X_i(\mu^2) X_i - \frac{\mu^2}{2} \mathrm{grad}_g(\mu^2).$$

Since $\phi : (M, \tilde{g}) \to (N, h)$ is an isometric immersion it follows that

$$\begin{aligned}(\nabla^N_{d\phi(X_i)} d\phi(X_i))^{-} &= \nabla^N_{d\phi(X_i)} d\phi(X_i) - d\phi(\tilde{\nabla}^M_{X_i} X_i) \\ &= \nabla^N_{d\phi(X_i)} d\phi(X_i) - d\phi(\nabla^M_{X_i} X_i) \\ &\quad + \mu^2 d\phi(\frac{1}{2} \mathrm{grad}_g(\mu^2) - X_i(\mu^2) X_i)\end{aligned}$$

for $i = 1, \dots, m$. Hence by summing over i we arrive at

$$mH = \tau(\phi) + \frac{\mu^{-2}}{2}(m-2) d\phi(\mathrm{grad}_g(\mu^2))$$

which immediately proves the theorem. □

The concept of weak conformality is dual to that of *horizontal weak conformality* which we will define in the next chapter. We will also derive a result corresponding to Theorem 2.19 for horizontally conformal maps.

Example 2.20. Let $\phi : M \to S^{n-1}$ be a map into the unit sphere in $\mathbb{R}^n$, $i : S^{n-1} \hookrightarrow \mathbb{R}^n$ be the inclusion map and $\hat{\phi} = i \circ \phi$. Then the composition law gives:

$$\tau(\hat{\phi}) = di(\tau(\phi)) + \mathrm{trace}(\hat{\nabla} di(d\phi, d\phi)).$$

Note that the first term on the right is tangent to the sphere and the second is orthogonal to it. Thus ϕ is harmonic if and only if $\hat{\phi}$ is parallel to $\tau(\hat{\phi})$ i.e. if the tangential part of $\tau(\hat{\phi})$ vanishes. If $f(x) = |x|^2$, $x \in \mathbb{R}^n$, then

$$\begin{aligned} 0 = \triangle^M(f \circ \hat{\phi}) &= df(\tau(\hat{\phi}) + \operatorname{trace}(\hat{\nabla} df(d\hat{\phi}, d\hat{\phi})) \\ &= 2\Big(\langle \operatorname{grad} f, \tau(\hat{\phi})\rangle + \operatorname{trace}(\hat{\nabla} df(d\hat{\phi}, d\hat{\phi}))\Big) \\ &= 2(\langle \hat{\phi}, \tau(\hat{\phi})\rangle + |d\phi|^2). \end{aligned}$$

Hence ϕ is harmonic if and only if $\tau(\hat{\phi}) = -|d\phi|^2\hat{\phi}$.

Example 2.21. If $\pi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ is the radial projection given by $\pi(x) = x/|x|$, we see, using the notation of Example 2.20, that

$$\tau(\hat{\pi}) = -(n-1)|x|^{-2}\hat{\pi}(x),$$

Hence π is a harmonic map and $|d\pi_x|^2 = (n-1)|x|^{-2}$.

Example 2.22. Let $\gamma : H^n \setminus \{0\} \to S^{n-1}$ be given by $\gamma(x) = x/|x|$, where

$$H^n = (\{\, x \in \mathbb{R}^n \mid |x| < 1 \,\}, \frac{4}{(1-|x|^2)^2}\langle \cdot\, , \cdot \rangle_{\mathbb{R}^n})$$

is the hyperbolic space of constant curvature -1. Then we get

$$\tau(\hat{\gamma})(x) = -(n-1)\frac{(1-|x|^2)^2}{4|x|^2}\hat{\gamma}(x).$$

Thus γ is a harmonic map with $|d\gamma_x|^2 = (n-1)(1-|x|^2)^2/4|x|^2$.

Example 2.23. Let $i : S^{m-1} \hookrightarrow \mathbb{R}^m$ be the inclusion and let $x \in S^{m-1}$. Choose a local orthonormal frame $\{e_1, \dots, e_{m-1}\}$ for TS^{m-1} around x and geodesics $\gamma_k : (-\epsilon, \epsilon) \to S^{m-1}$, $k = 1, \dots, n$, such that $\gamma_k(0) = x$ and $\gamma_k'(0) = e_k(x)$ for all k. Denoting by $\partial/\partial r$ the derivation in the direction normal to S^{m-1} and using that $\gamma_k'' = -\partial/\partial r$, gives us at the point x:

$$\tau(i) = \sum_k \gamma_k''(0) = -(m-1)\frac{\partial}{\partial r}.$$

If $F : \mathbb{R}^m \to \mathbb{R}$ is a function and f its restriction to S^{m-1} we get

$$\begin{aligned} \triangle^{S^{m-1}}(f) &= \triangle^{S^{m-1}}(F \circ i) \\ &= dF(\tau(i)) + \operatorname{trace}(\hat{\nabla} dF(di, di)) \\ &= -(m-1)\frac{\partial F}{\partial r} \circ i + \triangle^{\mathbb{R}^m}(F) \circ i - \frac{\partial^2 F}{\partial r^2} \circ i. \end{aligned}$$

In particular, if F is a harmonic polynomial, homogeneous of degree p then

$$\triangle^{S^{m-1}}(f) = -p(p+m-2)f. \tag{2.6}$$

Hence f is an eigenfunction of $\triangle^{S^{m-1}}$. It is well known that all eigenfunctions of $\triangle^{S^{m-1}}$ arise in this way (see [**12**], page 160).

Suppose that $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is a harmonic map with each of its component functions polynomials, homogeneous of degree p and that Φ restricts to a map $\phi : S^{m-1} \to S^{n-1}$. Denote by $i : S^{m-1} \hookrightarrow \mathbb{R}^m$ and $j : S^{n-1} \hookrightarrow \mathbb{R}^n$ the inclusions so that $\Phi \circ i = j \circ \phi$. By equation (2.6):

$$\tau(\Phi \circ i) = -p(p+m-2)\Phi \circ i.$$

From Example 2.20 we see that ϕ is harmonic and that

$$|d\phi|^2 = p(p+m-2).$$

Example 2.24. Recall that for two Kähler manifolds (M^{2m}, g, J^M) and (N^{2n}, h, J^N) a map $\phi : M \to N$ is said to be *+holomorphic* (or simply holomorphic) if $\bar{\partial}\phi = 0$ i.e. if $d\phi J^M = J^N d\phi$ and *−holomorphic* if $\partial\phi = 0$ i.e. if $d\phi J^M = -J^N d\phi$. If ϕ is either + or −holomorphic then ϕ is said to be ±holomorphic.

Since M and N are both Kähler, $\nabla^M J^M = J^M \nabla^M$ and similar for N. Thus if $\phi : M \to N$ is ±holomorphic and $X, Y \in C^\infty(TM)$ (for simplicity writing J for both J^M and J^N):

$$\begin{aligned} \hat{\nabla} d\phi(X, JY) &= \nabla^\phi_X d\phi(JY) - d\phi(\nabla^M_X JY) \\ &= \pm J(\nabla^\phi_X d\phi(Y) - d\phi(\nabla^M_X Y)) \\ &= \pm J \hat{\nabla} d\phi(X, Y) \\ &= \hat{\nabla} d\phi(JX, Y) \end{aligned}$$

using the symmetry of $\hat{\nabla} d\phi$. In particular

$$\hat{\nabla} d\phi(JX, JX) = -\hat{\nabla} d\phi(X, X).$$

Hence by choosing a local orthonormal frame $\{e_1, \dots, e_m, f_1, \dots, f_m\}$ for TM with $Je_i = f_i$ for all i, we get

$$\tau(\phi) = \sum_{i=1}^m (\hat{\nabla} d\phi(e_i, e_i) + \hat{\nabla} d\phi(Je_i, Je_i)) = 0.$$

Thus we see that every ±holomorphic map between Kähler manifolds is harmonic. This was first proved by Eells and Sampson in [**24**]. The converse is not true, take for instance $\phi : \mathbb{C}^2 \to \mathbb{C}$ defined by $\phi(z_1, z_2) =$

$z_1\bar{z}_2$. Then ϕ is not ±holomorphic with respect to the standard Kähler structures on $\mathbb{C}^2$ and $\mathbb{C}$ but harmonic since

$$\tau(\phi) = 4\Big(\frac{\partial^2\phi}{\partial z_1\partial\bar{z}_1} + \frac{\partial^2\phi}{\partial z_2\partial\bar{z}_2}\Big) = 0.$$

Example 2.25. [**24**] If G_1 and G_2 are Lie groups with bi-invariant Riemannian metrics g_1 and g_2, respectively, and

$$\phi : (G_1, g_1) \to (G_2, g_2)$$

is a Lie group homomorphism, then ϕ is a harmonic map, in fact; totally geodesic. To prove this denote by $\mathfrak{g}_1$ and $\mathfrak{g}_2$ the Lie algebras of G_1 and G_2, respectively. By identifying these spaces with the tangent spaces at the neutral elements we get an induced Lie algebra homomorphism

$$d\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$$

satisfying $d\phi(X)(\phi(x)) = d\phi_x(X_x)$ for any $X \in \mathfrak{g}_1$ and $x \in G_1$. Furthermore, it follows from the bi-invariance of the metrics that for left invariant vector fields X, Y on G_1, $\nabla^{G_1}_X Y = \frac{1}{2}[X,Y]$ and similarly on G_2. Thus for X, $Y \in \mathfrak{g}_1$ we get:

$$\begin{aligned}\hat{\nabla} d\phi(X,Y) &= \nabla^{\phi}_X d\phi(Y) - d\phi(\nabla^{G_1}_X Y)\\ &= \nabla^{G_2}_{d\phi(X)} d\phi(Y) - \frac{1}{2} d\phi([X,Y])\\ &= \frac{1}{2}[d\phi(X), d\phi(Y)] - \frac{1}{2} d\phi([X,Y])\\ &= 0\end{aligned}$$

which proves the statement.

The following very important result was proved by Sampson in [**51**] by applying the uniqueness theorem of Aronszajn in [**2**] to the local expression of the tension field:

Theorem 2.26 (The Unique Continuation)**.** [**51**] *Let $\phi : M \to N$ be a harmonic map. If at some point of M all the partial derivatives of ϕ to any order vanish, then ϕ is constant. In particular, if ϕ is constant on an open subset of M then ϕ is constant on the whole of M.*

3. Harmonic Functions

In this section we state some well known results on harmonic functions needed later on. Locally there exists a great variety of harmonic functions on a Riemannian manifold as the following result by Ishihara shows (see [**43**], Lemma 4.1). It is based on an extension of a lemma by Bers in [**13**] and will play a crucial role in the next chapter.

Theorem 2.27. [**43**] *Let x be a point in M^m, (x^k) be normal coordinates on M centered at x and $\{c_k, c_{ij}\}_{i,j,k=1}^m$ constants with $c_{ij} = c_{ji}$ and $\sum_i c_{ii} = 0$. Then there exists a neighbourhood U of x in M and a harmonic function $f : U \to \mathbb{R}$ such that*

$$\frac{\partial f}{\partial x^k}(x) = c_k, \quad \frac{\partial^2 f}{\partial x^i \partial x^j}(x) = c_{ij}$$

for all $i, j, k = 1, \ldots, m$.

Using Bers' lemma, Greene and Wu proved the next result.

Theorem 2.28. [**33**] *For any $x \in M$ there is a chart $(U, (x^k))$ around x such that all the coordinate functions x^k are harmonic.*

Definition 2.29. A (smooth) function $f : M \to \mathbb{R}$ is said to be *subharmonic* if

$$\triangle^M(f) \geq 0.$$

The function f is said to be *superharmonic* if $-f$ is subharmonic.

The following result is a direct consequence of the ellipticity of equation (2.5) (see [**31**], Chapter 3).

Theorem 2.30 (The Maximum Principle)**.** *If $f : M \to \mathbb{R}$ is a subharmonic function having a maximum in an open subset of M, then f is constant.*

Corollary 2.31. *If M is compact then every subharmonic function on M is constant.*

In the special case when M is orientable Corollary 2.31 is due to E. Hopf and may very well be proved without appealing to the Maximum Principle. For since $\triangle^M$ is a divergence, Stokes' theorem gives in this case that

$$\int_M \triangle^M(f)\nu = 0$$

for any function $f : M \to \mathbb{R}$. In particular, if f is subharmonic then this implies that f in fact is harmonic. Furthermore an easy calculation gives that

$$\triangle^M(f^2/2) = |\mathrm{grad}(f)|^2 + f\triangle^M(f) = |\mathrm{grad}(f)|^2.$$

Hence if f is subharmonic we get

$$0 = \int_M \triangle^M(f^2/2)\nu = \int_M |\mathrm{grad}(f)|^2\nu$$

so f is indeed constant.

Note that together with Theorem 2.18, the result of Corollary 2.31 implies that there are no compact minimal (even immersed) submanifolds of $\mathbb{R}^m$.

Example 2.32. In harmonic function theory in Euclidean spaces the following functions are very important:

$$\mathbb{R}^2 \setminus \{0\} \ni x \mapsto \log|x|$$

and

$$\mathbb{R}^m \setminus \{0\} \ni x \mapsto |x|^{2-m}$$

where $m \geq 3$. It is easy to see that they are both harmonic where defined. If $\Omega \subseteq \mathbb{R}^m$ is open and connected and $a \in \Omega$ then by a theorem of Bochner (see [**3**]), any function u which is harmonic in $\Omega \setminus \{a\}$ and positive near a is of the form

$$x \mapsto v(x) - c\log|x-a|, \text{ if } m = 2,$$

and

$$x \mapsto v(x) + c|x-a|^{2-m}, \text{ if } m \geq 3,$$

for some constant $c \geq 0$ and a harmonic function v in Ω. In particular, if u is harmonic and non-negative in $\mathbb{R}^m \setminus \{0\}$ with $n \geq 3$, it follows from an application of the Maximum Principle that v is in fact constant. Hence

$$u(x) = b + c|x|^{2-m}$$

for some constants b and c. Since $\log|x|$ is not bounded for large $|x|$ the same argument can not be applied to a function harmonic and non-negative in $\mathbb{R}^2 \setminus \{0\}$. Indeed, any such function f must be constant by the theorem of Liouville (see [**3**]) since the function $f(e^z)$, $z \in \mathbb{C}$, would be harmonic and non-negative in $\mathbb{R}^2$.

CHAPTER 3

Harmonic Morphisms

In this chapter we define harmonic morphisms and prove some basic facts on these. The key to the theory is Theorem 3.8 which states that harmonic morphisms constitute a certain subclass of the harmonic maps; having the additional property of being horizontally (weakly) conformal. We therefore begin with a description of this concept. In the last two sections we briefly discuss the existence and non-existence of harmonic morphisms and study their behaviour on polar sets. All the results given in this chapter can be found in [**38**], a regularly updated list of publications on harmonic morphisms.

1. Horizontal Conformality

A map between Riemannian manifolds of equal dimension is *conformal* if its differential at any point is a conformal linear map with respect to the Riemannian metrics. Horizontal conformality is a generalization of this concept to the case when the target manifold is of lower dimension than the domain.

Definition 3.1. Suppose that $\phi : M \to N$ is a map. At each point $x \in M$, the *vertical space* $\mathcal{V}_x$ of ϕ is the kernel $\ker d\phi_x$ of the differential $d\phi_x$ of ϕ at x. The *horizontal space* $\mathcal{H}_x$ is the orthogonal complement of $\mathcal{V}_x$ in T_xM with respect to the Riemannian metric g on M.

If the map ϕ is a submersion i.e. $d\phi$ is surjective at each point of M, then we may associate to ϕ two distributions on M; the *vertical* and the *horizontal* distribution, asigning to each point $x \in M$ the subspaces $\mathcal{V}_x$ and $\mathcal{H}_x$ of T_xM, respectively. A vector field X on M is then said to be *vertical* (*horizontal*) if it belongs to the vertical (horizontal) distribution.

By the Inverse Function Theorem, the fibre $\phi^{-1}(\phi(x))$ is a (possibly disconnected) submanifold of M for every $x \in M$, the tangent plane of which is the vertical space. Thus the vertical distribution $\mathcal{V}$ is integrable and the map ϕ determines a foliation of M which leaves are the fibres of ϕ. From the integrability of $\mathcal{V}$ one may easily deduce that the horizontal distribution $\mathcal{H}$ is in fact smooth on M.

Two vector fields $X \in C^\infty(TM)$ and $Y \in C^\infty(TN)$ are said to be *ϕ-related* if $d\phi_x(X_x) = Y_{\phi(x)}$ for every $x \in M$. A vector field $X \in C^\infty(TM)$ is called *basic* if it is ϕ-related to some vector field $Y \in C^\infty(TN)$. If in addition X is horizontal, then X is called *a horizontal lift of* Y. It follows easily that for a given $Y \in C^\infty(TN)$ there exists a unique horizontal lift of Y which in fact is smooth.

Definition 3.2. Let $\phi : M^m \to N^n$ be a map. The *critical set of* ϕ is the set $C_\phi = \{x \in M \mid d\phi_x = 0\}$. The map $\phi : M \to N$ is said to be *horizontally (weakly) conformal* if for each $x \in M \setminus C_\phi$, the restriction of $d\phi_x$ to the horizontal space $\mathcal{H}_x$ is surjective and conformal.

Example 3.3. Let $\phi : \mathbb{R}^8 \to \mathbb{R}^4$ be the multiplication of quaternions given by

$$\begin{aligned}\phi(x_1, x_2, \dots, x_8) = (&x_1x_5 - x_2x_6 - x_3x_7 - x_4x_8,\\ &x_1x_6 + x_2x_5 + x_3x_8 - x_4x_7,\\ &x_1x_7 - x_2x_8 + x_3x_5 + x_4x_6,\\ &x_1x_8 + x_2x_7 - x_3x_6 + x_4x_5).\end{aligned}$$

If we identify $\mathbb{R}^4$ with the space $\mathbb{H}$ of quaternions, then $\phi : \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is given by $\phi(q_1, q_2) = q_1q_2$ for $q_1, q_2 \in \mathbb{H}$. In the canonical bases we have

$$d\phi_{(q_1,q_2)} = \begin{pmatrix} x_5 & -x_6 & -x_7 & -x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & x_5 & x_8 & -x_7 & x_2 & x_1 & -x_4 & x_3 \\ x_7 & -x_8 & x_5 & x_6 & x_3 & x_4 & x_1 & -x_2 \\ x_8 & x_7 & -x_6 & x_5 & x_4 & -x_3 & x_2 & x_1 \end{pmatrix}.$$

Clearly the horizontal space is generated by the rows of this matrix. Note that these are orthogonal and of equal length $|q_1|^2 + |q_2|^2$ so if $(q_1, q_2) \neq (0, 0)$ and we express a horizontal vector v in the base given by these rows as $v = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, then

$$d\phi_{(q_1,q_2)}(v) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)(|q_1|^2 + |q_2|^2).$$

Thus for horizontal vectors v, w we see that

$$\langle d\phi_{(q_1,q_2)}(v), d\phi_{(q_1,q_2)}(w)\rangle = (|q_1|^2 + |q_2|^2)\langle v, w\rangle.$$

Hence ϕ is horizontally (weakly) conformal with the critical set consisting of the origin.

The adjective "weak" indicates that the critical set of ϕ may be non-empty. From now on we shall refer to a horizontally weakly conformal map as a horizontally conformal map, it being understood that it need not be a submersion.

The horizontal conformality of ϕ implies that there exists a function $\lambda : M \setminus C_\phi \to \mathbb{R}^+$ such that for all $x \in M \setminus C_\phi$ and $X, Y \in \mathcal{H}_x$:

$$\lambda^2(x) g(X, Y) = h(d\phi_x(X), d\phi_x(Y)).$$

For $x \in M \setminus C_\phi$ and an orthonormal basis (e_i) of $T_x M$ we have

$$n\lambda^2(x) = \sum_{i=1}^{n} h(d\phi_x(e_i), d\phi_x(e_i)) = |d\phi_x|^2, \tag{3.1}$$

where n is the dimension of N. Hence λ extends in a unique way to a continuous function on the whole of M also denoted by λ, satisfying equation (3.1). This extended function vanishes on C_ϕ and is called the *dilation* of ϕ. By equation (3.1) λ^2 is smooth on the whole of M.

Example 3.4. If $n \in \mathbb{N}$ then the Lie group $S^1 \subset \mathbb{C}$ acts freely on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ by left multiplication and the resulting quotient space S^{2n+1}/S^1 may be identified with the complex projective space $\mathbb{C}P^n$, the set of complex lines in $\mathbb{C}^{n+1}$. More precisely: S^{2n+1} is a *principal fibre bundle over $\mathbb{C}P^n$ with structure group S^1*. If $\pi : S^{2n+1} \to \mathbb{C}P^n$ is the canonical projection then π is a submersion, so $\mathbb{C}P^n$ may be given a Riemannian metric induced by π. With this metric π is a Riemannian submersion, i.e. horizontally conformal with constant dilation $\lambda = 1$. The fibres are of course the orbits under the action of S^1:

$$\pi^{-1}(\pi(x)) = \{e^{i\theta} x \mid \theta \in \mathbb{R}\}$$

for $x \in S^{2n+1}$. The tangent space of the fibre at x is the vertical space of π and so is the real line through x spanned by ix and the horizontal space is the orthogonal complement of this in $T_x S^{2n+1}$.

Proposition 3.5. [**21**]. *Let $\xi : V \to W$ be a non-zero linear map between finite-dimensional Euclidean vector spaces and $\xi^* : W \to V$ be its adjoint characterized by $\langle \xi^*(w), v\rangle = \langle w, \xi(v)\rangle$. Then ξ is horizontally conformal if and only if ξ^* is conformal. If in addition, dim V=dim W, then ξ is a conformal isomorphism.*

Proof. [**21**] Suppose ξ is horizontally conformal with dilation λ. Choose orthonormal bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ of V and W, respectively, such that $\xi(v_i) = \lambda w_i$ for $i = 1, \dots, n$ and $\xi(v_i) = 0$ if $i > n$. Obviously, $\langle \xi^*(w_i), v_j\rangle = \langle w_i, \xi(v_j)\rangle = \lambda\delta_{ij}$ for $i, j = 1, \dots, n$. Hence $\xi^*(w_i) = \lambda v_i$ for $i = 1, \dots, n$ so it is clear that ξ^* is conformal. The converse follows with the same choice of bases and the last statement is obvious. □

Example 3.6. If $\phi : (M^m, g) \to (N^n, h)$ is a map, $x \in M$, choose local coordinates (x^i) and (y^α) on M and N around x and $\phi(x) = y$, respectively. It is then easy to see that

$$(d\phi_x)^* = \sum_{\alpha,\beta} h_{\alpha\beta}(y) dy^\beta(y) \otimes (\mathrm{grad}(\phi^\alpha))(x).$$

Define a local frame on N by

$$e_\alpha = \sum_{\beta=1}^{n} h^{\alpha\beta} \frac{\partial}{\partial y^\alpha}, \quad \alpha = 1, \dots, n.$$

Then $h(e_\alpha, e_\beta) = h^{\alpha\beta}$ and $(d\phi)^*(e_\alpha) = \mathrm{grad}(\phi^\alpha)$. By Proposition 3.5, ϕ is horizontally conformal at x with dilation $\lambda(x)$ if and only if

$$\lambda(x)^2 h^{\alpha\beta} = \sum_{i,j} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}, \tag{3.2}$$

for $\alpha, \beta = 1, \dots, n$. It follows that ϕ is horizontally conformal with dilation λ if and only if this holds for any $x \in M$ and local coordinates around x and $\phi(x)$, respectively. In particular, if (y^α) are *normal* at $\phi(x)$, then ϕ is horizontally conformal at x with dilation $\lambda(x)$ if and only if $\mathrm{grad}(\phi^\alpha)$, $\alpha = 1, \dots, n$, are mutually orthogonal and of equal length $\lambda(x)$. When N^2 is a Riemann surface with a Hermitian metric, then by writing $\phi = \phi^1 + i\phi^2$ in a local holomorphic coordinate, we see that the horizontal conformality is given by

$$g(\mathrm{grad}(\phi), \mathrm{grad}(\phi)) = 0.$$

In local coordinates (x^i) on M this can be expressed by

$$\sum_{i,j} g^{ij} \frac{\partial \phi}{\partial x^i} \frac{\partial \phi}{\partial x^j} = 0. \tag{3.3}$$

In the interesting case when M is also a Riemann surface with a Hermitian metric and $z = x+iy$ a complex coordinate on M, then equation (3.3) reduces to

$$(\frac{\partial \phi}{\partial x})^2 + (\frac{\partial \phi}{\partial y})^2 = 4 \frac{\partial \phi}{\partial z} \frac{\partial \phi}{\partial \bar{z}} = 0. \tag{3.4}$$

In particular, if ϕ is ±holomorphic then ϕ is automatically horizontally conformal.

2. Harmonic Morphisms

We can now define harmonic morphisms and prove some of their elementary properties.

Definition 3.7. A map $\phi : M \to N$ is said to be a *harmonic morphism* if for every open subset U of N with $\phi^{-1}(U) \neq \emptyset$ and every harmonic function $f : U \to \mathbb{R}$, the composition $f \circ \phi : \phi^{-1}(U) \to \mathbb{R}$ is harmonic.

The above definition means that harmonic morphisms pull back germs of harmonic functions to germs of harmonic functions. It is therefore immediate that the composition of harmonic morphisms is a harmonic morphism. A useful characterization for harmonic morphisms is provided by the next theorem, which gives the link between horizontal conformality and harmonic morphisms.

Theorem 3.8 (The Fuglede-Ishihara Characterization)**.** [**28**][**43**] *A map $\phi : M \to N$ is a harmonic morphism if and only if it is harmonic and horizontally conformal.*

PROOF. [**21**] Suppose $\phi : M^m \to N^n$ is a harmonic morphism. If $x_0 \in M$, consider systems of local coordinates (x^i) and (y^α) around x_0 and $y_0 = \phi(x_0)$, respectively, where we assume that (y^α) are normal, centered at y_0. According to Theorem 2.27 we may for every $\gamma = 1, \dots, n$ choose a function f defined and harmonic in a neighborhood of y_0 with

$$\frac{\partial f}{\partial y^\alpha}(y_0) = \delta_{\alpha\gamma} \quad \text{and} \quad \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(y_0) = 0$$

for all $\alpha, \beta = 1, \dots, n$. By assumption, the function $f \circ \phi$ is harmonic in a neighbourhood of x_0, so by Proposition 2.9

$$0 = \triangle^M (f \circ \phi) = df(\tau(\phi)) + \hat{\nabla} df(d\phi, d\phi).$$

In particular, since at x_0:

$$\hat{\nabla} df = \sum_{\alpha,\beta} \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} dy^\alpha \otimes dy^\beta = 0 \quad \text{and} \quad df(\tau(\phi)) = \tau(\phi)^\gamma$$

the above relation implies that $\tau(\phi)^\gamma(x_0) = 0$. Since this holds for any $\gamma = 1, \dots, n$ and $x_0 \in M$ we conclude that the map ϕ is harmonic.

To prove the horizontal conformality of ϕ, we once more apply Theorem 2.27, by noting that we may for every sequence $(c_{\alpha\beta})_{\alpha\beta=1}^n$ with $c_{\alpha\beta} = c_{\beta\alpha}$ and $\sum_\alpha c_{\alpha\alpha} = 0$, choose a harmonic function f such that

$$\frac{\partial f}{\partial y^\alpha}(y_0) = 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(y_0) = c_{\alpha\beta}$$

for all $\alpha, \beta = 1, \dots, n$. Hence we obtain at x_0:

$$\begin{aligned} 0 &= \triangle^M (f \circ \phi) \\ &= \operatorname{trace}(\hat{\nabla} df(d\phi, d\phi)) \\ &= \sum_{\alpha,\beta,i,j} c_{\alpha\beta} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \qquad (3.5) \\ &= \sum_{\alpha,i,j} c_{\alpha\alpha} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\alpha}{\partial x^j} + \sum_{\alpha \neq \beta, i,j} c_{\alpha\beta} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}, \end{aligned}$$

where we as usual write ϕ^α for $y^\alpha \circ \phi$. We subtract

$$0 = \sum_{\alpha,i,j} c_{\alpha\alpha} g^{ij} \frac{\partial \phi^1}{\partial x^i} \frac{\partial \phi^1}{\partial x^j}$$

from equation (3.5) and obtain:

$$0 = \sum_{\alpha,i,j} c_{\alpha\alpha} g^{ij} \left(\frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\alpha}{\partial x^j} - \frac{\partial \phi^1}{\partial x^i} \frac{\partial \phi^1}{\partial x^j} \right) + 2 \sum_{\alpha < \beta, i,j} c_{\alpha\beta} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j}.$$

It now follows that

$$\sum_{i,j} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} = 0 \quad \text{for all } \alpha \neq \beta \text{ and}$$

$$\sum_{i,j} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\alpha}{\partial x^j} = \sum_{i,j} g^{ij} \frac{\partial \phi^1}{\partial x^i} \frac{\partial \phi^1}{\partial x^j} \quad \text{for all } \alpha.$$

This can be summarized as

$$\sum_{i,j} g^{ij} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} = \lambda^2(x_0) \delta_{\alpha\beta} \quad \text{for all } \alpha, \beta = 1, \dots, n.$$

The last system of equations is equivalent to the statement that the components ϕ^α all have orthogonal gradients of equal length, so by Example 3.6, the map ϕ is horizontally conformal.

Conversely, if ϕ is harmonic and horizontally conformal with dilation λ, U an open subset of N with $\phi^{-1}(U) \neq \emptyset$ and $f : U \to \mathbb{R}$ a harmonic function, then we have $\triangle^M (f \circ \phi) = \operatorname{trace}(\hat{\nabla} df(d\phi, d\phi))$ on $\phi^{-1}(U)$. If $x_0 \in \phi^{-1}(U)$, consider once again systems (x^i) of local and (y^α) of normal coordinates around x_0 and $\phi(x_0)$, respectively.

Then at x_0:

$$\begin{aligned}
\operatorname{trace}(\hat{\nabla} df(d\phi, d\phi)) &= \sum_{\alpha,\beta,i,j} g^{ij} \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} \frac{\partial \phi^\alpha}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \\
&= \sum_{\alpha,\beta} g(\operatorname{grad}(\phi^\alpha), \operatorname{grad}(\phi^\beta)) \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} \\
&= \sum_{\alpha,\beta} \lambda^2(x_0) \delta_{\alpha\beta} \frac{\partial^2 f}{\partial y^\alpha \partial y^\beta} \\
&= \lambda^2(x_0) \triangle^M(f) \\
&= 0.
\end{aligned}$$

Thus ϕ is a harmonic morphism. □

Corollary 3.9. [**28**] *If $\phi : M^m \to N^n$ is a non-constant harmonic morphism then ϕ is a submersion on an open dense subset of M, so $m \geq n$.*

PROOF. [**21**] We see that for $x \in M$, if rank $d\phi_x < n$, then $d\phi_x = 0$. Hence the set of $x \in M$ with rank $d\phi_x = n$ is open and non-empty. It is also dense, for if on an open subset we had $d\phi = 0$, then by Theorem 2.26 we have $d\phi = 0$ in the whole of M so ϕ would be constant on M. □

Corollary 3.10. [**21**] *A harmonic morphism preserves harmonic maps, i.e. if $\phi : M \to N$ is a harmonic morphism, $\psi : N \to P$ a harmonic map, then $\psi \circ \phi : M \to P$ is harmonic.*

PROOF. Let $n = \dim N \leq \dim M$. At a point $x \in M$, choose orthonormal bases (e_i) for T_xM and (f_j) for $T_{\phi(x)}N$, such that $d\phi_x(e_i) = \lambda(x) f_i$, $1 \leq i \leq n$ and $d\phi_x(e_i) = 0$ for $i > n$. This is possible since ϕ is horizontally conformal. Thus at x we get for *any* map $\psi : N \to P$:

$$\begin{aligned}
\tau(\psi \circ \phi) &= \operatorname{trace} \hat{\nabla} d\psi(d\phi, d\phi) \\
&= \sum_{i=1}^{n} \hat{\nabla} d\psi(d\phi(e_i), d\phi(e_i)) \qquad (3.6) \\
&= \lambda^2(x) \tau(\psi)
\end{aligned}$$

which immediately proves the statement. □

A submersion is always an open map as follows from the Implicit Function Theorem. In view of Corollary 3.9 the following result is therefore not surprising. For two different proofs see [**28**] and [**21**].

Theorem 3.11. [**28**] *Every non-constant harmonic morphism is an open mapping.*

Corollary 3.12. [**28**] *If $\phi : M \to N$ is a non-constant harmonic morphism and M is compact then so is N and $\phi(M) = N$.*

Proof. By Theorem 3.11 the image $\phi(M)$ is both open and closed in N. The result now follows from the connectivity of N. $\square$

Example 3.13. The result of Theorem 3.8 gives us simple means to determine whether a given map is a harmonic morphism or not. From Example 3.6 we see that $\phi : U \subseteq \mathbb{R}^m \to \mathbb{C}$, U open, is a harmonic morphism if and only if it satisfies the following system of differential equations:

$$\triangle^{\mathbb{R}^m}(\phi) = 0, \quad \sum_i (\frac{\partial \phi}{\partial x^i})^2 = 0.$$

For a map $\phi : M \to \mathbb{C}$ a necessary and sufficient condition for ϕ to be a harmonic morphism is that this holds for every local coordinate (x^i) on M, with $\triangle^{\mathbb{R}^m}$ replaced with $\triangle^M$ and the second equation replaced with (3.3).

From Example 2.24 we know that any ±holomorphic map from a Kähler manifold to a Riemann surface with a Hermitian metric is harmonic. Since the differential of such a map will either commute or anti-commute with the almost complex structure, it is not hard to see that it will be horizontally conformal (see [**21**], Corollary 8.17). Hence any ±holomorphic map from a Kähler manifold to a Riemann surface with a Hermitian metric is a harmonic morphism. Example 2.24 shows that the converse is not true in general.

Example 3.14. A Riemann surface N^2 can be defined as an orientable surface with a conformal class of Riemannian metrics. A metric belonging to that class is then said to be *compatible* with N^2.

If $\phi : (M, g) \to (N^2, h)$ is a harmonic morphism to a surface, it follows from Theorem 3.8 that if $\hat{h}$ is a Riemannian metric on N^2 which is conformally equivalent to h, then $\phi : (M, g) \to (N^2, \hat{h})$ is a harmonic morphism. In particular, the concept of a harmonic morphism from (M, g) to a *Riemann surface* is well defined.

Example 3.15. Let M_1, M_2 and N be Riemannian manifolds and assume that

$$\phi : M_1 \times M_2 \to N$$

is a map, where $M_1 \times M_2$ is given the product structure and product metric. Define for $x \in M_1$ and $y \in M_2$ the map $\phi_x : M_2 \to N$ by

$\phi_x(y) = \phi(x, y)$ and similarly $\phi_y : M_1 \to N$. From the definition of the product metric it follows that the inclusions i_1 of $M_1 \times \{y\}$ and i_2 of $\{x\} \times M_2$ into $M_1 \times M_2$, respectively, are totally geodesic embeddings. Using Proposition 2.9 gives

$$\begin{aligned}\tau(\phi)(x, y) &= \mathrm{trace}\hat{\nabla} d\phi(x, y) \\ &= \mathrm{trace}\hat{\nabla} d\phi(di_1, di_1)(x) + \mathrm{trace}\hat{\nabla} d\phi(di_2, di_2)(y) \\ &= \tau(\phi_y)(x) + \tau(\phi_x)(y).\end{aligned}$$

It follows that if ϕ_x and ϕ_y are harmonic for every $x \in M_1$ and $y \in M_2$ then ϕ is harmonic. Furthermore, for a function $f : M_1 \times M_2 \to \mathbb{R}$:

$$\mathrm{grad}(f)(x, y) = \mathrm{grad}(f_y)(x) + \mathrm{grad}(f_x)(y)$$

and the two terms on the right are orthogonal. Thus we see that if for every $x \in M_1$ and $y \in M_2$ the maps ϕ_x and ϕ_y are both horizontally conformal with dilations λ_x and λ_y, respectively, then ϕ is horizontally conformal with dilation satisfying

$$\lambda^2(x, y) = \lambda_x^2(y) + \lambda_y^2(x).$$

In particular, if ϕ is a harmonic morphism in each variable separately then ϕ is a harmonic morphism.

Definition 3.16. A map $\phi : M \to N$ is said to be *horizontally homothetic* if ϕ is horizontally conformal and the gradient $\mathrm{grad}(\lambda^2)$ of the square of the dilation λ is vertical everywhere.

The horizontal homothety implies that the dilation is constant along horizontal curves. The concept of horizonal homothety is actually more natural than it may seem at first and it is in many ways a more suitable generalization of a Riemannian submersion than that of a horizontally conformal map. For a further investigation on the geometry of horizontally homothetic maps see Chapter 2 of [**35**]. By a result of Fuglede we have the following:

Theorem 3.17. [**29**] *A non-constant horizontally homothetic harmonic morphism is a submersion.*

The following result gives the theory nice geometric features and is dual to that of Theorem 2.19. It was first proved by Baird and Eells but the proof presented here was given by Gudmundsson in the context of semi-Riemannian manifolds.

Theorem 3.18. [**6**] *Let $\phi : M^m \to N^n$ be a horizontally conformal submersion. Then*
a) if $n = 2$, its fibres are minimal if and only if it is harmonic.
b) if $n \neq 2$, two of the following conditions imply the other:

1) *ϕ is harmonic,*
2) *ϕ has minimal fibres,*
3) *ϕ is horizontally homothetic.*

PROOF. [**36**] Let $\{Z_1, \dots, Z_n\}$ be a local orthonormal frame for TN and for $i = 1, \dots, n$ let X_i be the horizontal lift of Z_i. If λ is the dilation of ϕ, then $\{\lambda X_1, \dots, \lambda X_n\}$ is a local orthonormal frame for the horizontal distribution $\mathcal{H}$. If X, Y are basic vector fields on M which are ϕ-related to $\hat{X}$ and $\hat{Y}$, respectively, and if $\mathcal{H}$ denotes the projection on the horizontal space, then we get:

$$\begin{aligned}
\mathcal{H}(\nabla^M_X Y) &= \sum_k g(\nabla^M_X Y, \lambda X_k)\lambda X_k \\
&= \lambda^2 \sum_k g(\nabla^M_X Y, X_k) X_k \\
&= \frac{\lambda^2}{2} \sum_k \Big(X g(Y, X_k) + Y g(X_k, X) - X_k g(X, Y) \\
&\quad + g([X,Y], X_k) + g([X_k, X], Y) - g([Y, X_k], X) \Big) X_k \\
&= \frac{\lambda^2}{2} \sum_k \Big(h(\hat{Y}, Z_k) X(\lambda^{-2}) + \lambda^{-2} \hat{X} h(\hat{Y}, Z_k) + h(Z_k, \hat{X}) Y(\lambda^{-2}) \\
&\quad + \lambda^{-2} \hat{Y} h(Z_k, \hat{X}) - h(\hat{X}, \hat{Y}) X_k(\lambda^{-2}) - \lambda^{-2} Z_k h(\hat{X}, \hat{Y}) \\
&\quad + \lambda^{-2} \Big(h([\hat{X}, \hat{Y}], Z_k) + h([Z_k, \hat{X}], \hat{Y}) - h([\hat{Y}, Z_k], \hat{X}) \Big) \Big) X_k
\end{aligned}$$

Hence

$$\begin{aligned}
d\phi(\nabla^M_X Y) &= \nabla^N_{\hat{X}} \hat{Y} \\
&\quad + \frac{\lambda^2}{2} \Big(X(\lambda^{-2}) \hat{Y} + Y(\lambda^{-2} \hat{X}) - h(\hat{X}, \hat{Y}) \sum_k X_k(\lambda^{-2}) Z_k \Big).
\end{aligned}$$

In particular, for $X = Y = X_k$ we obtain

$$\begin{aligned}
d\phi(\nabla^M_{X_k} X_k) &= \lambda^2 X_k(\lambda^{-2}) Z_k - \frac{1}{2} d\phi(\mathrm{grad}_{\mathcal{H}}(\lambda^{-2})) + \nabla^N_{Z_k} Z_k \\
&= d\phi(\lambda X_k(\lambda^{-2}) \lambda X_k) - \frac{1}{2} d\phi(\mathrm{grad}_{\mathcal{H}}(\lambda^{-2})) + \nabla^N_{Z_k} Z_k.
\end{aligned}$$

This gives us

$$\begin{aligned}\tau_{\mathcal{H}}(\phi) &= \sum_{k=1}^{n} \hat{\nabla} d\phi(\lambda X_k, \lambda X_k) \\ &= \lambda^2 \sum_{k=1}^{n} \hat{\nabla} d\phi(X_k, X_k) \\ &= \lambda^2 \sum_{k=1}^{n} (\nabla^N_{Z_k} Z_k - d\phi(\nabla^M_{X_k} X_k)) \\ &= \lambda^2 \sum_{k=1}^{n} \Big(\frac{1}{2} d\phi(\mathrm{grad}_{\mathcal{H}}(\lambda^{-2})) - d\phi(\lambda X_k(\lambda^{-2})\lambda X_k) \Big) \\ &= \frac{\lambda^{-2}}{2}(2-n) d\phi(\mathrm{grad}_{\mathcal{H}}(\lambda^2)).\end{aligned}$$

If $\{V_{n+1}, \dots, V_m\}$ is a local orthonormal frame for the vertical distribution, we finally come to the conclusion

$$\begin{aligned}\tau(\phi) &= \tau_{\mathcal{H}}(\phi) + \sum_{k=n+1}^{m} \hat{\nabla} d\phi(V_k, V_k) \\ &= \tau_{\mathcal{H}}(\phi) - \sum_{k=n+1}^{m} d\phi(\nabla^M_{V_k} V_k) \\ &= \tau_{\mathcal{H}}(\phi) - d\phi(\sum_{k=n+1}^{m} \nabla^M_{V_k} V_k) \\ &= \frac{\lambda^{-2}}{2}(2-n) d\phi(\mathrm{grad}_{\mathcal{H}}(\lambda^2)) - (m-n) d\phi(H),\end{aligned}$$

where H is the mean curvature vector field of the fibres. The theorem now immediately follows from the last equation. □

As a direct consequence of Theorem 3.18 we obtain the following classical result of Eells and Sampson.

Corollary 3.19. [**24**] *A Riemannian submersion has minimal fibres if and only if it is harmonic.*

Example 3.20. A simple example of a harmonic morphism is the orthogonal projection followed by a multiplication of a scalar:

$$\mathbb{R}^m \ni (x_1, \dots, x_m) \mapsto a(x_1, \dots, x_n) \in \mathbb{R}^n$$

for some $a \in \mathbb{R} \setminus \{0\}$. It has trivially constant dilation $|a|$, totally geodesic fibres and integrable horizontal distribution.

Example 3.21. Both the maps π and γ from Examples 2.21 and 2.22 are harmonic morphisms since they are horizontally conformal with dilations $x \mapsto |x|^{-1}$ and $x \mapsto (1-|x|^2)/2|x|$. Both have totally geodesic fibres and integrable horizontal distributions.

Example 3.22. Let G be a Lie group with a bi-invariant metric and K be a Lie subgroup of G. Denote by $\pi : G \to G/K$ the canonical projection onto the homogeneous space G/K. We may in a canonical way turn G/K into a Riemannian manifold with a G-invariant metric such that π is a Riemannian submersion. For left-invariant vector fields X, Y on G it is well known that

$$\nabla^G_X Y = \frac{1}{2}[X, Y].$$

In particular, if X and Y are left-invariant vector fields on K then the horizontal component of $\nabla^G_X Y$ is zero. Thus K is a totally geodesic submanifold of G and it follows that π has totally geodesic fibres. By Theorem 3.18 the canonical projection $\pi : G \to G/K$ is a harmonic morphism.

Example 3.23. If $\phi : M^m \to N^m$ is a conformal submersion between Riemannian manifolds of equal dimensions m then we get from the proof of Theorem 3.18

$$\tau_{\mathcal{H}}(\phi) = \tau(\phi) = \frac{\lambda^{-2}}{2}(2-m)d\phi(\mathrm{grad}(\lambda^2)). \tag{3.7}$$

It follows that a map $\phi : M^2 \to N^2$ is a harmonic morphism if and only if ϕ is conformal. If M and N are Riemann surfaces we know from Example 3.13 that every ±holomorphic map is a harmonic morphism.

Conversely, if ϕ is a harmonic morphism between Riemann surfaces, we see from equation (3.4) that for every $x \in M$ either $\partial\phi/\partial z(x) = 0$ or $\partial\phi/\partial\bar{z}(x) = 0$. Now ϕ and hence also $\partial\phi/\partial z$ and $\partial\phi/\partial\bar{z}$ are real analytic by the ellipticity of equation (2.5), so ϕ will be ±holomorphic on M. Thus a map between (connected) Riemann surfaces is a harmonic morphism if and only if it is ±holomorphic.

We also deduce the following.

Theorem 3.24. *If $\phi : M \to N$ is a map between Riemannian manifolds of equal dimensions $m \geq 3$, then ϕ is a harmonic morphism if and only if ϕ is a homothety, i.e. conformal with constant dilation.*

Proof. If ϕ is a harmonic morphism with dilation λ which is not constantly zero on M, set $M' = M \setminus C_\phi$ which is an open dense subset of M. The restriction of ϕ to M' is a harmonic morphism and from equation (3.7) it follows that $\mathrm{grad}(\lambda^2) = 0$ on M' and then by continuity on the whole of M. Hence λ is constant. Conversely, if ϕ is conformal with constant dilation then ϕ is harmonic by equation (3.7). $\square$

Example 3.25. Represent $S^m \setminus S^0$ in $\mathbb{R}^{m+1}$ as

$$S^m \setminus S^0 = \{(\cos t, \sin t \cdot e) \mid t \in (0,\pi),\ e \in S^{m-1}\}.$$

Define $\eta : S^m \setminus S^0 \to S^{m-1}$ as the projection onto the equatorial hypersphere along the longitudes:

$$\eta(\cos t, \sin t \cdot e) = e.$$

The horizontal curves for η are those for which t is constant so η is horizontally conformal with dilation

$$(\cos t, \sin t \cdot e) \mapsto 1/\sin t.$$

Hence η is horizontally homothetic and since it has minimal fibres it follows from Theorem 3.18 that η is a harmonic morphism. It has integrable horizontal distribution with small spheres $(\sin t)S^{m-1}$ as integral submanifolds. It is clear that η cannot be extended continuously to the whole of S^m.

Example 3.26. Let $\mathbb{D}$ be any of the normed division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or Cay of real dimension $d = 1, 2, 4$ or 8, respectively. The *Hopf polynomials* are then defined as the maps $\mathbb{R}^{2d} \to \mathbb{R}^{d+1}$ given by

$$(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\bar{z}_2)$$

for $(z_1, z_2) \in \mathbb{D}^2$. These are all harmonic morphisms, defined by homogeneous polynomials of degree 2, with dilation $x \mapsto 2|x|^{d-1}$. Their fibres are spheres of dimension $d-1$ so they cannot be minimal in $\mathbb{R}^{2d}$ for $d > 1$. This shows that there is no equivalence in the last part of Theorem 3.18.

We see that the Hopf polynomials all restrict to maps

$$S^{2d-1} \to S^d$$

called the *Hopf fibrations*. They are horizontally conformal with constant dilation (see the proof of Theorem 5.5) and they are harmonic by Example 2.23 or by Theorem 3.18 since their fibres are totally geodesic in S^{2d-1}. Hence they are submersive harmonic morphisms, surjective by Corollary 3.12.

There are several ways of describing the Hopf fibrations (see [**32**]) and they have been of great importance for the development of homotopy theory and fibre spaces. As we shall see in the next chapter the Hopf fibrations and the Hopf polynomials are also of great interest in the study of harmonic morphisms.

If we have a look at the case of $d = 2$, denoting by $\phi : S^3 \to S^2$

the corresponding Hopf fibration, we make the standard identification of S^2 with $\mathbb{C}P^1$. Considered as a map $S^3 \to \mathbb{C}P^1 = S^3/S^1$, ϕ is easily seen to be the quotient map taking (z_1, z_2) to the complex line in $\mathbb{C}^2$ through this point. This is the way the Hopf fibrations generally are presented.

If we identify S^3 with $SU(2)$ via the map

$$S^3 \ni (z_1, z_2) \mapsto \begin{pmatrix} z_1 & \bar{z_2} \\ -z_2 & \bar{z_1} \end{pmatrix} \in SU(2),$$

we note that the horizontal space of ϕ at the identity of $SU(2)$ is spanned by

$$X = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and the vertical space by

$$V = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Since

$$[X, Y] = XY - YX = 2V$$

the horizontal distribution of ϕ is not integrable. This is actually true for all but the first of the Hopf fibrations as will be motivated in the following section.

3. The Existence Problem

There is no general existence theory for harmonic morphisms; indeed, by Theorem 3.8, harmonic morphisms are solutions to an overdetermined non-linear system of partial differential equations, thus making the question of existence rather difficult. In most cases, the only known way of proving existence of non-constant harmonic morphisms between given Riemannian manifolds is by a direct construction of examples. Another way is to give harmonic morphisms as implicitly defined solutions to certain non-singular equations, a method that goes back to Jacobi (see Chapter 1). This was used by Gudmundsson in [**36**] to construct complex valued harmonic morphisms defined locally in several irreducible Riemannian symmetric spaces. In some non-compact cases even globally defined solutions were found. Similar methods have been used by Baird and Wood in [**9**] and [**10**] to construct both globally and locally defined complex valued harmonic morphisms from Euclidean spaces. We shall see in the next chapter that interesting existence theorems for polynomial harmonic morphisms have been achieved. In contrast, to find globally defined harmonic morphisms between *compact* Riemannian manifolds has offered great difficulties. Only a few examples are known, among them the Hopf fibrations of Example 3.26 (see The Atlas of Harmonic Morphisms [**39**]). When the co-domain is a compact Riemann surface though, we have the following interesting existence result by Gudmundsson.

Theorem 3.27. [**37**] *Let N^2 be a compact Riemann surface and ω a homotopy class of S^n-bundles over N^2. Then there exists a bundle $\pi : M \to N^2$ in ω and a Riemannian metric g on M such that the bundle map $\pi : (M, g) \to N^2$ is a harmonic morphism.*

More is known about the non-existence of harmonic morphisms. For instance, the existence of a non-constant harmonic morphism $\phi : M^m \to N^n$ immediately gives necessary conditions on the dimensions m and n, namely that $m \geq n$. In the next chapter we shall derive some further necessary conditions on the dimensions for the existence of solutions to the problem. It is also clear that Corollary 3.12 constitutes a simple but important non-existence result.

Wood observed in [**59**] that if $\phi : M \to N$ is a horizontally conformal submersion with integrable horizontal distribution $\mathcal{H}$, then the associated foliation $\mathcal{F}_{\mathcal{H}}$ is *totally umbilic* in M. This means that for a horizontal unit vector field X, the vertical field $\mathcal{V}(\nabla^M_X X)$ does not depend on the choice of X. For horizontal vector fields X and Y we have in this case (Theorem 2.1.3 of [**35**])

$$\mathcal{V}(\nabla^M_X Y) = -\frac{\lambda^2}{2} g(X, Y)\mathcal{V}(\mathrm{grad}(\lambda^{-2})).$$

If in addition $m > n \geq 3$ and ϕ is a harmonic morphism with totally geodesic fibres, then $\mathcal{F}_{\mathcal{H}}$ is *spherical* i.e. for every leaf $L \in \mathcal{F}_{\mathcal{H}}$ the mean curvature vector H_L of L in M is parallel in the normal bundle of the leaf. Gudmundsson generalized O'Neill's fundamental equations for a submersion (see [**47**]) to the horizontally conformal case. Using this and a characterization of the totally umbilic spherical foliations on manifolds with constant sectional curvature, he proved in [**35**] (Corollary 3.3.4) the following non-existence result.

Theorem 3.28. [**35**] *If $m \geq n \geq 3$ and $(M^m, N^n) = (S^m, S^n)$, $(\mathbb{R}^m, S^n)$ or (H^m, S^n), then there are no non-constant harmonic morphisms from M to N with totally geodesic fibres and integrable horizontal distribution.*

The last result implies that none but the first of the classical Hopf fibrations of Example 3.26 have integrable horizontal distribution.

A natural way to impose necessary curvature conditions on the manifolds for the existence of non-constant harmonic morphisms is by a

Bochner technique. This method can briefly be described as developing an equation relating the Laplacian of bundle valued sections to the curvature of the bundle in question, known as a Weitzenböck formula (see [**21**]). From this formula, vanishing results involving the curvature can be derived. This was done for harmonic maps by Eells and Sampson in [**24**] and for harmonic morphisms by Mustafa in [**45**]. Using this Mustafa then proved the following important theorem.

Theorem 3.29. [**45**] *Let M^m be a compact Riemannian manifold with non-negative Ricci curvature and let N^2 be a compact Riemann surface of genus $\gamma \geq 2$. Then any harmonic morphism from M^m to N^2 is constant.*

By choosing suitable Riemannian metrics on the space M and arguing as in Example 3.14, Mustafa obtained the following:

1. There are no non-constant harmonic morphisms from a compact irreducible Riemannian symmetric space to a compact Riemann surface of genus $\gamma \geq 1$.
2. There are no non-constant harmonic morphisms from a compact connected Lie group with a bi-invariant metric to a compact Riemann surface of genus $\gamma \geq 1$.

In [**46**] Mustafa developed the Bochner technique further to non-compact domains in the language of moving frames and derived the following astonishing result:

Theorem 3.30. [**46**] *There is no non-constant globally defined harmonic morphism from $\mathbb{R}^m$ into a Riemannian manifold with scalar curvature bounded above by a negative number.*

As a direct consequence there are no non-constant harmonic morphisms from $\mathbb{R}^m$ into the hyperbolic space H^n.

4. Polar Sets

In this section we shall return to potential theory for some elementary definitions needed in the next chapter. Note that the functions and maps are here unless otherwise stated, not assumed to be smooth. Recall that if $\mathcal{X}$ is a harmonic space (see Chapter 1) and $\mathcal{H}$ its sheaf of harmonic functions, then for every regular set V in $\mathcal{X}$ and continuous function $f : \partial V \to \mathbb{R}$, there is a unique harmonic function H_f^V on V, which can be extended to $\bar{V}$ so that the extension equals f on the boundary ∂V. Furthermore if $f \geq 0$ then $H_f^V \geq 0$. From this it easily follows that for every $x \in V$, the map

$$f \mapsto H_f^V(x)$$

is a positive linear functional on the set of continuous functions on ∂V. By the Riesz Representation Theorem there exists a unique Radon measure ρ_x^V on ∂V so that

$$H_f^V(x) = \int f(y) d\rho_x^V(y)$$

for every continuous function f on ∂V. The measure ρ_x^V is called *the harmonic measure for V at the point x.*

Example 3.31. Let σ denote the normalized volume measure on the unit sphere S^{m-1}. If B is the unit ball in $\mathbb{R}^m$ and f a continuous function on the boundary $\partial B = S^{m-1}$, then it is well known that the Poisson integral (see [**3**])

$$B \ni x \mapsto \int \frac{1 - |x|^2}{|x - y|^m} f(y) d\sigma(y)$$

is harmonic and may be extended continuously to $\bar{B}$ so that the extension equals f on S^{m-1}. Thus the harmonic measure for B at x is

$$d\rho_x^B(y) = \frac{1 - |x|^2}{|x - y|^m} d\sigma(y).$$

We now extend Definition 2.29 to a much more general situation.

Definition 3.32. Given an open set U in a harmonic space $\mathcal{X}$, a function $f : U \to]-\infty, \infty]$ is said to be *superharmonic* if:

i) f is lower semi-continuous,
ii) f is not identically ∞ on any component of U and
iii) for every regular set $V \subseteq \bar{V} \subseteq U$ and every $x \in V$, we have

$$f(x) \geq \int f(y) d\rho_x^V(y).$$

It was shown by Hervé (see Chapter 7 of [**40**]) that this definition is consistent with the one given in Chapter 2. The following concept is classical in potential theory.

Definition 3.33. Let $\mathcal{X}$ be a harmonic space. A subset $E \subseteq \mathcal{X}$ is said to be *polar* if for every point $x \in \mathcal{X}$ there exists a neighbourhood U and a superharmonic function f in U such that $f = \infty$ on $U \cap E$.

Although polar sets are generally small enough to be ignored, we will need some basic facts on these in the next chapter.

Example 3.34. If $n \geq 3$, then any point of $\mathbb{R}^n$ is polar. More generally if $n \geq 3$, every affine subspace $A \subset \mathbb{R}^n$ of codimension at least 2 is polar. Furthemore, the points of any relatively compact subset U

of $\mathbb{R}^2$ are polar in U. These results can be found in any elementary book on potential theory (see e.g. [**14**]).

Example 3.35. For a non-constant harmonic morphism ϕ between Riemannian manifolds, we know that its critical set C_ϕ is closed and nowhere dense. It was proved by Fuglede in [**28**] that it is actually polar.

If $\phi : M \to N$ is a harmonic morphism between Riemannian manifolds and f is a smooth superharmonic function on N, then we have by equation (3.6):

$$\triangle^M(f \circ \phi) = \lambda^2 \triangle^N(f) \leq 0.$$

Hence $f\circ\phi$ is superharmonic. Furthermore, Constantinescu and Cornea proved that a general harmonic morphism between harmonic spaces pulls back superharmonic functions to superharmonic functions (see [**18**], Corollary 3.2). This implies the following result needed for one of our main results of the next chapter:

Proposition 3.36. [**18**] *The pull-back of a polar set by a non-constant harmonic morphism is again a polar set.*

CHAPTER 4

Polynomial Harmonic Morphisms

In this chapter we study polynomial harmonic morphisms between Euclidean spaces. By a *polynomial* map $\mathbb{R}^m \to \mathbb{R}^n$ we mean a map with polynomial components. Its *degree* is the maximal degree of these components. In Section 1 we study general globally defined harmonic morphisms between Euclidean spaces. Why this is done in the context of polynomial harmonic morphisms is motivated by Theorem 4.3, which states that if $n \geq 3$, every globally defined harmonic morphism $\mathbb{R}^m \to \mathbb{R}^n$ is polynomial. Section 2 is devoted to a classification of the homogeneous polynomial harmonic morphisms of degree 2 due to Ou and finally in Section 3 we discuss some homogeneous polynomial harmonic morphisms of higher degree.

1. Globally Defined Harmonic Morphisms

Globally defined harmonic morphisms between Euclidean spaces have some very characteristic features, at least if the dimensions are large enough. We shall here discuss some striking results on such maps all of which were presented by Ababou, Baird and Brossard in the very recent manuscript [**1**].

Theorem 4.1. [**1**] *If $n \geq 3$ and $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a non-constant harmonic morphism, then ϕ is surjective.*

Proof. [**1**] For any $a \in \mathbb{R}^n$ the function $x \mapsto |x-a|^{2-n}$, $x \in \mathbb{R}^n \setminus \{a\}$ is harmonic. If there was some $a \in \mathbb{R}^n$ not in the image of ϕ then

$$x \mapsto |\phi(x) - a|^{2-n}$$

would be a positive harmonic function on the whole of $\mathbb{R}^m$ and therefore constant by the theorem of Liouville (see [**3**]). Hence ϕ would be constant which leads to a contradiction. $\square$

Example 4.2. The map

$$\mathbb{C} \ni z \mapsto e^z \in \mathbb{C}$$

is obviously not surjective but a harmonic morphism since it is holomorphic. This simple example shows that the statement of Theorem 4.1 is not true for $n = 2$.

Next we show a Liouville type of theorem for harmonic morphisms.

Theorem 4.3. [**1**] *Every globally defined harmonic morphism* $\phi : \mathbb{R}^m \to \mathbb{R}^n$ *with* $n \geq 3$ *is a polynomial map of degree* $\leq \frac{m-2}{n-2}$.

To prove Theorem 4.3 we need the following potential theoretic lemma. For a proof of this using probabilistic arguments, see Lemma 1.2 of [**1**].

Lemma 4.4. [**1**] *Suppose that* $m \geq 3$ *and* P *is a closed polar set in* $\mathbb{R}^m$. *If* $f : \mathbb{R}^m \setminus P \to \mathbb{R}$ *is a positive harmonic function, then there is a constant* $c > 0$ *such that*

$$f(x) \geq c(1 + |x|^{m-2})^{-1}$$

for all $x \in \mathbb{R}^m \setminus P$.

PROOF OF THEOREM 4.3. [**1**] If $m < n$ then by Corollary 3.9 the map ϕ is constant and we are done. Assume $m \geq n$. If $h : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is the harmonic function $h(x) = |x|^{2-n}$, then the composition

$$h \circ \phi : \mathbb{R}^m \setminus \phi^{-1}(0) \to \mathbb{R}$$

is a positive harmonic function and by Propositon 3.36 defined off a closed polar set. By Lemma 4.4 there is a constant $c > 0$ such that

$$|\phi(x)|^{n-2} \leq c^{-1}(1 + |x|^{m-2})$$

which obviously will hold for all $x \in \mathbb{R}^m$. If we write $\phi = (\phi^1, \dots, \phi^n)$ then this implies that for any $1 \leq i \leq n$ we will have

$$|\phi^i(x)| \leq a(1 + |x|^{m-2})^{\frac{1}{n-2}} \leq a(1 + |x|^{\frac{m-2}{n-2}})$$

for all $x \in \mathbb{R}^m$ and some positive constant a. Each ϕ^i is harmonic so by the Cauchy estimates (see [**3**]) and the above inequality it follows that the power series expansion of ϕ^i around 0 is a polynomial of degree $\leq \frac{m-2}{n-2}$. □

The following example shows that the condition $n \geq 3$ on the dimension of the target manifold in Theorem 4.3 can not be removed; Let $\psi : \mathbb{C}^m \to \mathbb{C}$ be any holomorphic function and define

$$F(z) = \cos(\psi(z))$$

for $z \in \mathbb{C}^m$. Since F is holomorphic it is a harmonic morphism and not polynomial except in trivial cases.

For obvious reasons the next result is together with Theorem 4.3 very important in the study of globally defined harmonic morphisms between Euclidean spaces.

Theorem 4.5. [**1**] *Every horizontally conformal polynomial* $F : \mathbb{R}^m \to \mathbb{R}^n$ *with* $n \geq 2$ *is harmonic.*

Here we give our corrected version of the proof contained in the original manuscript of [**1**].

PROOF. It is enough to prove the statement for $n = 2$. For if the result holds in this case and $F = (F_1, \dots, F_n) : \mathbb{R}^m \to \mathbb{R}^n$ is a horizontally conformal polynomial where $n \geq 2$, then (F_i, F_j) is a horizontally conformal polynomial $\mathbb{R}^m \to \mathbb{R}^2$ for any indices $i \neq j$. Hence F_i is harmonic for every i so F is harmonic.

Assume that F is a complex valued horizontally conformal polynomial on $\mathbb{R}^m$. If F is constant we are done. Otherwise, denote the part of F that is homogeneous of degree i by Q_i, hence

$$F = \sum_{i=0}^{d} Q_i,$$

where d is the degree of F. By choosing suitable coordinates (x) in $\mathbb{R}^m$ we may assume that 0 is not a singular point of F and that

$$Q_1(x_1, \dots, x_m) = x_1 + ix_2.$$

We write $z = x_1 + ix_2$. From the horizontal conformality of F we get

$$\begin{aligned}
0 &= \sum_{i=1}^{m} (\frac{\partial F}{\partial x_i})^2 \\
&= 4\frac{\partial F}{\partial z}\frac{\partial F}{\partial \bar{z}} + \sum_{i\geq 3}(\frac{\partial F}{\partial x_i})^2 \\
&= 4\sum_{r,s}\frac{\partial Q_r}{\partial z}\frac{\partial Q_s}{\partial \bar{z}} + \sum_{i\geq 3}\sum_{r,s}\frac{\partial Q_r}{\partial x_i}\frac{\partial Q_s}{\partial x_i} \\
&= 2\sum_{r,s}(\frac{\partial Q_r}{\partial z}\frac{\partial Q_s}{\partial \bar{z}} + \frac{\partial Q_r}{\partial \bar{z}}\frac{\partial Q_s}{\partial z}) + \sum_{r,s}\langle \nabla_x Q_r, \nabla_x Q_s \rangle,
\end{aligned}$$

where we have written

$$\nabla_x Q_r = (\frac{\partial Q_r}{\partial x_3}, \dots, \frac{\partial Q_r}{\partial x_m}).$$

Changing the order of summation implies that

$$\sum_{p\geq 2}\Big(2\sum_{r=1}^{p}(\frac{\partial Q_r}{\partial z}\frac{\partial Q_{p+1-r}}{\partial \bar{z}} + \frac{\partial Q_r}{\partial \bar{z}}\frac{\partial Q_{p+1-r}}{\partial z}) + \sum_{r=2}^{p-1}\langle \nabla_x Q_r, \nabla_x Q_{p+1-r}\rangle\Big) = 0$$

and from the homogeneity we conclude that for $p \geq 2$ we have

(4.1)

$$2\sum_{r=1}^{p}(\frac{\partial Q_r}{\partial z}\frac{\partial Q_{p+1-r}}{\partial \bar{z}} + \frac{\partial Q_r}{\partial \bar{z}}\frac{\partial Q_{p+1-r}}{\partial z}) + \sum_{r=2}^{p-1}\langle \nabla_x Q_r, \nabla_x Q_{p+1-r}\rangle = 0.$$

Claim 1: For $p \geq 2$, Q_p is of degree $\leq p-2$ in $\bar{z}$.
Proof of Claim 1: For $p = 2$, equation 4.1 reduces to

$$4\frac{\partial Q_2}{\partial \bar{z}} = 0 \tag{4.2}$$

so this is certainly true for $p = 2$. If it is assumed to hold for $Q_2, \dots,$ Q_{p-1}, it then follows from equation (4.1) that it also holds for Q_p. Hence our claim follows by induction. □

Now let $\hat{L}_r(z, x)$ denote the coefficient of $\bar{z}^{r-2}$ in Q_r for $r \geq 2$, where $x = (x_3, \dots, x_m)$. From equation (4.2) we see that $Q_2 = \hat{L}_2$. Furthermore, the coefficient of $\bar{z}^{p-3}$ in equation (4.1) must be

$$4(p-2)\hat{L}_p + \sum_{r=2}^{p-1}\langle \nabla_x \hat{L}_r, \nabla_x \hat{L}_{p+1-r}\rangle = 0.$$

In particular, if $L_r(x) = \hat{L}_r(0, x)$ we see that

$$4(p-2)L_p + \sum_{r=2}^{p-1}\langle \nabla_x L_r, \nabla_x L_{p+1-r}\rangle = 0. \tag{4.3}$$

We write $L_2(x) = x^t A x$, for some symmetric matrix A.
Claim 2: For $p \geq 2$ we have

$$L_p(x) = (-1)^p x^t A^{p-1} x.$$

Proof of Claim 2: This holds by definition for $p = 2$. If it is assumed to hold for $L_2, \dots, L_{p-1}$, then we have

$$(\nabla_x L_r)(x) = 2(-1)^r A^{r-1} x$$

for $r = 2, \dots, p-1$. Hence from equation (4.3) we see that

$$L_p(x) = (-1)^p x^t A^{p-1} x$$

and our claim follows by induction. □

Since d is the degree of F it follows that

$$x^t A^p x = 0$$

for $p \geq d$. Since A is assumed to be symmetric this implies that A is nilpotent so $\text{trace}(A) = 0$. Thus we get at the point 0

$$\tau(F)(0) = 4\frac{\partial^2 Q_2}{\partial z \partial \bar{z}} + 2\ \text{trace}(A) = 0.$$

The origin was arbitrarily chosen among the non-singular points of F, so it follows that

$$\tau(F)(p) = 0$$

for every non-singular point p of F. Since F is a horizontally conformal *polynomial*, its set of critical points is nowhere dense. Hence F is harmonic. □

Following Theorem 4.3 and Theorem 4.5 the study of globally defined harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$ with $n \geq 3$ is reduced to the study of horizontally conformal polynomials of degree $\leq \frac{m-2}{n-2}$.

Corollary 4.6. [$*$] *If $n \leq m \leq 2n-3$ and $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a non-constant harmonic morphism, then $\phi - \phi(0)$ is an orthogonal projection followed by a homothetic isomorphism.*

PROOF. We assume for simplicity that $\phi(0) = 0$. Since $n \leq 2n-3$ it follows that $n \geq 3$ and $(m-2)/(n-2) < 2$. Hence ϕ is linear by Theorem 4.3. Choose an orthonormal basis $\{f_1, \dots, f_n\}$ of the horizontal space $(\ker \phi)^-$ so that $\phi(f_i) = \lambda e_i$ for $i = 1, \dots, n$, where $\{e_1, \dots, e_n\}$ is the canonical base of $\mathbb{R}^n$ and λ the dilation of ϕ. Then ϕ is easily seen to be the orthogonal projection onto the horizontal space followed by a homothetic isomorphism. □

Example 4.7. If $m \leq 3n-5$ then any harmonic morphism $\mathbb{R}^m \to \mathbb{R}^n$ is polynomial of degree no more than 2. We have in this case the following result which we state without proof:

Theorem 4.8. [**1**] *If $F : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial harmonic morphism of degree 2, then there is an orthonormal basis for $\mathbb{R}^m$ such that $F - F(0)$ is either homogeneous of degree 2 or of the form*

$$(x_1, \dots, x_m) \mapsto \alpha(x_1, \dots, x_n) + Q(x_{n+1}, \dots, x_m)$$

for some $\alpha \in \mathbb{R}$ and some polynomial harmonic morphism

$$Q : \mathbb{R}^{m-n} \to \mathbb{R}^n$$

homogeneous of degree 2.

Theorem 4.8 together with the results of Section 2 give a complete classification of all harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$ with $m \leq 3n-5$.

2. The Classification of Ou

In this section we study harmonic morphisms defined by homogeneous polynomials of degree 2. Due to a connection with the representation theory of Clifford algebras we obtain a complete classification of these. In our presentation of this classification we follow the work of Ou and Wood in [**50**] and Ou in [**49**]. We also derive some results needed in the next chapter.

Definition 4.9. A map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is said to be *quadratic* if each of its components are quadratic forms on $\mathbb{R}^m$. The set of all quadratic harmonic morphisms $\mathbb{R}^m \to \mathbb{R}^n$ will be denoted by $H_2(m, n)$. Two quadratic maps ϕ, ψ are said to be *domain-equivalent* if there is an $A \in O(\mathbb{R}^m)$ such that $\phi = \psi \circ A$ and *bi-equivalent* if there is a $B \in O(\mathbb{R}^m)$ and a $C \in O(\mathbb{R}^n)$ such that $\phi = C \circ \psi \circ B$.

We see that domain-equivalence amounts to an orthonormal change of coordinates of $\mathbb{R}^m$ and bi-equivalence to an orthonormal change of coordinates in both the domain and the co-domain.

If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a quadratic map we may write

$$\phi^i(x) = \langle A_i x, x\rangle, \qquad x \in \mathbb{R}^m$$

for $i = 1, \dots, n$ and some symmetric $A_i \in \mathrm{End}(\mathbb{R}^m)$. Henceforth we make no distinction between $\mathrm{End}(\mathbb{R}^m)$ and the set of $m \times m$-matrices over $\mathbb{R}$ thus refering to A_i as the *i-th component matrix of* ϕ. We also equip $\mathrm{End}(\mathbb{R}^m)$ with the standard inner product

$$\langle A, B\rangle = \frac{1}{m}\mathrm{trace}(A^t B).$$

Definition 4.10. A quadratic map $\phi : \mathbb{R}^m \to \mathbb{R}^n$ with component matrices A_i, $i = 1, \dots, n$ is said to be *separable* if it is possible to write $\mathbb{R}^m$ as a direct sum of non-trivial subspaces, invariant under A_i for every i. Otherwise ϕ is said to be *non-separable.*

From Theorem 3.8 we may now derive the following equations.

Proposition 4.11. [**50**] *A quadratic map* $\phi : \mathbb{R}^m \to \mathbb{R}^n$ *with component matrices* A_i, $i = 1, \dots, n$, *is a harmonic morphism if and only if*

a) $A_i^2 = A_j^2$ *for all i,j,*
b) $A_i A_j + A_j A_i = 0$ *for all* $i \neq j$, *and*
c) $\mathrm{trace}(A_i) = 0$ *for all i.*

PROOF. [**50**] For every i we have

$$\operatorname{grad}(\phi^i)(x) = 2A_i x \quad \text{and} \quad \triangle^{\mathbb{R}^m}(\phi^i) = 2\ \operatorname{trace}(A_i).$$

Thus c) is immediate and a) and b) follow from Example 3.6. Observe that for b) A_iA_j is not symmetric in general. □

Example 4.12. For $t \in \mathbb{R}$ we see that $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\begin{aligned}\phi(x,y) =&(x^2\cos t - y^2\cos t + 2xy\sin t,\\ &- x^2\sin t + y^2\sin t + 2xy\cos t)\end{aligned}$$

has component matrices

$$\begin{pmatrix}\cos t & \sin t\\ \sin t & -\cos t\end{pmatrix}, \begin{pmatrix}-\sin t & \cos t\\ \cos t & \sin t\end{pmatrix}.$$

They do satisfy the equations of Proposition 4.11 so ϕ is a quadratic harmonic morphism.

Definition 4.13. For manifolds M, N and maps $f : M \to \mathbb{R}^n$ and $g : N \to \mathbb{R}^n$ we define their *direct sum* as the map $f \oplus g : M \times N \to \mathbb{R}^n$ by $(f \oplus g)(x, y) = f(x) + g(y)$, where $M \times N$ is given the usual product structure.

It follows directly from Proposition 4.11 that a direct sum of quadratic harmonic morphisms gives new quadratic harmonic morphisms. The direct sum construction can of course be applied to more than two terms.

Lemma 4.14 (The Rank Lemma). [**50**] *Let $\phi \in H_2(m, n)$ have component matrices $A_1, \dots, A_n$. Then*

1. *all the component matrices have the same rank which is an even number,*
2. *all the component matrices have the same eigenvalues, and*
3. *the map ϕ is domain-equivalent to a quadratic harmonic morphism ψ with component matrices $F_1, \dots, F_n$, with*

$$F_1 = \begin{pmatrix}D & 0 & 0\\ 0 & -D & 0\\ 0 & 0 & 0\end{pmatrix}, \quad F_i = \begin{pmatrix}0 & B_i & 0\\ B_i^t & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad i = 2, \dots, n,$$

where D is a diagonal matrix with only positive diagonal entries and B_i matrices satisfying

$$\begin{aligned}&B_iD = DB_i, \quad B_iB_i^t = D^2 \quad \textit{for all } i = 1, \dots, n, \textit{ and}\\ &B_i^tB_j + B_j^tB_i = 0 \quad \textit{for all } i, j = 1, \dots, n, \ i \neq j.\end{aligned}$$

The map ψ is said to be of the normal form.

The proof of this lemma is a simple application of the equations in Proposition 4.11, using well-known facts about symmetric matrices. We omit this and refer to page 47 of [**50**].

Example 4.15. The Hopf polynomials of Example 3.26 are all quadratic harmonic morphisms

$$\mathbb{R}^{2d} \to \mathbb{R}^{d+1}$$

for $d = 1, 2, 4$ or 8. They are all of normal form in the canonical basis of $\mathbb{R}^{2d}$. For $d = 1$, $\phi(x, y) = (x^2 - y^2, 2xy)$ has component matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $d = 2$ the corresponding Hopf polynomials are given by

$$\begin{aligned}\phi(x_1, x_2, x_3, x_4) =&(x_1^2 + x_2^2 - x_3^2 - x_4^2,\\ &2x_1x_3 + 2x_2x_4, -2x_1x_4 + 2x_2x_3),\end{aligned}$$

with component matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

The next one with $d = 4$ is given by

$$\begin{aligned}\phi(x_1, \dots, x_8) =&(x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2,\\ &2x_1x_5 + 2x_2x_6 + 2x_3x_7 + 2x_4x_8,\\ &- 2x_1x_6 + 2x_2x_5 - 2x_3x_8 + 2x_4x_7,\\ &- 2x_1x_7 + 2x_2x_8 + 2x_3x_5 - 2x_4x_6,\\ &- 2x_1x_8 - 2x_2x_7 + 2x_3x_6 + 2x_4x_5)\end{aligned}$$

so it is also of the normal form. It is easy to see that this is also the case for $d = 8$.

Definition 4.16. If $\phi \in H_2(m, n)$, then the common rank of its component matrices is called the *Q-rank* of ϕ. If the Q-rank of ϕ is m, then ϕ is said to be *Q-non-singular.* If $\phi \in H_2(m, n)$ is Q-non-singular and the positive eigenvalues of its component matrices are the same then ϕ is said to be *umbilical.* The set of all umbilical elements of $H_2(m, n)$ with positive eigenvalue 1 is denoted by $H_2^1(m, n)$.

Example 4.17. We see that the Hopf polynomials are all umbilical with positive eigenvalue 1 and so are the maps of Example 4.12. Note that if $\phi_1 : \mathbb{R}^{m_1} \to \mathbb{R}^n$ and $\phi_2 : \mathbb{R}^{m_2} \to \mathbb{R}^n$ are umbilical quadratic harmonic morphisms with positive eigenvalue 1, then for $\alpha, \beta \in \mathbb{R}$:

$$\alpha\phi_1 \oplus \beta\phi_2 : \mathbb{R}^{m_1+m_2} \to \mathbb{R}^n$$

will in general not be umbilical. We shall later see that this is the only way to construct non-umbilical quadratic harmonic morphisms.

Corollary 4.18. [**50**] *Any quadratic harmonic morphism is the composition of an orthogonal projection followed by a Q-non-singular quadratic harmonic morphism from an even-dimensional space.*

PROOF. The statement is clearly true for a quadratic harmonic morphism of the normal form. The general case follows from Lemma 4.14. □

We may compare Corollary 4.18 with a result of Baird and Wood (see Theorem 4.1 of [**8**]) stating that any non-constant harmonic morphism $\mathbb{R}^3 \to \mathbb{R}^2$ is the composition of an orthonormal projection $\mathbb{R}^3 \to \mathbb{R}^2$ followed by a weakly conformal map. By Corollary 4.18 we see that it is enough to study Q-non-singular quadratic harmonic morphisms from even-dimensional spaces.

Corollary 4.19. [**50**] *If $\phi \in H_2(m, n)$ is umbilical then ϕ is domain-equivalent to a $\psi \in H_2(m, n)$ given by*

$$\psi(x) = \lambda(\langle P_1 x, x\rangle, \dots, \langle P_n x, x\rangle),$$

for some real constant λ and $P_i \in Sym(\mathbb{R}^m)$ satisfying

$$P_i P_j + P_j P_i = 2\delta_{ij} I_m \quad \text{for all } i, j,$$

where I_m is the identity endomorphism of $\mathbb{R}^m$.

PROOF. This follows immediately from Lemma 4.14. □

Definition 4.20. An n-tuple $(P_1, \dots, P_n)$ of symmetric endomorphisms on $\mathbb{R}^m$ satisfying

$$P_i P_j + P_j P_i = 2\delta_{ij} I_m$$

for all $i, j = 1, \dots, n$, is called an *n-dimensional Clifford system on $\mathbb{R}^m$*. The set of all n-dimensional Clifford systems on $\mathbb{R}^m$ is denoted by $C(m, n)$.

We note that from the last definition the only possible eigenvalues for any P_i is ± 1. Since P_j with $i \neq j$ defines an isomorphism between the eigenspaces E_1 and E_{-1} of P_i, these spaces have the same dimension. From the symmetry of P_i, we conclude that $m = 2\ \dim(E_1)$. Hence $C(m, n) = \emptyset$ for m odd. It also follows that $\text{trace}(P_i) = 0$ for all

i, so that every Clifford system defines a quadratic umbilical harmonic morphism with positive eigenvalue 1.

Corollary 4.21. [**50**] *Up to a homothetic change of coordinates of* $\mathbb{R}^m$ *and domain-equivalence, every umbilical quadratic harmonic morphism is given by a Clifford system as in Corollary 4.19. In particular, if* m *is odd, then there are no umbilical quadratic harmonic morphisms*

$$\mathbb{R}^m \to \mathbb{R}^n.$$

We now focus our attention on Clifford systems.

Definition 4.22. Two Clifford systems $(P_1, \dots, P_n)$ and $(Q_1, \dots, Q_n)$ on $\mathbb{R}^{2m}$ are said to be *algebraically equivalent* if there exists an $A \in O(2m)$ such that

$$A^t P_i A = Q_i$$

for all $i = 1, \dots, n$. They are said to be *geometrically equivalent* if there exists a $B \in O(\operatorname{span}_{\mathbb{R}}\{P_1, \dots, P_n\})$ such that $(B(P_1), \dots, B(P_n))$ and $(Q_1, \dots, Q_n)$ are algebraically equivalent.

It follows (see [**49**], Theorem 2.6) that two Clifford systems are algebraically (geometrically) equivalent if and only if the corresponding quadratic harmonic morphisms are domain-equivalent (bi-equivalent).

Definition 4.23. If $(P_1, \dots, P_n) \in C(2m_1, n)$ and $(Q_1, \dots, Q_n) \in C(2m_2, n)$ are two n-dimensional Clifford systems on $\mathbb{R}^{2m_1}$ and $\mathbb{R}^{2m_2}$, respectively, then their *direct sum* is the n-dimensional Clifford system on $\mathbb{R}^{2(m_1+m_2)}$ given by $(P_1 \oplus Q_1, \dots, P_n \oplus Q_n)$. A Clifford system $(P_1, \dots, P_n)$ on $\mathbb{R}^{2m}$ is said to be *irreducible* if it is not possible to write $\mathbb{R}^{2m}$ as a direct sum of non-trivial subspaces invariant under all P_i.

It is easy to see that a Clifford system is irreducible if and only if it is not algebraically equivalent to a direct sum of two Clifford systems. Furthermore, irreducible Clifford systems correspond to non-separable quadratic harmonic morphisms.

As noted in [**27**] there is a natural connection between Clifford systems and the representation of the Clifford algebras C_m. From this connection the following may be deduced:

Theorem 4.24. [**27**] *The following facts hold for Clifford systems:*

1. *Every Clifford system is algebraically equivalent to a direct sum of irreducible Clifford systems.*
2. *Irreducible Clifford systems* $(P_1, \dots, P_n) \in C(2m, n)$ *exist precisely for the values of* n *and* $m = m(n)$ *listed in Table 1.*

3. *For* $n \not\equiv 1$*(mod* 4*) all Clifford systems in* $C(2m(n), n)$ *are algebraically equivalent.*
4. *For* $n \equiv 1$*(mod* 4*) there are in* $C(2m(n), n)$ *two equivalence classes under algebraic equivalence and one under geometrical equivalence.*

n	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	...	*n+8*	...
m(n)	*1*	*2*	*4*	*4*	*8*	*8*	*8*	*8*	...	*16m(n)*	...

Table 1.

From Corollary 4.21 and Theorem 4.24 we may now prove existence of *umbilical* quadratic harmonic morphisms.

Theorem 4.25. [**49**] *For any* $n \in \mathbb{N}$ *there exist non-separable umbilical quadratic harmonic morphisms* $\mathbb{R}^{2m} \to \mathbb{R}^n$ *for exactly the values of* $(m, n) = (m(n), n)$ *listed in Table 1. Other umbilical quadratic harmonic morphisms into* $\mathbb{R}^n$ *exist exactly in the cases* $\mathbb{R}^{2km} \to \mathbb{R}^n$*, when* $(m, n) = (m(n), n)$ *is contained in Table 1 and* $k \in \mathbb{N}$*.*

PROOF. [**49**] The statement follows directly from the fact that by Corollary 4.21, any umbilical quadratic harmonic morphism corresponds to a Clifford system, which is algebraically equivalent to a direct sum of irreducible Clifford systems. ☐

Example 4.26. The Hopf polynomials are clearly given by Clifford systems in $C(2m(n), n)$ with $n = 2, 3, 5$ or 9. Since every Clifford system in $C(2m(n), n)$ is irreducible, the Hopf polynomials are all non-separable.

In order to give a complete characterization of quadratic harmonic morphisms we next prove that umbilical quadratic harmonic morphisms are the building blocks for the general case:

Lemma 4.27 (The Splitting Lemma)**.** [**49**] *If* $\phi : \mathbb{R}^{2m} \to \mathbb{R}^n$ *is a Q-non-singular quadratic harmonic morphism, then* ϕ *is domain-equivalent to a direct sum of umbilical quadratic harmonic morphisms.*

PROOF. [**49**] If ϕ is umbilical we are done. If not, let $\lambda_1, \dots, \lambda_m$ be the positive eigenvalues of the component matrices of ϕ counted by multiplicity, with $\lambda_1 = \lambda_2 = \cdots = \lambda_k \neq \lambda_l$, for $k < l \leq m$. Then ϕ is by Lemma 4.14 domain-equivalent to ψ, where ψ is of the normal form with D diagonal and diagonal entries $\lambda_1, \dots, \lambda_m$. From the fact that $DB_i = B_iD$, we see that

$$B_i = \begin{pmatrix} b_i & 0 \\ 0 & c_i \end{pmatrix}$$

for some $b_i \in GL(\mathbb{R}^k)$, $c_i \in GL(\mathbb{R}^{m-k})$, $i = 2, \ldots, n$. Using the isometry

$$H = \begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & 0 & I_{m-k} & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & 0 & I_{m-k} \end{pmatrix}$$

we see that ϕ is domain-equivalent to $\varphi = \psi \circ H$ and that

$$\varphi = \varphi_1 \oplus \varphi_2,$$

where

$$\varphi_1 : \mathbb{R}^{2k} \to \mathbb{R}^n \quad \text{and} \quad \varphi_2 : \mathbb{R}^{2(m-k)} \to \mathbb{R}^n.$$

By Proposition 4.11 the maps φ_1 and φ_2 are both quadratic harmonic morphisms and φ_1 is umbilical by construction. Repeating the procedure completes the proof in a finite number of steps. □

We may now give the characterization theorem for Q-non-singular quadratic harmonic morphisms. By Corollary 4.18 this also covers the general case.

Theorem 4.28. [**49**] *For $n \in \mathbb{N}$, Q-non-singular quadratic harmonic morphisms with values in $\mathbb{R}^n$ exist precisely in the cases*

$$\mathbb{R}^{2km(n)} \to \mathbb{R}^n$$

where $k \in \mathbb{N}$. Each such map is domain-equivalent to a direct sum of the kind

$$\lambda_1\phi_1 \oplus \ldots \oplus \lambda_k\phi_k$$

for some $\phi_i \in H_2^1(2m(n), n)$ given by an irreducible Clifford system.

PROOF. This follows directly from Theorem 4.25 and Lemma 4.27. □

Example 4.29. [**49**] Fix $k \in \mathbb{N}$. By Theorem 4.28 any Q-non-singular quadratic harmonic morphism

$$\phi : \mathbb{R}^{2km(n)} \to \mathbb{R}^n$$

is, up to domain-equivalence, of the form

$$\phi = \lambda_1\phi_1 \oplus \cdots \oplus \lambda_k\phi_k$$

for some k-tuple $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$ and some $\phi_i \in H_2^1(2m(n), n)$, $i = 1, \ldots, k$, given by irreducible Clifford systems. If $n \not\equiv 1(\text{mod } 4)$, it follows from the properties of Clifford systems that every two elements of $H_2^1(2m(n),n)$ are domain-equivalent. Hence up to domain-equivalence, ϕ is in this case of the form

$$\phi = \lambda_1\varphi \oplus \cdots \oplus \lambda_k\varphi$$

where $\varphi \in H_2^1(2m(n),n)$ is given by irreducible Clifford systems. In particular, we see that for a fixed k-tuple $(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$ any two quadratic harmonic morphisms with these eigenvalues are domain--equivalent.

If $n \equiv 1 (\mathrm{mod}\ 4)$ the situation is more complicated since in this case there are two algebraic equivalence classes in $C(2m(n),n)$. But as noted by Ou (see [**49**], Corollary 4.12), two non-equivalent Clifford systems will only differ by a sign of one, say of the last component. Hence there are in this case two domain-equivalence classes in $H_2^1(2m(n),n)$ and two non-equivalent quadratic harmonic morphisms in this set will differ only by a sign of one, say the last of their components. This implies that for a fixed k-tuple $(\lambda_1,\ldots,\lambda_k) \in \mathbb{R}^k$, we get 2^k possibilities of constructing our Q-non-singular quadratic harmonic morphism of the form

$$\lambda_1\phi_1 \oplus \cdots \oplus \lambda_k\phi_k$$

for $\phi_i \in H_2^1(2m(n),n)$. Half of these possibilities can be obtained from the other by an orthonormal change of coordinates in $\mathbb{R}^n$ as one easily checks. Hence there are in this case 2^{k-1} bi-equivalent classes. In particular for $k = 1$ and $n \in \mathbb{N}$, there is just one bi-equivalent class in $H_2^1(2m(n),n)$.

3. Polynomial Harmonic Morphisms of Higher Degree

So far we have not seen any explicit examples of polynomial harmonic morphisms of degree higher than 2, except when the target manifold is $\mathbb{C}$. Of course by composing quadratic harmonic morphisms we may construct examples of even degree, but it was an open question for several years whether there exist polynomial harmonic morphisms of higher degree that do not arise in this way. For instance, do there exist polynomial harmonic morphisms of odd degreeΓ This question has recently been given an affirmative answer in some cases in [**7**] by a surprisingly simple construction. We shall in this section describe this method and show how it provides a multitude of examples.

Definition 4.30. A multilinear map

$$\Phi : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k} \to \mathbb{R}^n$$

is said to be *norm-preserving* if

$$|\Phi(x_1, \dots, x_k)| = |x_1| \cdots |x_k|$$

for every $(x_1, \dots, x_k) \in \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k}$.

Example 4.31. [**7**] It is easily seen that the following maps are norm-preserving:

$$\mathbb{C} \times \mathbb{C}^2 \ni (z_1, z_2, z_3) \mapsto \frac{1}{2} z_1 (z_2 + \bar{z}_2, -i(z_2 - \bar{z}_2), 2z_3) \in \mathbb{C}^3$$

and

$$\begin{aligned} \mathbb{C} \times \mathbb{C}^3 \ni &(z_1, z_2, z_3, z_4) \mapsto \\ &\qquad (z_1 z_2, z_1 z_3, \frac{1}{2} z_1 (z_4 + \bar{z}_4), \frac{i}{2} z_1 (z_4 - \bar{z}_4)) \in \mathbb{C}^4. \end{aligned}$$

Example 4.32. For any $k \in \mathbb{N}$ and $d = 1, 2, 4$ or 8, let

$$\Phi : \underbrace{\mathbb{R}^d \times \cdots \times \mathbb{R}^d}_{k} \to \mathbb{R}^d, \ (x_1, \dots, x_k) \mapsto (x_1 \cdots (x_{k-2}(x_{k-1} x_k)) \cdots)$$

be the multiplication of real, complex, quaternionic or Cayley numbers. Then Φ is norm-preserving. Actually these are the only possible dimensions for the existence of a multilinear norm-preserving map

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R}^n$$

with $k \geq 2$. For if we fix $(x_3, \dots, x_k) \in S^{n-1} \times \cdots \times S^{n-1}$ then the resulting map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ will turn $\mathbb{R}^n$ into a *normed division algebra*, hence we must have $n = 1, 2, 4$ or 8 by a classical result of Hurwitz (see [**42**]). We shall make use of this fact in the proof of the following theorem.

Theorem 4.33. [**7**] *A norm-preserving multilinear map*

$$\Phi : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k} \to \mathbb{R}^n$$

is a harmonic morphism if and only if $p_1 = \cdots = p_k = n = 1, 2, 4$ *or* 8.

PROOF. [**7**] If $p_1 = \cdots = p_k = n$ then Φ is a harmonic morphism by Theorem 4.5 and Example 3.15 since Φ is a homothetic linear transformation in each variable separately.

Conversely, assume that $\Phi : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k} \to \mathbb{R}^n$ is a multilinear norm-preserving harmonic morphism. Since Φ is norm-preserving it is clear that

$$p_i \leq n \tag{4.4}$$

for $i = 1, \dots, n$. If λ is the dilation of Φ we see that for $x = (x_1, \dots, x_k) \in S^{p_1-1} \times \cdots \times S^{p_k-1}$ we have

$$n\lambda^2(x_1, \dots, x_k) = \operatorname{trace}(\langle d\Phi_x, d\Phi_x \rangle) = p_1 + p_2 + \cdots + p_k. \tag{4.5}$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$ we get from equation (3.6) together with Example 3.15:

$$\lambda^2(x_1, \dots, x_k) \triangle^{\mathbb{R}^n}(f) \circ \Phi(x_1, \dots, x_k) = \sum_{i=1}^{k} \triangle^{\mathbb{R}^{p_i}}(f \circ \Phi_i)(x_i), \tag{4.6}$$

where $\Phi_i = \Phi(x_1, \dots, \hat{x}_i, \dots, x_k)$. If we choose $f(y) = |y|^4$ so that $\triangle^{\mathbb{R}^n}(f)(y) = 4n|y|^2 + 8|y|^2$, then for $(x_1, \dots, x_k) \in S^{p_1-1} \times \cdots \times S^{p_k-1}$ we have

$$\triangle^{\mathbb{R}^{p_i}}(f \circ \Phi_i)(x_i) = 4p_i + 8.$$

From equations (4.6) and (4.5) it then follows that

$$\frac{p_1 + p_2 + \cdots + p_k}{k} = n$$

and comparing with equation (4.4) we obtain $p_1 = p_2 = \cdots = p_k = n$. The theorem now follows from the previously mentioned result of Hurwitz. $\square$

Using K-theory, Tang has recently improved Theorem 4.33 and obtained the following:

Theorem 4.34. [**54**] *If the map $F : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k} \to \mathbb{R}^n$ is multilinear and non-singular (i.e. $F(x_1, \dots, x_k) = 0$ implies that $x_i = 0$ for some i), then F is a harmonic morphism if and only if $p_1 = \cdots = p_k = n = 1, 2, 4$ or 8.*

From Theorem 4.33 we may now for any $k \in \mathbb{N}$ and $n = 1, 2, 4$ or 8 construct polynomial homogeneous harmonic morphisms of degree k

$$\underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \to \mathbb{R}^n.$$

Example 4.35. [$*$] For the non-homogeneous case we have the following construction: For Riemannian manifolds $M_1, \dots, M_k$ and a family $\phi_i : M_i \to \mathbb{R}^n$, $i = 1, \dots, k$, of harmonic morphisms, their direct sum

$$\phi_1 \oplus \cdots \oplus \phi_k : M_1 \times \cdots \times M_k \to \mathbb{R}^n$$

is given by

$$\phi_1 \oplus \cdots \oplus \phi_k(x_1, \dots, x_k) = \phi_1(x_1) + \cdots + \phi_k(x_k)$$

for $(x_1, \dots, x_k) \in M_1 \times \cdots \times M_k$. Since this is a harmonic morphism in each variable separately it is a harmonic morphism by Example 3.15.

Now assume that $A_i : \mathbb{R}^{p_i} \to \mathbb{R}^n$ is a homogeneous polynomial harmonic morphism for $i = 1, \dots, k$. Define the map

$$F : \mathbb{R}^{p_1} \times \cdots \times \mathbb{R}^{p_k} \to \mathbb{R}^n$$

as the direct sum of the A_i's:

$$F(x_1, \dots, x_k) = A_1(x_1) + \cdots + A_k(x_k).$$

The map F is a harmonic morphism and if the A_i's are not all of the same degree, then F will be non-homogeneous.

After Example 4.35 was produced it was pointed out by Ou that a special case of this construction has appeared in [**48**]. But since this is written in Chinese it is our hope of not being accused of plagiarism. Example 4.35 motivates the following definition:

Definition 4.36. A harmonic morphism $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is said to be *separable* if, up to isometries, Φ can be written as a direct sum of harmonic morphisms from spaces of strictly lower dimensions. Otherwise Φ is said to be *non-separable.*

The map F constructed in Example 4.35 is separable and by Theorem 4.8 any polynomial harmonic morphism of degree 2 is either homogeneous (up to an additive constant) or separable. We make the following conjecture:

Conjecture 4.37. [$*$] *If $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a non-separable polynomial harmonic morphism then, up to an additive constant, ϕ is homogeneous.*

CHAPTER 5

Applications

In this chapter we present a new way of proving two well-known theorems of Eells and Yiu. The first states that the Hopf polynomials are essentially the only homogeneous polynomial harmonic morphisms preserving spheres. The second states that these are essentially the only harmonic homogeneous polynomials that restricts to harmonic morphisms between spheres. For this we use results from Chapter 3 together with some important results of algebraic and differential topology. This is then used to derive information on the singularities of general harmonic morphisms.

1. The Theorems of Eells and Yiu

The results needed from algebraic topology originate in the following remarkable result of W. Browder (see Spanier [**52**] for the definitions):

Theorem 5.1. [**16**] *Let F be a connected polyhedron and $p : S^m \to B$ a weak fibration over B with fibre F. If B is not a single point, then F must be homotopic to S^1, S^3 or S^7.*

Timourian used Theorem 5.1 to obtain the following result ([**55**], Lemma 2.7), more suitable for our purposes.

Theorem 5.2. [**55**] *If $m > n \geq 1$ and T^m is a homotopy m-sphere and $\phi : T^m \to S^n$ is a fibre bundle with compact $(m-n)$-dimensional fibres, then $(m, n) = (3, 2)$, $(7, 4)$ or $(15, 8)$.*

Note that these are exactly the dimensions of the Hopf fibrations of Example 3.26 with $m > n$. It is well known that all the Hopf fibrations are fibre bundle maps and they are indeed surjective submersions between compact manifolds. The following result is due to Ehresmann (see also Wolf [**58**]).

Theorem 5.3. [**26**] *If the map $\phi : M^m \to N^n$ is a proper submersion i.e. a submersion that pulls back compact sets to compact sets, then $\phi : M \to \phi(M)$ is a fibre bundle.*

Using the results of Ehresmann, Wolf and Timourian, Hsu gave a complete classification of all horizontally conformal maps $S^m \to S^n$ with constant dilation. Her result is as follows:

Theorem 5.4. [**41**] *If $H_n^m(\lambda)$ is the set of all horizontally conformal C^1-maps $S^m \to S^n$ with constant dilation λ, then $H_n^m(\lambda) = \emptyset$ unless*

a) $m = n$, $\lambda = 1$ *and* $H_n^m(\lambda) = O(m+1)$,
b) $m = n = 1$, $\lambda \in \mathbb{N}$ *and* $H_n^m(\lambda) = \{g_\lambda \circ \psi \mid g_\lambda(z) = z^\lambda, \psi \in O(2)\}$,
c) $(m,n) = (3,2)$, $(7,4)$ *or* $(15,8)$ *and* $\lambda = 2$.

All the maps of cases *a*) and *b*) are clearly harmonic morphisms; the harmonicity follows from Example 2.23 and the horizontal conformality will follow from the proof of Theorem 5.5. Case *b*) with $\lambda = 2$ and ψ the identity is the first Hopf fibration and by the classification of the quadratic harmonic morphisms it follows that any non-constant quadratic harmonic morphism $\mathbb{R}^2 \to \mathbb{R}^2$ must be domain-equivalent to this Hopf polynomial. As mentioned earlier a similar result is valid in higher dimensions. This is one of the theorems of Eells and Yiu:

Theorem 5.5. [**25**] *If $m > n$ and $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is a non-constant harmonic morphism defined by polynomials homogeneous of degree p, with $|\Phi|$ constant on S^{m-1}, then $p = 2$ and Φ is bi-equivalent to a constant multiple of one of the Hopf polynomials.*

To prove Theorem 5.5 we need the following lemma, which is a strenghtening of Theorem 4.3 for homogeneous polynomial harmonic morphisms.

Lemma 5.6. [**5**] *If $\Phi : \mathbb{R}^m \to \mathbb{R}^n$ is a non-constant polynomial harmonic morphism, homogeneous of degree p, then $p(n-2) \leq m-2$ with equality if and only if $|\Phi|$ is constant on S^{m-1} in which case the dilation of Φ is constant on S^{m-1}.*

PROOF. [**5**] Without loss of generality we may assume that Φ is normalized so that

$$\sup_{x \in S^{m-1}} |\Phi(x)|^2 = 1.$$

Let

$$\Gamma = \{x \in S^{m-1} |\ |\Phi(x)|^2 = 1\}$$

and define $F : \mathbb{R}^m \to \mathbb{R}$ by $F = |\Phi|^2$ and $f = F|_{S^{m-1}}$. Then for $x \in S^{m-1}$ (with ∇ denoting the gradient):

$$\begin{aligned}\nabla f(x) &= \nabla F(x) - \frac{\partial F}{\partial r}(x)x \\ &= 2\sum_k \Phi^k(x)\nabla\Phi^k(x) - 2pf(x)x.\end{aligned} \tag{5.1}$$

Since f attains its maximum in Γ, $\nabla f = 0$ and $\triangle^{S^{m-1}}(f) \leq 0$ there. Thus for $x \in \Gamma$ we have

$$\sum_k \Phi^k(x)\nabla\Phi^k(x) = px. \tag{5.2}$$

Taking the norm squared of this gives $\lambda^2(x) = p^2$ for $x \in \Gamma$, where λ is the dilation of Φ. From the hypothesis on Φ we get

$$\triangle^{\mathbb{R}^m}(F) = 2\sum_k \operatorname{div}(\Phi^k\nabla\Phi^k) = 2\sum_k |\nabla\Phi^k|^2 = 2n\lambda^2.$$

If $i : S^{m-1} \hookrightarrow \mathbb{R}^m$ is the inclusion map, then Example 2.23 implies:

$$\begin{aligned}\triangle^{S^{m-1}}(f) &= \triangle^{\mathbb{R}^m}(F) - 2p(2p+m-2)f \\ &= 2n\lambda^2 - 2p(2p+m-2)f.\end{aligned} \tag{5.3}$$

Since this is non-positive on Γ and $\Gamma \neq \emptyset$, we obtain $(n-2)p \leq m-2$.

If f is constant then $\Gamma = S^{m-1}$ so by equation (5.2), λ is constantly equal to p on S^{m-1}. Furthermore $\triangle^{S^{m-1}}(f) = 0$ which implies that $(n-2)p = m-2$.

Conversely, if $(n-2)p = m-2$, then from equation (5.3):

$$\triangle^{S^{m-1}}(f) = 2n(\lambda^2 - p^2 f).$$

From this we get that if $f(x) = 0$ then $\triangle^{S^{m-1}}(f)(x) \geq 0$. Taking the norm squared of both sides of equation (5.1) and using the homogeneity of Φ implies

$$|\nabla f|^2 = 4f(\lambda^2 - p^2 f) = \frac{4f}{2n}\triangle^{S^{m-1}}(f).$$

Hence $\triangle^{S^{m-1}}(f) \geq 0$ on the whole of S^{m-1} and since S^{m-1} is compact, f must be constant by Corollary 2.31. $\square$

PROOF OF THEOREM 5.5. [*] We may assume that $|\Phi| = 1$ on S^{m-1}. It follows from Lemma 5.6 that the dilation λ of Φ is constant on S^{m-1}, hence the sphere contains no critical points of Φ. For any $y \in S^{n-1}$ and $x \in \mathbb{R}^m \setminus \{0\}$ with $\Phi(x) = y$ we have

$$1 = |\Phi(x/|x|)| = |y|/|x|^p = |x|^{-p}$$

so it follows that $\Phi^{-1}(y) \subset S^{m-1}$. In particular, the kernel $\ker d\Phi_x = T_x\Phi^{-1}(y)$ is contained in T_xS^{m-1}. Thus if $\phi : S^{m-1} \to S^{n-1}$ is the restriction of Φ to the sphere, then $\ker d\Phi_x = \ker d\phi_x$ for any $x \in S^{m-1}$ and the horizontal space of ϕ at x is the intersection of the horizontal space of Φ at x with T_xS^{m-1}. Hence ϕ is horizontally conformal with constant dilation λ. By Theorem 5.4, $\lambda = 2$ and $(m, n) = (4, 3)$, $(8, 5)$ or $(16, 9)$ which are the dimensions of the Hopf polynomials. From Lemma 5.6 we get $p = 2$, so Φ is a quadratic harmonic morphism and hence bi-equivalent to one of the Hopf polynomials by Example 4.29. □

The other theorem of Eells and Yiu is the following:

Theorem 5.7. [**25**] *Let $m > n$ and $\phi : S^{m-1} \to S^{n-1}$ be the restriction of a non-constant harmonic homogeneous polynomial map $\Phi : \mathbb{R}^m \to \mathbb{R}^n$. Then ϕ is a harmonic morphism if and only if Φ is bi-equivalent to one of the Hopf polynomials.*

PROOF. [$*$] If Φ is bi-equivalent to one of the Hopf polynomials it follows from the proof of Theorem 5.5 that ϕ is a harmonic morphism.

Conversely, let λ be the dilation of ϕ. If Φ is of degree p, then it follows from Example 2.23 that

$$\lambda^2(x) = \frac{|d\phi_x|^2}{n-1} = \frac{p(p+m-2)}{n-1} \tag{5.4}$$

for $x \in S^{m-1}$. Thus ϕ is horizontally conformal with constant dilation so by Theorem 5.4 we must have $\lambda = 2$ and $(m, n) = (4, 3)$, $(8, 5)$ or $(16, 9)$. By equation (5.4) we see that $p = 2$ so Φ is a harmonic homogeneous polynomial of degree 2 in the dimensions of the Hopf polynomials. Once we have shown that Φ is horizontally conformal, Theorem 5.5 will complete the proof. Since Φ is of degree 2, $d\Phi_0 = 0$. The homogeneity of Φ now implies that what remains is to prove the horizontal conformality at points of S^{m-1}. Choose for that purpose $x \in S^{m-1}$ and denote by $\mathcal{H}_x(\Phi)$ and $\mathcal{H}_x(\phi)$ the horizontal space of Φ and ϕ at x, respectively. Once again by homogeneity, $d\Phi_x$ will map every non-zero vector orthogonal to T_xS^{m-1} to a non-zero vector orthogonal to $T_{\Phi(x)}S^{n-1}$. Hence $\ker d\Phi_x \subseteq T_xS^{m-1}$ so

$$\mathcal{H}_x(\Phi) = \mathcal{H}_x(\phi) \oplus [x],$$

where $[x]$ denotes the line spanned by x. Since the spaces on the right hand side are orthogonal it follows that for $v, w \in \mathcal{H}_x(\phi)$ and $\alpha, \beta \in \mathbb{R}$:

$$\begin{aligned}\langle d\Phi_x(v + \alpha x), d\Phi_x(w + \beta x)\rangle &= \langle d\phi_x(v) + \alpha d\Phi_x(x), d\phi_x(w) + \beta d\Phi_x(x)\rangle \\ &= \langle d\phi_x(v), d\phi_x(w)\rangle + 4\alpha\beta|\Phi(x)|^2 \\ &= 4\langle v, w\rangle + 4\alpha\beta \\ &= 4\langle v + \alpha x, w + \beta x\rangle,\end{aligned}$$

where we have used the homogeneity of Φ. Thus Φ is horizontally conformal at any point of S^{m-1}. This proves the statement. □

Thus the only harmonic homogeneous polynomials to spaces of strictly lower dimensions which restrict to harmonic morphisms between spheres are those which are bi-equivalent to the Hopf polynomials. It should be noted that these are the only known non-trivial examples of globally defined harmonic morphisms between spheres of constant curvature +1 (see The Atlas of Harmonic Morphisms [**39**]).

2. The Symbol of Harmonic Morphisms

We shall now show how the above results can be applied to give information on the singularities of harmonic morphisms. Recall that for a function $f : \mathbb{R}^m \to \mathbb{R}$ the *p-th differential of f at $x \in \mathbb{R}^m$* is the homogeneous polynomial

$$d^p f_x : T_x\mathbb{R}^m \cong \mathbb{R}^m \to \mathbb{R}$$

given by

$$d^p f_x(\xi^1, \dots, \xi^m) = \sum_{|k|=p} \frac{p!}{k!}\partial^k f(x)(\xi^1)^{k_1} \cdots (\xi^m)^{k_m}.$$

Thus the p-th differential $d^p f$ of f is a natural generalization of the (first) differential df of f. We recall Taylor's formula for a C^{p+1} function f:

$$f(x + \xi) = f(x) + df_x(\xi) + \frac{1}{2!}d^2 f_x(\xi) + \dots + \frac{1}{p!}d^p f_x(\xi) + O(|\xi|^{p+1})$$

This is generalized to maps between arbitrary Riemannian manifolds as follows:

Definition 5.8. Assume that M and N are Riemannian manifolds and $\phi : M \to N$ a differentiable map. The *p-th differential* of $\phi^\alpha =$

$y^\alpha \circ \phi$ at $x \in M$ is the function $d^p\phi^\alpha_x : T_xM \to \mathbb{R}$ defined in terms of local coordinates (x^k) around x and (y^α) around $y = \phi(x)$ by

$$d^p\phi^\alpha_x(\xi) = \sum_{|k|=p} \frac{p!}{k!}\partial^k\phi^\alpha(x)(\xi^1)^{k_1}\cdots(\xi^m)^{k_m},$$

where $\xi = (\xi^1, \dots, \xi^m)$ are the components of the vector $\xi \in T_xM$ in the chosen coordinates. The *order* of ϕ at x is the smallest integer $p \geq 1$ such that for some k, ϕ^k has a non-vanishing p-th differential at x. The *symbol* of ϕ at x is the map $\sigma_x(\phi) : T_xM \to T_yN$ whose contravariant components $\sigma^\alpha_x(\phi)$, in the chosen coordinate systems, are given by

$$\sigma^\alpha_x(\phi)(\xi) = \frac{1}{p!}d^p\phi^\alpha_x(\xi),$$

where p is the order of ϕ at x.

It is easy to see that the order and the symbol of a map are both well defined and independent of the choice of the local coordinates.

Theorem 5.9. [**28**] *Assume that $\phi : M \to N$ is a horizontally conformal map between Riemannian manifolds. If ϕ is of finite order at a point $x \in M$, then the symbol of ϕ at x is a harmonic morphism.*

PROOF. [**28**],[**1**] We may choose normal coordinates (x^k) and (y^α) centered around $x \in M$ and $y = \phi(x)$, respectively, thus identifying the tangent spaces T_xM and T_yN with $\mathbb{R}^m$ and $\mathbb{R}^n$ equipped with their standard Euclidean metrics. The order of ϕ at x, the symbol and the horizontal conformality will be invariant under these identifications. Hence it will be enough to prove the theorem at the origin for a horizontally conformal map $\phi : U \to \mathbb{R}^n$ where U is an open neighbourhood of $0 \in \mathbb{R}^m$, $\phi(0) = 0$ and the order of ϕ at 0 is finite and equal to p.

If $p = 1$ then the gradients of the symbol will coincide with the gradients of the components of ϕ itself. From Example 3.6 it follows that the symbol, in this case, is horizontally conformal.

Assume $p \geq 2$. Obviously for every k:

$$\frac{\partial}{\partial\xi^i}\frac{1}{k!}d^k\phi^\alpha_0(\xi) = \frac{1}{(k-1)!}d^{k-1}(\frac{\partial\phi^\alpha}{\partial x^i})_0(\xi).$$

Since $\partial\phi^\alpha/\partial x^i$ is of order $\geq p-1$ at 0, Taylor's formula gives that

$$\begin{aligned}\frac{\partial\phi^\alpha}{\partial x^i}(\xi) &= \frac{1}{(p-1)!}d^{p-1}(\frac{\partial\phi^\alpha}{\partial x^i})_0(\xi) + O(|\xi|^p)\\ &= \frac{\partial}{\partial\xi^i}(\frac{1}{p!}d^p\phi^\alpha_0(\xi)) + O(|\xi|^p).\end{aligned}$$

If λ is the dilation of ϕ we will therefore have

$$\begin{aligned}\lambda^2(\xi)\delta_{\alpha\beta} &= \sum_i \frac{\partial\phi^\alpha}{\partial x^i}(\xi)\frac{\partial\phi^\beta}{\partial x^i}(\xi)\\ &= \sum_i \frac{\partial}{\partial\xi^i}(\frac{1}{p!}d^p\phi_0^\alpha(\xi))\frac{\partial}{\partial\xi^i}(\frac{1}{p!}d^p\phi_0^\beta(\xi)) + O(|\xi|^{2p-1}).\end{aligned}$$

Note that the sum in the last expression is a homogeneous polynomial in ξ of degree $2p-2$. It follows that

$$\frac{1}{(2p-2)!}\delta_{\alpha\beta}d^{2p-2}(\lambda^2)_0 = \sum_i \frac{\partial\sigma_0^\alpha(\phi)}{\partial\xi^i}\frac{\partial\sigma_0^\beta(\phi)}{\partial\xi^i}.$$

By Example 3.6, the symbol of ϕ is horizontally conformal, and hence a harmonic morphism by Theorem 4.5. □

A non-constant harmonic morphism $\phi : M^m \to N^n$ is by Theorem 2.26 of finite order everywhere. We therefore have to every point $x \in M$ associated a harmonic morphism $\mathbb{R}^m \to \mathbb{R}^n$ homogeneous of degree p, where p is the degree of the zero of $d\phi$ at x. From Theorem 4.3 we now get the following important theorem:

Theorem 5.10. [**5**] *If* $m < 2n-2$ *and* $\phi : M^m \to N^n$ *is a non-constant harmonic morphism, then* ϕ *is a submersion.*

Proof. Assume that for some $x \in M$ we have $d\phi_x = 0$, then the symbol of ϕ at x is a polynomial harmonic morphism of degree ≥ 2, contradicting the statement of Theorem 4.3. □

It is well known that for $n \geq 3$ the sphere S^{n+1} cannot be a fibre bundle over S^n (see page 147 of [**53**]). From the previously mentioned result of Ehresmann it follows that there are no non-constant harmonic morphisms

$$S^{n+1} \to S^n$$

for $n \geq 4$. Theorem 5.2 now leads to an improvement of this result.

Corollary 5.11. [**25**],[**5**] *If* $n < m < 2n-2$ *and* $\phi : S^m \to S^n$ *is a harmonic morphism, then* ϕ *is constant.*

Corollary 5.12. [**5**] *If* $m = 2n-2$ *and* $\phi : M^m \to N^n$ *is a non-constant and non-submersive harmonic morphism, then* $n = 2, 3, 5$ *or* 9.

Proof. Since $(m-2) = 2(n-2)$, either $m = n = 2$ or $m > n \geq 3$ in which case the statement follows from Lemma 5.6 together with Theorem 5.5. □

Finally we derive the following non-existence result:

Theorem 5.13. [**1**] *If* $m \leq 3n - 5$ *and* $\phi : M^m \to N^n$ *is a non-constant non-submersive harmonic morphism, then one of the following conditions must hold:*

1. $n = 3$ *and* $m = 4$,
2. $n = 5$ *and* $m \in \{8, 9, 10\}$,
3. $n = 7$ *and* $m = 16$,
4. $n = 8$ *and* $m \in \{16, 17, 18, 19\}$,
5. $n = 9$ *and* $m \in \{16, \dots, 22\}$.

PROOF. Since ϕ is non-submersive its symbol at some point of M is of degree > 1 and since $m \leq 3n - 5$ this degree can not exceed 2 by Theorem 4.3. Hence there must exist a non-constant quadratic harmonic morphism $\mathbb{R}^m \to \mathbb{R}^n$. By the classification of the quadratic harmonic morphisms in Chapter 3, this will be an orthogonal projection followed by a Q-non-singular quadratic harmonic morphism

$$\mathbb{R}^{2km(n)} \to \mathbb{R}^n$$

for some $k \in \mathbb{N}$. By hypothesis we must have

$$2km(n) \leq 3n - 5$$

and it follows easily from Table 1 that these are the only possibilities. $\square$

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