SYMMETRIC SPACES

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Abstract

In this text we study the differential geometry of symmetric spaces. We describe how a symmetric space (M, g) can be seen as a homogeneous space G/K, the quotient of its isometry group G and a isotropy group K at a point. We study the one-to-one correspondence between symmetric spaces and symmetric pairs. Furthermore we investigate the expressions for curvature on a symmetric space. Finally we describe the notion of dual symmetric spaces.

To illustrate how well symmetric spaces lend themselves to explicit calculations we calculate the curvature of the real Grassmann manifold and find their dual space.

Keywords: homogeneous spaces, symmetric spaces, symmetric pairs, the Killing form, curvature of symmetric spaces, dual symmetric spaces.

Throughout this text it has been my intention to give reference to all the sources that have been used.

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Overview

History[4]

At the end of the nineteenth century, after studying spaces of constant curvature, mathematicians wanted to classify all locally symmetric Riemannian manifolds, i.e. Riemannian manifolds whose curvature tensor is parallel i.e. satisfies $\nabla R = 0$. The whole issue was settled by Elie Cartan in 1932. Previously in his thesis he had classified all simple complex Lie algebras and he also classified the simple real Lie algebras. Using this he gave a complete classification of all symmetric spaces which by the way were introduced by himself in 1926.

Symmetric Spaces

A symmetric space is a Riemannian manifold (M, g) such that for every point $p \in M$ there exist an isometry σ_p of (M, g) called an involution such that

(1) $\sigma_p(p) = p$ and
(2) $d\sigma_p = -id_{T_pM}$.

By composing involutions one gets translations along geodesics, which can be used to extend geodesics to the whole of $\mathbb{R}$ i.e. M is geodesically complete. By the Hopf-Rinow theorem any two points in a geodesically complete Riemanninan manifold can be connected by a geodesic. Therefore the translations along the geodesics makes the isometry group G acting on M transitive. Using the theory of homogeneous spaces one can identify M with G/K where K is the isotropy group at a point $p \in M$, i.e $K = \{k \in G : k(p) = p\}$.

Symmetric Pairs

The description of a symmetric space, in terms of a Lie group G, a closed subgroup K and an involution σ, leads to the concept of a symmetric pair which is defined as a Lie group G with a closed subgroup K and an involutive automorphism s on G satisfying

(1) $(G_s)_0 \subseteq K \subseteq G_s$,

(2) Ad K is compact

where G_s is the set of elements left invariant by s. We show that symmetric pairs lead to symmetric spaces.

Curvature on a Symmetric Space

We show that left invariant vector fields on the isometry group G are mapped to Killing fields in the symmetric space (M, g), which generate Jacobi fields and therefore provide the connection to the curvature tensor. This gives a very simple formula for the curvature tensor on a symmetric space G/K, namely

$$R(X,Y)Z = -[[X,Y],Z] \quad \text{for } X, Y, Z \in \mathfrak{p}$$

where $\mathfrak{p}$ is the linear complement of $\mathfrak{k}$ which is the Lie algebra of K.

Each symmetric space M has a dual space $M^\dagger$ which is defined by the existence of maps

(1) A Lie algebra isomorphism $\varphi : \mathfrak{k} \to \mathfrak{k}^\dagger$ such that
$$g^\dagger(\varphi(X), \varphi(Y)) = -g(X,Y)$$

(2) A linear isometry $\hat{\varphi} : \mathfrak{p} \to \mathfrak{p}^\dagger$ such that
$$\varphi\big([X,Y]\big) = -[\hat{\varphi}(X), \hat{\varphi}(Y)]^\dagger$$

where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is a linear complement of $\mathfrak{k}$ in the Lie algebra $\mathfrak{g}$ of G. Dual spaces have opposite curvature and if one is compact its dual is noncompact and vice versa.

Although the path chosen in this paper is more along minimal differential geometry, the standard description is more in the language of Lie groups and Lie algebras.

For a reader who wants a more thorough description we refer to [**5**] Helgason, which is the standard reference on the theory of symmetric spaces.

CHAPTER 1

Homogeneous Spaces

In this chapter we create a manifold structure for the quotient of a Lie group and a closed subgroup.

1. Group Actions

Definition 1.1. Let M be a smooth manifold and G be a Lie group with identity element $e \in G$. A smooth map $\nu : G \times M \to M$ is called a *group action* on M if

$$\begin{aligned} \nu(g_1, \nu(g_2, p)) &= \nu(g_1 g_2, p) \\ \nu(e, p) &= p \end{aligned}$$

for all $g_1, g_2 \in G$ and $p \in M$. The action ν is said to be *effective* if

$$\nu(g, p) = p \text{ for all } p \in M \text{ implies that } g = e.$$

The action ν is said to be *transitive* if for all $p, q \in M$ there exists a $g \in G$ such that $\nu(g, p) = q$. The *isotropy group* K_{p_0} of ν at $p_0 \in M$ is given by

$$K_{p_0} = \{g \in G : \nu(g, p_0) = p_0\}.$$

If $\nu : G \times M \to M$ is a group action on M, then the group G is said to *act on* M and it is customary to write gp for $\nu(g, p)$.

Example 1.2. Let $M = S^n$ be the unit sphere in $\mathbb{R}^{n+1}$ and $G = \mathbf{SO}(n+1)$ be the special orthogonal group. Then we have the action

$$\nu : \mathbf{SO}(n+1) \times S^n \to S^n, (A, p) \mapsto A \cdot p$$

For $p_0 = (1, 0, \ldots, 0)$ we get $K_{p_0} = \mathbf{SO}(1) \times \mathbf{SO}(n)$.

2. Homogeneous Spaces

We need the following lemma to prove that G/K can get a manifold structure.

Lemma 1.3. [**10**] *Let G be a Lie group and let K be a closed subgroup of G. Denote the Lie algebra of G by $\mathfrak{g}$ and the Lie algebra of K by $\mathfrak{k}$. If $\mathfrak{m}$ is a linear complement to $\mathfrak{k}$, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and we give G/K the quotient topology, let $\pi : G \to G/K$ be the canonical projection onto the quotient, then the map*

$$\pi \circ exp : \mathfrak{m} \to G/K$$

is a local homeomorphism at 0.

PROOF. [**10**] Let φ be the map

$$\varphi : \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m} \to G, \quad \varphi(X, Y) = \exp X \exp Y$$

then $d\varphi_{(0,0)}(X, Y) = X + Y$, so by the inverse function theorem φ is a diffeomorphism from an open neighborhood $U_0 \times V_0$ of $(0, 0)$.

We define some sets that will help to show that $\pi \circ \varphi$ is one-to-one. The set $\exp(V_0)$ is an open neighborhood of e in K with the subspace topology so $\exp(V_0) = H \cap K$ for some open set H in G. So there exists an open set

$$U_1 \times V_1 \subseteq U_0 \times V_0$$

such that

$$\varphi(U_1 \times V_1) \cap K \subseteq H \cap K = \exp V_0$$

If $X \in U_1$, $Y \in V_1$ and $\varphi(X, Y) \in K$ then

$$\exp X \exp Y = \exp Y' \text{ for some } Y' \in V_0$$

But φ is a diffeomorphism on $U_1 \times V_1$ so $X = 0$ and $Y = Y'$, thus

$$\varphi(U_1 \times V_1) \cap K = \exp(V_1).$$

Now we will show the injectivity. Let $U_2 \subseteq U_1$ be a neighborhood of 0 in $\mathfrak{m}$ such that

$$\exp(-U_2)\exp(U_2) \subseteq \varphi(U_1 \times V_1)$$

Then $\pi \circ \exp_{|U_2}$ is injective, since if $X', X'' \in U_2$ and

$$\pi(\exp(X')) = \pi(\exp(X''))$$

then

$$\exp(-X')\exp(X'') \in \varphi(U_1 \times V_1) \cap K \text{ so } X' = X''.$$

The surjectivity on a neighborhood follows since φ is surjective from $U_2 \times \{0\}$ onto $\varphi(U_2, 0)$.

By definition $\pi \circ \exp_{|\mathfrak{m}}$ is continuous.

If N is an open subset of U_2 then $\pi(\exp(N)) = \pi \circ \varphi(N, V_1)$ which is open since φ is a diffeomorphism here and we have the quotient topology. So the inverse is continuous. Thus $\pi \circ \exp : \mathfrak{m} \to G/K$ is a local homeomorphism at 0 in the quotient topology of G/K. □

Theorem 1.4. [**10**] *Let G be a Lie group and K a closed subgroup of G, then the quotient space G/K has a unique manifold structure such that*

(1) *the projection $\pi : G \to G/K, g \mapsto gK$ is smooth and*
(2) *π has smooth local lifts such that for every $gK \in G/K$ there is a neighborhood U and a map $l : U \to G$ such that $\pi \circ l = id$.*

With this manifold structure a map

$$f : G/K \to N$$

is smooth if and only if

$$f \circ \pi : G \to N$$

is smooth. Furthermore the action

$$G \times G/K \to G/K, \quad (g, g'K) \mapsto gg'K$$

is smooth.

PROOF. [**10**] If one chooses the quotient topology on G/K then we get a Hausdorff space by the following. Consider the map

$$\varphi : G \times G \to G, \quad \varphi(g_1, g_2) = g_1^{-1} g_2$$

which is continuous, and since K is closed the inverse image $\varphi^{-1}(K)$ is closed. Now if $g_1 K \neq g_2 K$ then $g_1^{-1} g_2 \notin K$ so there are open sets W_1, W_2 such that $(g_1, g_2) \in W_1 \times W_2$ and $W_1 \times W_2 \cap \varphi^{-1}(K) = \emptyset$. Now if $gK \in W_i K$ then there is a $k_i \in K$ such that $gk_i \in W_i$. So if $gK \in W_1 K \cap W_2 K$ then $gk_1 \in W_1$ and $gk_2 \in W_2$. Therefore $(gk_1, gk_2) \in W_1 \times W_2$ is mapped to $k_1^{-1} k_2 \in K$ which contradicts that $W_1 \times W_2 \cap \varphi^{-1}(K) = \emptyset$. So G/K is Hausdorff.

To get coordinates we need a linear complement $\mathfrak{m}$ to $\mathfrak{k}$. If we use the same sets as in Lemma 1.3, the maps $U_2 \to G/K, X \mapsto g\exp(X)K$, for $g \in G$ are homeomorphisms. Then we can choose local coordinates on G/K at gK as the inverses of these maps. We identify a chart with g. Suppose two charts g, g' intersect then $g\exp(X_0)K = g'\exp(X_0')K$ for $X_0, X_0' \in U_2$. Then $\exp(X_0) = g^{-1} g' \exp(X_0') k_0$ and $\exp(X_0) \subseteq \varphi(U_2 \times \{0\}) \subseteq \varphi(U_2 \times V_1)$. So if X' is close to X_0' then there must be an $X \in U_2$ and $Y \in V_1$ such that $g^{-1} g' \exp(X') k_0 = \varphi(X, Y)$ and X depends smoothly on X' since φ is a diffeomorphism on $U_2 \times V_1$. Which finally gives $g'\exp(X')K = g\exp(X)K$, which shows that points in chart g are mapped smoothly to points in chart g' and vice versa. So G/K has a manifold structure.

(1) The projection π is smooth since
$\pi(g\exp(X)\exp(Y)) = g\exp(X)K$.

(2) The lifts l are given by $l(g\exp(X)K) = g\exp(X)$, which are smooth.

Uniqueness: If $(G/K)'$ and $(G/K)''$ satisfy the above then $\pi \circ l' : U' \to (G/K)''$ is the identity, also $\pi \circ l'' : U'' \to (G/K)'$ is the identity. Therefore one can make the identity mapping $(G/K)' \to (G/K)''$ a diffeomorphism, so they are identical as manifolds.

If $f \circ \pi$ is smooth then f is locally given by $f \circ \pi \circ l$ and is smooth. The reverse is trivial.

The map $(g, g'K) \mapsto gg'K$ is smooth since it is $\pi \circ$ (group op. on G) $\circ\, l$ which is a composition of smooth maps. □

The following result tell us important things about the kernel of $d\pi$.

Corollary 1.5. [**10**] *Let G be a Lie group, K be a closed subgroup in G, $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be the Lie algebra of K. If $\mathfrak{m}$ is a linear complement to $\mathfrak{k}$ in $\mathfrak{g}$ then*

$$d\pi_e : \mathfrak{g} \to T_{eK}G/K$$

is a surjective vector space homomorphism with ker $d\pi_e = \mathfrak{k}$.

PROOF. By the proof of Theorem 1.4 the map $\pi \circ \exp_{|\mathfrak{m}} : \mathfrak{m} \to G/K$ is local diffeomorphism, so the differential $d(\pi \circ \exp_{|\mathfrak{m}}) = d\pi_{|\mathfrak{m}}$ is an isomorphism. Therefore $d\pi_e$ is surjective.

Since $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and $d\pi_{|\mathfrak{k}} = 0$ we get that $\ker d\pi = \mathfrak{k}$. □

We now give some examples that illustrates the use of the previous theory.

Example 1.6. Following Example 1.2 let $G = \mathbf{SO}(n+1)$, $K = \mathbf{SO}(n)$ then the quotient $\mathbf{SO}(n+1)/\mathbf{SO}(n)$ has manifold structure such that the action

$$\begin{array}{rcl} \mathbf{SO}(n+1) \times \big(\mathbf{SO}(n+1)/\mathbf{SO}(n)\big) & \to & \mathbf{SO}(n+1)/\mathbf{SO}(n) \\ (A_1, A_2K) & \mapsto & A_1A_2K \end{array}$$

is smooth.

The Lie algebra $\mathfrak{g} = \mathfrak{so}(n+1)$ of $\mathbf{SO}(n+1)$ consists of the skew symmetric matrices. The Lie algebra of $\mathfrak{k}$ can be identified with the matrices

$$\begin{pmatrix} 0 & \mathbf{0}^{\mathbf{T}} \\ \mathbf{0} & A \end{pmatrix}$$

where $A \in \mathfrak{so}(n)$ and $\mathbf{0}$ is the $n \times 1$ matrix of zeros. The linear complement $\mathfrak{m}$ corresponds to the matrices

$$\begin{pmatrix} 0 & \mathbf{v}^{\mathbf{T}} \\ -\mathbf{v} & \mathbf{0} \end{pmatrix}$$

where $\mathbf{v}$ is a $n \times 1$ matrix. By Lemma 1.5 the trivial map $\mathfrak{m} \to T_{eK}G$ is an isomorphism.

Example 1.7. For $x, y \in \mathbb{R}^{n+1}$ define

$$f(x,y) = x_1y_1 - \sum_{i=2}^{n+1} x_iy_i$$

using matrix multiplication we write this as

$$f(x,y) = x^TQy \quad \text{where } Q = \begin{pmatrix} 1 & \mathbf{0}^{\mathbf{T}} \\ \mathbf{0} & -I \end{pmatrix}$$

where $\mathbf{0}$ is the $n \times 1$ matrix of zeros and I is the $n \times n$ identity matrix.

Now define

$$\mathbf{O}(1,n) = \{A \in \mathbf{GL}_{n+1}(\mathbb{R}) : A^T Q A = Q\}$$

We find the Lie algebra $\mathfrak{g} = \mathfrak{o}(1,n)$ of $\mathbf{O}(1,n)$ by taking the derivative of a curve at e, in the defining expression for $\mathbf{O}(1,n)$. So if X=$\dot{A}(0)$ where $A(t) \in \mathbf{O}(1,n)$ then

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(A(t)^T Q A(t)_{|t=0} = X^T Q + QX \tag{1.1}$$

if we write X as

$$X = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

we get for Equation (1.1)

$$\begin{pmatrix} E^T + E & -G^T + F \\ F^T - G & -H^t - H \end{pmatrix} = 0$$

So

$$\mathfrak{o}(1,n) = \left\{ \begin{pmatrix} 0 & F^T \\ F & H \end{pmatrix} : F \in M_{(n,1)},\ \ H \in \mathfrak{o}(n) \right\}$$

If $A \in \mathbf{O}(1,n)$ then $\det(\mathrm{A^T Q A}) = \det(\mathrm{Q})$ which implies $\det(\mathrm{A})^2 = 1$ so $\det(\mathrm{A}) = \pm 1$. We turn our attention to the subgroup $\mathbf{SO}(1,n)$ with determinant 1. The hyperboloid

$$\mathcal{H}_{1,n} = \{x \in \mathbb{R}^{n+1} : f(x,x) = 1\}$$

has two connected components $\mathcal{H}^+_{1,n}$ with $x_1 > 0$ and $\mathcal{H}^-_{1,n}$ with $x_1 < 0$. There are $A \in \mathbf{SO}(1,n)$ which interchange the two components of $\mathcal{H}_{1,n}$, so we restrict our attention to the subgroup

$$\mathbf{Lor}(1,n) = \{A \in \mathbf{SO}(1,n) : A\mathcal{H}^+_{1,n} = \mathcal{H}^+_{1,n}\}$$

If we consider the vector $e_1 = (1, 0, \ldots, 0)$ which is left unchanged by matrices of the form

$$\begin{pmatrix} 1 & \mathbf{0^T} \\ \mathbf{0} & B \end{pmatrix} \quad \text{where } B \in \mathbf{SO}(n)$$

and therefore correspond to the isotropy group of e_1. Therefore $\mathbf{Lor}(1,n)/\mathbf{SO}(n)$ has a manifold structure.

We will now show that $\mathbf{Lor}(1,n)$ acts transitively on $\mathcal{H}^+_{1,n}$. Let $u \in \mathcal{H}^+_{1,n}$ and consider the vectors $v \in \mathbb{R}^{n+1}$ such that

$$0 = u^T Q v = u_1 v_1 - \sum_{i=2}^{n+1} u_i v_i$$

This is an n-dimensional subspace V, since by choosing $v_i, i = 2 \ldots n+1$ arbitrarily we get v_1

$$v_1 = \frac{\mathbf{u} \cdot \mathbf{v}}{u_1} \quad \text{, this is well defined since } u \in \mathcal{H}^+_{1,n} \text{ have } u_1 > 0$$

It also follows that the vector v satisfies $f(v, v) < 0$ since

$$\begin{aligned}
f(v, v) &= \frac{|\mathbf{u} \cdot \mathbf{v}|^2}{u_1^2} - |\mathbf{v}|^2 \\
&= \frac{|\mathbf{u} \cdot \mathbf{v}|^2}{1 + |u|^2} - |\mathbf{v}|^2 \\
&< \frac{|\mathbf{u} \cdot \mathbf{v}|^2}{|u|^2} - |\mathbf{v}|^2 \quad \text{by Cauchy-Schwartz} \\
&\leq |v|^2 - |v|^2 = 0
\end{aligned}$$

So $-f_V$ is a positive definite inner product on V and by the Gram-Schmidt process we can construct a basis $\{w^i\}_{i=2}^{n+1}$ such that

$$f(w^i, w^j) = \delta^{ij} \quad , f(u, w^i) = 0$$

Then the matrix

$$A = \begin{pmatrix} u_1 & w_1^1 & \ldots & w_1^n \\ \mathbf{u} & \mathbf{w^1} & \ldots & \mathbf{w^n} \end{pmatrix}$$

is in $\mathbf{Lor}(1, n)$ and $A(e_1) = u$. We decompose the Lie algebra of $\mathbf{O}(1, n)$ as

$$\mathfrak{k} = \left\{ \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0} & A \end{pmatrix} : A \in \mathfrak{o}(n) \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A^T \\ A & \mathbf{0} \end{pmatrix} : A \in M_{(n,1)} \right\}$$

Example 1.8. Let $G_k(\mathbb{R}^n)$ denote the set of all k-dimensional subspaces in $\mathbb{R}^n$. This space is called the real Grassman manifold. The orthogonal group $\mathbf{O}(n)$ acts transitively on $G_k(\mathbb{R}^n)$, since if V is spanned by the k first elements of the canonical basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ let $e'_1, \ldots, e'_n$ be another basis of $\mathbb{R}^n$. The matrix $A \in \mathbf{O}(n)$ with $e'_1, \ldots e'_k$ as the k first columns has the effect

$$AV = W$$

where W is the space spanned by $e'_1, \ldots e'_k$. The isotropy group of V consists of matrices of the form

$$\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix} \quad \text{where } B \in \mathbf{O}(k), C \in \mathbf{O}(n-k)$$

so we have that

$$\mathbf{O}(n)/(\mathbf{O}(k) \times \mathbf{O}(n-k))$$

is a manifold. Since the Lie algebra of $\mathbf{O}(n)$ consists of the skew-symmetric matrices we get

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0} & D \end{pmatrix} : A \in \mathfrak{o}(k),\ \ D \in \mathfrak{o}(n-k) \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} \mathbf{0} & B \\ -B^T & \mathbf{0} \end{pmatrix} : B \in M_{(k,n-k)} \right\}$$

Example 1.9. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, then if one considers $\mathbb{K}^{n+1}$ as a left $\mathbb{K}$-vector space with the action of

$$\mathbb{K}^* = \{a \in \mathbb{K} : a \neq 0\}$$

on $\mathbb{K}_0^{n+1} = \mathbb{K}^{n+1} - 0$ given by

$$\nu : \mathbb{K}^* \times \mathbb{K}_0^{n+1} \to \mathbb{K}_0^{n+1}, \nu_z(x) = xz^{-1}$$

Then set of orbits is denoted by $\mathbb{K}\mathrm{P}^n$ and is called the n-dimensional $\mathbb{K}$-projective space. The orbit is denoted as

$$[x] = \{xz^{-1} : z \in \mathbb{K}^*\}$$

Let us now restrict ourselves to $\mathbb{C}\mathrm{P}^n$. Let $A \in \mathbf{U}(n+1)$ then

$$\nu_A[x] = [Ax]$$

is well defined since $A(xz^{-1}) = (Ax)z^{-1}$. Now $[e_1]$ is stabilized by

$$\left\{ \begin{pmatrix} e^{i\vartheta} & \mathbf{0} \\ \mathbf{0} & B \end{pmatrix} : \quad \vartheta \in \mathbb{R} \text{ and } B \in \mathbf{U}(n) \right\}$$

So $\mathbf{U}(n+1)/(\mathbf{U}(1) \times \mathbf{U}(n))$ has a manifold structure. Similarly to the case of the sphere the Lie algebra decomposes as

$$\mathfrak{k} = \left\{ \begin{pmatrix} ir & 0 \\ 0 & B \end{pmatrix} : \quad r \in \mathbb{R},\ \ B \in \mathfrak{u}(n) \right\}$$

and

$$\mathfrak{m} = \left\{ \begin{pmatrix} 0 & A \\ -A^* & \mathbf{0} \end{pmatrix} : \quad A \in M_{(1,n)} \right\}.$$

CHAPTER 2

Symmetric Spaces

In this section we define the notion of a symmetric space and study some of their properties.

1. Definitions

Here it is assumed that the reader is familiar with the concept of a geodesically complete Riemannian manifold and the Hopf-Rinow theorem. If not, the reader is referred to Appendix A.

Definition 2.1. A Riemannian manifold (M, g) is said to be a *symmetric space* if for every point $p \in M$ there exists an isometry σ_p of (M, g) such that

(1) $\sigma_p(p) = p$, and
(2) $d\sigma_p = -id_{T_pM}$.

Such an isometry is called an *involution* at $p \in M$.

Lemma 2.2. [**6**] *Let (M, g) be a symmetric space and let $\sigma_p : (M, g) \to (M, g)$ be an involution at $p \in M$. Then σ_p reverses the geodesics through p, i.e. $\sigma_p(\gamma(t)) = \gamma(-t)$ for all geodesics $\gamma \in M$ such that $\gamma(0) = p$.*

PROOF. [**6**] A geodesic $\gamma : I \to M$ is uniquely determined by the initial data $\gamma(0)$ and $\dot\gamma(0)$. Both the geodesics $t \mapsto \sigma_p(\gamma(t))$ and $t \mapsto \gamma(-t)$ take the value $\gamma(0)$ and have the tangent $-\dot\gamma(0)$ for $t = 0$. □

The following lemma entails the core features of a symmetric space.

Lemma 2.3. [**6**] *Let (M, g) be a symmetric space. If $\gamma : I \to M$ is a geodesic with $\gamma(0) = p$ and $\gamma(x) = q$ then $\sigma_q \circ \sigma_p(\gamma(t)) = \gamma(t + 2x)$. For $v \in T_{\gamma(t)}M$, $d\sigma_q(d\sigma_p(v)) \in T_{\gamma(t+2x)}M$ is the vector at $\gamma(t + 2x)$ obtained by parallel transport of v along γ.*

PROOF. [**6**] Let $\tilde\gamma(t) = \gamma(t + x)$ then $\tilde\gamma$ is a geodesic with $\tilde\gamma(0) = q$. So by Lemma 2.2 it follows that

$$\begin{aligned}\sigma_q(\sigma_p(\gamma(t))) &= \sigma_q(\gamma(-t)) \\ &= \sigma_q(\tilde\gamma(-t - x)) \\ &= \tilde\gamma(t + x) \\ &= \gamma(t + 2x).\end{aligned}$$

If $v \in T_pM$ and V is a parallel vector field along γ with $V(p) = v$, then $d\sigma_p(V)$ is parallel, since σ_p is an isometry. Also

$$d\sigma_q \circ d\sigma_p(V(\gamma(t))) = V(\gamma(t + 2\varkappa))$$

by the above and since $d\sigma$ applied twice cancels direction reversals. □

As a display of the power of Lemma 2.3 we have the following

Corollary 2.4. [**6**] *Every symmetric space (M, g) is geodesically complete and thus any two points $p, q \in M$ in the same path component of M can be connected by a geodesic.*

PROOF. [**6**] By repeatedly composing as in Lemma 2.3 we can get to $\gamma(2^l\varkappa)$ until $2^l\varkappa$ greater than any real number. This shows that (M, g) is geodesically complete, so by the Hopf-Rinow theorem any two points can be connected by a geodesic. □

Definition 2.5. Let (M, g) be a symmetric space and let $\gamma : \mathbb{R} \to M$ be a geodesic with $p = \gamma(0)$ and $v = \dot\gamma(0)$. Then for $t \in \mathbb{R}$ the isometries

$$\tau_{tv} : (M, g) \to (M, g)$$

given by

$$\tau_{tv} = \sigma_{\gamma(t/2)} \circ \sigma_{\gamma(0)}$$

are called *transvections*.

It is easily seen, by using Lemma 2.3, that for the particular geodesic $\gamma : \mathbb{R} \to M$ in Definition 2.5 the transvection τ_{tv} satisfies

$$\tau_{tv}(\gamma(s)) = \gamma(s + t)$$

Proving things about isometries are facilitated by

Lemma 2.6. [**10**] *Let (M, g) be a connected Riemannian manifold and $p \in M$. If $\psi, \psi' : (M, g) \to (M, g)$ are isometries such that $\psi(p) = \psi'(p)$ and $d\psi_p = d\psi'_p$ then $\psi = \psi'$.*

PROOF. [**10**] Let $B_\delta(p)$ be a normal ball around $p \in M$. Then if $\gamma_v(t) = \mathrm{Exp}_p(tv)$ is a geodesic then $\psi(\gamma_v(t)) = \psi'(\gamma_v(t))$ are the same geodesics since geodesics are given by initial data. So $\psi = \psi'$ on an open set but, this set is also closed, hence it is M. □

Here we have some nice properties of transvections

Proposition 2.7. [**10**] *Let (M, g) be a symmetric space, $p \in M, v \in T_pM$ and define $\gamma : \mathbb{R} \to M$ by $\gamma(t) = Exp_p(tv)$ as the unique geodesic with $\gamma(0) = p, \dot\gamma(0) = v$. Let τ_{bv} be the transvections corresponding to the involution σ_p, then*

(1) $\tau_{av} = \sigma_{\gamma(t+\frac{a}{2})} \circ \sigma_{\gamma(t)}$ *for all $t \in \mathbb{R}$*

(2) τ_{av} *depends only on av*
(3) $\tau_{av}(\gamma(t)) = \gamma(t+a)$
(4) $\tau_{av} \circ \tau_{bv} = \tau_{(a+b)v}$
(5) $\sigma_p \circ \tau_{av} \circ \sigma_p = \tau_{-av}$

PROOF. [**10**] (1) Note that τ is the composition of two isometries so it is actually an isometry.
For $\gamma(s) \in M$, if $\tilde{\gamma}(r) = \gamma(r+t)$ then $\gamma(t) = \tilde{\gamma}(0)$ and $\gamma(s) = \tilde{\gamma}(s-t)$ so

$$\begin{aligned} \sigma_{\gamma(t+\frac{a}{2})} \circ \sigma_{\gamma(t)}(\gamma(s)) &= \sigma_{\tilde{\gamma}(\frac{a}{2})} \circ \sigma_{\tilde{\gamma}(0)}(\tilde{\gamma}(s-t)) = \tilde{\gamma}(s-t+a) \\ &= \gamma(s+a) \end{aligned}$$

Now we also have $\sigma_{\gamma(\frac{a}{2})} \circ \sigma_{\gamma(0)}(\gamma(s)) = \gamma(s+a)$. So the isometries agree at $\gamma(s)$.
If $w \in T_{\gamma(s)}M$ and V is the parallel vector field along γ such that $V(\gamma(s)) = w$.
Then $\mathrm{d}(\sigma_{\gamma(t+\frac{a}{2})} \circ \sigma_{\gamma(t)})w$ is the parallel transport of w along γ from $\gamma(s)$ to $\gamma(s+a)$. The same is true for $\mathrm{d}\tau_{av}w = \mathrm{d}(\sigma_{\gamma(\frac{a}{2})} \circ \sigma_{\gamma(0)})w$.
So since their initial data agree we have $\tau_{av} = \sigma_{\gamma(t+\frac{a}{2})} \circ \sigma_{\gamma(t)}$ by Lemma 2.6.
(2) $\tau_{av} = \sigma_{\gamma(\frac{a}{2})} \circ \sigma_{\gamma(0)} = \sigma_{\mathrm{Exp}_{p_0}(\frac{av}{2})} \circ \sigma_{p_0}$, so it depends only on the value of av
(3) This follows directly from the proof of 1
(4) By (1) $\tau_{av} \circ \tau_{bv} = \sigma_{\gamma(\frac{a}{2}+\frac{b}{2})} \circ \sigma_{\gamma(\frac{b}{2})} \circ \sigma_{\gamma(0)} = \sigma_{\gamma(\frac{a}{2}+\frac{b}{2})} \circ \sigma_{\gamma(0)} = \tau_{(a+b)v}$
(5) $\sigma_{\gamma(0)} \circ \sigma_{\gamma(\frac{a}{2})} \circ \sigma_{\gamma(0)} \circ \sigma_{\gamma(0)} = \sigma_{\gamma(0)} \circ \sigma_{\gamma(\frac{a}{2})} = \tau_{-av}$ ☐

Here are some concrete examples of involutions and transvections

Example 2.8. At $p_0 \in \mathbb{R}^n$ define the involution

$$\sigma_{p_0} : x \mapsto -(x-p_0) + p_0$$

In $\mathbb{R}^n$ geodesics are straight lines, so consider the geodesic

$$\gamma(t) : \mathbb{R} \to \mathbb{R}^n, t \mapsto p_0 + t\mathbf{v}$$

where $\mathbf{v} \in T_{p_0}\mathbb{R}^n = \mathbb{R}_n$
Then we get the transvection

$$\begin{aligned} \tau_{t\mathbf{v}}(x) &= \sigma_{\gamma(t/2)} \circ \sigma_{\gamma(0)}(x) \\ &= \sigma_{\gamma(t/2)}(-(x-\gamma(0)) + \gamma(0)) \\ &= -(-(x-\gamma(0)) + \gamma(0) - \gamma(t/2)) + \gamma(t/2) \\ &= -(-(x-p_0) + p_0 - (p_0 + t\mathbf{v})) + (p_0 + t\mathbf{v}) \\ &= x + t\mathbf{v}. \end{aligned}$$

Now we can verify the translational properties of the transvection

$$\tau_{t\mathbf{v}}\gamma(s) = (p_0 + s\mathbf{v}) + t\mathbf{v} = p_0 + (s+t)\mathbf{v} = \gamma(s+t)$$

Example 2.9. If we have a submanifold M in $(\mathbb{R}^m, g)$ for some $m \in \mathbb{Z}^+$ and $M = \{x \in \mathbb{R}^m : g(x,x) = r\}$ for example S^n or $\mathcal{H}^+_{1,n}$, then we have the involution at p_0

$$\begin{aligned} \sigma &: M \to M \\ x &\mapsto 2\frac{g(p_0, x)}{r} p_0 - x \end{aligned}$$

then

$$\sigma_{p_0}(p_0) = 2\frac{g(p_0, p_0)}{r} p_0 - p_0 = p_0$$

since $T_{p_0}M \perp p_0$ we get for $v \in T_{p_0}M$

$$d\sigma_{p_0}(v) = 2\frac{g(p_0, v)}{r} p_0 - v = -v$$

so we have an involution on the submanifold M.

2. The Isometry Group

Definition 2.10. Let $I(M)$ be the isometry group of the symmetric space (M, g), and let G be the connected component of $I(M)$ containing the neutral element $e \in G$, i.e. G is the identity component of $I(M)$. Furthermore let K_{p_0} be the isotropy group of G at $p_0 \in M$.

It should be noted that by continuity, all the transvections belong to the group G.

Now we can prove the a very important thing concerning the isometry group.

Theorem 2.11. [**6**] *Let (M, g) be a symmetric space and G the identity component of $I(M)$, then G acts transitively on M*

PROOF. [**6**] By Corollary 2.4 any $p, q \in M$ can be connected by a geodesic γ. If $\gamma(0) = p$, $\dot\gamma(0) = v$ and $\gamma(s) = q$ and τ_{tv} is the family of translations along γ then $q = \tau_{sv}(p)$. So the action is transitive. □

Lemma 2.12. [**10**] *Let N be a smooth manifold and let $f : N \to I(M)$ be a map such that $f(n)(p)$ depends smoothly on (n, p) for $n \in N$ and $p \in M$ is in a neighborhood of $p_0 \in M$. Then the dependence is smooth for all $p \in M$.*

PROOF. [**10**] If we prove the claim for $p \in M$ in a normal ball around $q \in M$, then we can cover M with intersecting balls from p_0 to an arbitrary point in M. So if $p \in B_\delta(q)$ then there exists a $v \in T_qM$ such that $p = \mathrm{Exp}_q v$ and v depends smoothly on p. Then

$$f(n)(p) = f(n)(\mathrm{Exp}_q v) = \mathrm{Exp}_{f(n)(q)}(df(n)(q))v$$

where the right side depends smoothly on (n, p). □

Now we have a very important lemma for showing that charts and actions on our manifold are smooth.

Lemma 2.13. [**10**] *Let (M, g) be a symmetric space with involutions σ_p, transvections τ_v and let K_{p_0} be the isotropy group of the isometries $I(M)$ acting at a point $p_0 \in M$, then*

(1) *$\sigma_p(q)$ depends smoothly on $(p, q) \in M \times M$.*
(2) *$\tau_v(q)$ depends smoothly on $(v, q) \in T_pM \times M$.*
(3) *$\tau_p(q)$, depends smoothly on $(p, q) \in M \times M$, where τ_p is τ_v such that $\tau_v(p_0) = p$.*
(4) *$k(q)$ depends smoothly on $(k, q) \in K_{p_0} \times M$.*

PROOF. [**10**] (1) By Lemma 2.12 we only need to show the claim for a point q in a neighborhood of p. Then $q = \mathrm{Exp}_p(v)$ where v depends smoothly on (p, q). But $\mathrm{Exp}_p(tv)$ goes through p so $\sigma_p(q) = \sigma_p(\mathrm{Exp}_p(tv))_{|t=1} = \mathrm{Exp}_p(-v)$ which again depends smoothly on (p, q).

(2) The claim follows from the definition of τ_v and 1)

(3) Since v in the geodesic $Exp_{p_0}(tv)$ connecting p_0 to p depends smoothly on p close to p_0, then τ_v depends smoothly on p close to p_0. Then by Lemma 2.12 this is true for all $p \in M$.

(4) By Lemma 2.12 we show it for q in a neighborhood of p_0. Then $q = Exp_{p_0}(v)$ so $k(q) = k(Exp_{p_0}(v)) = Exp_{k(p_0)}(\mathrm{d}k)v = Exp_{p_0}(\mathrm{d}k)v$ which depends smoothly on (k, q). □

Note that K_{p_0} immediately can be defined as a Lie group using Lemma 2.6 and the fact that for all $f \in K_{p_0}$, $f(p_0) = p_0$, i.e. an isometry on a connected manifold is identified by its value at one point and its differential at the same point. So since $\mathrm{d}f(p_0)$ is an element of the Lie group of isometries $T_{p_0}M \to T_{p_0}M$ which is $\mathbf{O}(\dim M)$. Since K_0 is closed we can identify K_0 with a Lie-subgroup of $\mathbf{O}(\dim M)$.

Theorem 2.14. [**10**] *Let (M, g) be a symmetric space. Then the isometry group $I(M)$ has the structure of a Lie group such that*

(1) *The map $I(M) \times M \to M$, with $\quad (g, p) \mapsto g(p)$ is smooth.*
(2) *If N is a manifold then $f : N \to I(M)$ is smooth if and only if*

$$N \times M \to M, \quad (n, q) \mapsto f(n)(q)$$

is smooth.

PROOF. [**10**] We will define charts in a neighborhood around $g_0 \in I(M)$. Let U be a neighborhood of $p_0 \in M$. We'll define a map that will help us with the charts. Let $U \times K_{p_0} \to I(M), \quad (p, k) \mapsto g_0\tau_p k$, where τ_p is defined as in Lemma 2.13. This map is injective since we can retrieve both p, v by

$$g_0^{-1} g_0 \tau_p k p_0 = \tau_p p_0 = p$$

$$\tau_p^{-1} g_0^{-1} g_0 \tau_p k = k$$

So we can talk about the inverse image of this map around g_0.

If we choose local coordinates for M by $p \mapsto (x_1, \ldots x_m)$ at p_0 and local coordinates $k \mapsto (y_1, \ldots y_l)$ at k_0 in K_{p_0}, then we get local coordinates around g_0 in $I(M)$

$$g_0 \tau_p k \mapsto (x_1(p), \ldots, x_m(p), y_1(k), \ldots, y_l(k)).$$

These coordinates are smooth, since if the image of two such charts intersect we get $g_1 \tau_{p_1} k_1 = g_2 \tau_{p_2} k_2$ so

$$\begin{aligned} p_1 &= g_2^{-1} g_2 \tau_{p_2} k_2(p_0) \\ dk_1(p_0) &= (\mathrm{d}\tau_{p_1}^{-1}(p_1))\mathrm{d}(g_1^{-1} g_2)(p_2)(\mathrm{d}\tau_{p_2}(p_0))(\mathrm{d}k_2(p_0)) \end{aligned}$$

these depend smoothly on each other by Lemma 2.13.

The topology is Hausdorff since if ψ, ψ' are two different isometries then we either have $\psi(p_0) \neq \psi'(p_0)$ or $(d\psi(p_0) \neq (d\psi(p_0))$. If the first one holds we can separate the isometries since M is Hausdorff. If the second hold we must have that $k_1 \neq k_2$ so we can separate in $O(\dim M)$.

Now the group operation is smooth since if $f, g \in I(M)$ then $f^{-1} g = (g_1 \tau_{p_1} k_1)^{-1} (g_2 \tau_{p_2} k_2) = (g_3 \tau_{p_3} k_3)$ in local coordinates. So

$$p_3 = \tau_{p_3} k_3(p_0) = g_3^{-1} k_1^{-1} \tau_{p_1}^{-1} g_1^{-1} g_2 \tau_{p_2} k_2(p_0)$$

and

$$\mathrm{d}k_3 = \mathrm{d}\tau_{p_3}^{-1} \mathrm{d}g_3^{-1} \mathrm{d}k_1^{-1} \mathrm{d}\tau_{p_1}^{-1} \mathrm{d}g_1^{-1} \mathrm{d}g_2 \mathrm{d}\tau_{p_2} \mathrm{d}k_2$$

which is smooth by Lemma 2.13. Next

(1) Since g is an isometry then $(g, p) \mapsto g(p)$ is smooth

(2) Suppose $f(n)$ is smooth. Let $n_0 \in N$ then $f(n) = g_0 \tau_p k$ where (p, k) depends smoothly on n. So $f(n)(q) = g_0 \tau_p k(q)$ depends smoothly on (n, q).

Conversely if the assignment $(n, q) \mapsto f(n)(q)$ is smooth then

$$f(n) = g_0 \tau_p k$$

where $p = g_0^{-1} f(n)(p_0)$ and $\mathrm{d}k = \mathrm{d}\tau_p^{-1} \mathrm{d}g_0^{-1} \mathrm{d}f(n)$. So (p, k) depends smoothly on n. □

Example 2.15. If we consider the isometry groups in Examples 1.6-1.9 of Section 2 then we see that

$\mathbf{SO}(n+1)$ acts smoothly on S^n,
$\mathbf{Lor}(1, n)$ acts smoothly on $\mathcal{H}_{1,n}^+$,
$\mathbf{O}(n)$ acts smoothly on $G_k(\mathbb{R}^n)$ and
$\mathbf{U}(n+1)$ acts smoothly on $\mathbb{CP}^n$

3. G/K is Diffeomorphic to M

Here we show that the quotient of the isometry group and the isotropy group can be identified with the symmetric space.

Theorem 2.16. [**10**] *Let (M, g) be a symmetric space and let $G = I(M)$ act transitively on M. Let K be the isotropy group of G at $p_0 \in M$. Then the map*

$$\psi : G/K \to M \text{ with } gK \mapsto g(p_0)$$

is a diffeomorphism such that

$$\psi \circ \mu(g) = g \circ \psi$$

where $\mu(g')gK = g'(gK) = g'gK$ as in Theorem 1.4.

If the action is not effective one can consider the groups G/N and K/N instead, where N is the kernel of the action. So if M is a symmetric space then $I_0(M)/K_0$ is diffeomorphic to M.

PROOF. [**10**] Clearly ψ is well defined and bijective since $\nu_k(p_0) = p_0$ for $k \in K$. Also

$$\psi \circ \mu(g')gK = \psi(g'gK) = \nu_{g'g}(p_0) = \nu_{g'}\nu_g(p_0) = \nu_{g'}(\psi(gK))$$

ψ is smooth since $\psi \circ \pi(g) = \nu_g$ and π are smooth.
We will show that ψ^{-1} is smooth. This is done by showing that dim $(G/K) \geq$ dim M. Then we show $d\psi$ is injective, so $d\psi$ is an isomorphism and by the inverse function theorem locally ψ^{-1} is smooth. But ψ^{-1} exists globally so by some form of gluing lemma ψ^{-1} is smooth.

Now $M = \cup_{j=1}^{\infty} U_j$, where U_j are the images of the charts on G/K. Since M is locally compact, at least one $\bar{U}_j$ contains an open subset of M by the Baire category theorem. This open subset on M is in bijective correspondence with an open set V in G/K since G/K acts smoothly. So using the charts we would get an onto map $\mathbb{R}^n \to V \to U_j \to \mathbb{R}^m$, so we cannot have $n < m$ i.e. dim $(G/K) \geq$ dim M.

Thus we only have to show the injectivity which will imply that dim $(G/K) \leq$ dim M. Thus dim $(G/K) =$ dim M. But $\psi \circ \mu(g) = \nu_g \circ \psi$ so $\psi = \nu_g \circ \psi \circ \mu(g^{-1})$. If we differentiate at g' we get

$$(d\psi)_{g'K} = (d\nu_g \circ d\psi)_{g^{-1}g'K} \circ (d\mu(g^{-1}))_{g'K}$$

So if $d\psi$ is injective at aK then it is injective at $g'K$ by choosing g such that $g^{-1}g'K = aK$ in $\psi = \nu_g \circ \psi \circ \mu(g^{-1}$. Therefore it is enough to show the injectivity at eK. Let $v \in T_{eK}(G/K)$ with $d\psi(v) = 0$. By Corollary 1.5 $d\pi$ is surjective and has kernel $\mathfrak{k}$. So there is a $X \in \mathfrak{g}$ such that $d\pi(X) = v$. The curve $\gamma(t) = \nu_{\exp(tX)}(p_0)$ satisfies

$$\frac{d\gamma(t)}{dt} = \frac{d}{dt}\Big(\psi \circ \pi(\exp(tX))\Big)$$

$$
\begin{aligned}
&= d\psi \circ d\pi_{\exp(tX)}(X) \\
&= d\psi \circ d\pi \circ dL_{\exp(tX)}(X(e)) \\
&= d\psi \circ d\mu(\exp(tX)) \circ d\pi(X(e)) \\
&= d\nu_{\exp(tX)} \circ d\psi \circ d\pi(X(e)) \\
&= d\nu_{\exp(tX)} \circ d\psi(v) \\
&= 0
\end{aligned}
$$

so $\gamma \in K$ and $X \in \mathfrak{k}$, i.e. $v = 0$. □

Example 2.17. We have seen in Examples 1.6-1.9 of section 2 groups G that act transitively on manifolds and their isotropy groups K at certain points. Therefore we have the following diffeomorphisms

$$
\begin{aligned}
\mathbf{SO}(n+1)/\mathbf{SO}(n) &\cong S^n \\
\mathbf{Lor}(1,n)/\mathbf{SO}(n) &\cong \mathcal{H}^+_{1,n} \\
\mathbf{O}(n)/\mathbf{O}(k) \times \mathbf{O}(n-k) &\cong G_k(\mathbb{R}^n) \\
\mathbf{U}(n+1)/\mathbf{U}(1) \times \mathbf{U}(n) &\cong \mathbb{C}P^n
\end{aligned}
$$

CHAPTER 3

Symmetric Pairs

In this chapter we introduce the concept of a symmetric pair and show how they lead to symmetric spaces.

Definition 3.1. A pair (G, K) is said to be a Riemannian symmetric pair if G is a Lie group, K a closed subgroup of G, and s_{p_0} an involutive automorphism on G such that

(1) $(G_{s_{p_0}})_0 \subseteq K \subseteq G_{s_{p_0}}$

(2) $Ad(K)$ is a compact subset of $GL(\mathfrak{g})$.

Here $G_{s_{p_0}}$ are the elements of G that are left invariant by s_{p_0}, i.e

$$G_{s_{p_0}} = \{g \in G : s_{p_0} g = g\}.$$

The involutive automorphism s_{p_0} is often uniquely defined, so it is common to only write (G, K) for a symmetric pair.

1. From a Symmetric Space to a Symmetric Pair

As before let G, K_{p_0} denote the identity component of the isometry and isotropy groups on M. Then by Theorem 2.16 there is a bijective correspondence

$$G/K \leftrightarrow M, \quad gK \mapsto g(p_0)$$

We define an involution corresponding to σ_{p_0} on G by

$$s_{p_0} : G \to G \quad s_{p_0}(g) = \sigma_{p_0} \circ g \circ \sigma_{p_0} = \sigma_{p_0} \circ g \circ \sigma_{p_0}^{-1}$$

Then $s_{p_0}^2 = id$ and $s_{p_0}(g)p = \sigma_{p_0} \circ g \circ \sigma_{p_0}^{-1}(p)$, so by Lemma 2.13 it depends smoothly on (g, p). Thus s_{p_0} is an involutive Lie group automorphism of G. Note also that

$$(\mathrm{d}s_{p_0})_e \quad : \quad \mathfrak{g} \to \mathfrak{g}$$

is a involutive Lie algebra automorphism.

Let $G_{s_{p_0}} = \{g \in G : s_{p_0} g = g\}$, and let $\mathfrak{g}_{s_{p_0}}$ be its Lie algebra. Then $G_{s_{p_0}}$ is a Lie group since it is a closed subgroup.

We have the following theorem

Theorem 3.2. [**10**] *Let (M, g) be a symmetric space with a fixed point p_0, G be the identity component of the isometry group and let K be the isotropy group of G at p_0. Then the map $G/K \to M$ with $K \mapsto g(p_0)$ is a bijection. The group G has an involutive automorphism s_{p_0} given by*

$$s_{p_0}(g) = \sigma_{p_0} \circ g \circ \sigma_{p_0}$$

with stabilizer $G_{s_{p_0}}$ such that

$$(G_{s_{p_0}})_0 \subseteq K \subseteq G_{s_{p_0}}$$

This says that (G, K) is a symmetric pair because $Ad(K)$ is compact, since K is closed and bounded and Ad is a homeomorphism.

PROOF. [**10**] It only remains to prove the last statement.

If $X \in \mathfrak{g}_{p_0}$ then $\exp(tX) \in G_{p_0}$ so $s_{p_0}(\exp(tx)) = \exp(tX)$ therefore $\mathrm{d}s_{p_0} = X$. Also if $\mathrm{d}s_{p_0}X = X$ then $\exp(tX) = \exp(t\mathrm{d}s_{p_0}X) = s_{p_0}\exp(tX)$, the implications work in the opposite direction as well, so

$$\mathfrak{g}_{p_0} = \{X \in g : \mathrm{d}s_{p_0}X = X\}.$$

Let $k \in K$ then

$$\begin{aligned} s_{p_0}(k)p_0 &= \sigma_{p_0} \circ k \circ \sigma_{p_0}(p_0) = p_0 = k(p_0) \quad \text{and} \\ \mathrm{d}s_{p_0}(k) &= \mathrm{d}\sigma_{p_0} \circ \mathrm{d}k_{p_0} \circ \mathrm{d}\sigma_{p_0} \\ &= (-id_{T_{p_0}M}) \circ \mathrm{d}k_{p_0} \circ (-id_{T_{p_0}M}) \\ &= \mathrm{d}k_{p_0}. \end{aligned}$$

This means that Lemma 2.6 implies that $s_{p_0}(k) = k$ so $K \subseteq G_{p_0}$. Further if $X \in \mathfrak{g}_{p_0}$ then $\sigma_{p_0}(\exp(tX))p_0 = \sigma_{p_0}(\exp(tX))\sigma_{p_0}(p_0) = s_{p_0}(\exp(tX))p_0 = \exp(tX)p_0$, so $\exp(tX)p_0$ is a fixed point of σ_{p_0}. But if t is small then $\exp(tX)p_0$ is close to p_0 and this is the only fixed point so $\exp(tX)p_0 = p_0$. Thus $\exp(tX) \in K_0 = K$ and $X \in \mathfrak{k}$. So $\mathfrak{g}_{p_0} \subseteq \mathfrak{k} \Rightarrow (G_{p_0})_0 \subseteq K$.

□

2. The Tangent Space of G/K

Here we will split up the Lie algebra $\mathfrak{g}$ so that the complement of $\mathfrak{k}$ can be identified with the tangent space of M. We know that

$$\mathfrak{k} = \{X \in \mathfrak{g} : \mathrm{d}s_{p_0}(X) = X\}$$

and define

$$\mathfrak{p} = \{X \in \mathfrak{g} : \mathrm{d}s_{p_0}(X) = -X\}$$

Then since $\mathrm{d}s_{p_0}$ is an automorphism $\mathfrak{k} \cap \mathfrak{p} = \{0\}$. We also have that $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for all $X \in \mathfrak{g}$ since

$$X = \frac{1}{2}(X + \mathrm{d}s_{p_0}(X)) + \frac{1}{2}(X - \mathrm{d}s_{p_0}(X))$$

where the first term is in $\mathfrak{k}$ and the second is in $\mathfrak{p}$. Therefore

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

Since $\mathrm{d}s_{p_0}$ is a Lie algebra automorphism, i.e.

$$\mathrm{d}s_{p_0}[X, Y] = [\mathrm{d}s_{p_0}X, \mathrm{d}s_{p_0}Y]$$

we have

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$$
$$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$$
$$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$$

We now reveal the connection between the isometry group and the symmetric space.

Theorem 3.3. [**10**] *Let (M, g) be a symmetric space, G be the identity component of $I(M)$, K the isotropy group of G at $p_0 \in M$, $\mathfrak{g}$ the Lie algebra of G, $\mathfrak{k}$ the Lie algebra of K and $\mathfrak{p}$ a linear complement of $\mathfrak{k}$ in $\mathfrak{g}$. As usual let $\nu : G \to M, g \mapsto \nu_g(p_0)$ then*

$$\begin{aligned} \mathrm{d}\nu_{|\mathfrak{k}}(p_0) &= 0 \\ \mathrm{d}\nu_{|\mathfrak{p}}(p_0) &\cong T_{p_0}M \end{aligned}$$

If $X \in \mathfrak{p}$ then $exp(X) = \tau_{\mathrm{d}\nu_X(p_0)}$ and $\nu_{exp(X)}(p_0) = Exp_{p_0}(\mathrm{d}\nu_X(p_0))$.

PROOF. [**10**] $\mathrm{d}\nu_{|\mathfrak{k}}(p_0) = 0$ since $\nu(p_0)$ is constant on K. Now let $v \in T_{p_0}M$ then the map $t \mapsto \tau_{tv}$ is a smooth group homomorphism $\mathbb{R} \to G$. There is a unique $X \in \mathfrak{g}$ such that $\tau_{tv} = \exp(tX)$. For this X we have

$$\exp(t\mathrm{d}s_{p_0}X) = s_{p_0}(\exp(tX)) = \sigma_{p_0} \circ \tau_{tv} \circ \sigma_{p_0} = \tau_{-tv}.$$

This means that $\mathrm{d}s_{p_0}(x)) = -X$ so $X \in \mathfrak{p}$. Also

$$\begin{aligned} \mathrm{d}\nu_X(p_0) &= \frac{\mathrm{d}}{\mathrm{d}t}\nu_{\exp(tX)_{|t=0}}(p_0) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\exp(tX)(p_0)_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\tau_{tv}(p_0)_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Exp}_{p_0}(tv)_{|t=0} = v \end{aligned}$$

This shows that $\mathrm{d}\nu(p_0) : \mathfrak{p} \to T_{p_0}M$ is a surjective map.

Since the coordinates of G were the coordinates for K and T_{p_0},

$$\dim G = \dim K + \dim M$$

Also since $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we get dim M=dim $\mathfrak{p}$. Therefore $d\nu(p_0) : \mathfrak{p} \to T_{p_0}M$ must be injective as well.

Now the last formulas of the claim follow immediately

$$\exp X = \tau_v = \tau_{d\nu_X(p_0)}$$

and

$$\nu_{\exp X}(p_0) = \exp(X)p_0 = \tau_{d\nu_X(p_0)}p_0 = \operatorname{Exp}_{p_0}(d\nu_X(p_0))$$

□

So the map $\psi : G/K \to M$ in Theorem 2.16 defined by $gK \mapsto g(p)$ is a submersion. We apply this theorem in some of the symmetric spaces we know.

Example 3.4. By Theorem 3.3 for $G = \mathbf{SO}(3)$, $K = \mathbf{SO}(2)$ acting on S^2 at

$$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in S^2$$

we have for

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p}$$

$$\nu_{\exp tX}p_0 = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}$$

is the geodesic in S^2 starting at p_0 in direction

$$d\nu_X p_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

To get a geodesic going in an arbitrary direction we apply any element of K to the above geodesic which gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & \sin\vartheta \\ 0 & -\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos t \\ -\sin t \\ 0 \end{pmatrix}$$

Example 3.5. By Theorem 3.3 for $G = \mathbf{Lor}(1,2)$, $K = \mathbf{SO}(2)$ acting on $\mathcal{H}^+_{1,2}$ at

$$p_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{H}^+_{1,2}$$

we have for

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \mathfrak{p}$$

$$\nu_{\exp tX} p_0 = \begin{pmatrix} \cosh\, t & \sinh\, t & 0 \\ \sinh\, t & \cosh\, t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cosh\, t \\ \sinh\, t \\ 0 \end{pmatrix}$$

is the geodesic in $\mathcal{H}^+_{1,2}$ starting at p_0 in direction

$$d\nu_X p_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

To get a geodesic going in an arbitrary direction we apply any element of K to the above geodesic which gives

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\,\vartheta & \sin\,\vartheta \\ 0 & -\sin\,\vartheta & \cos\,\vartheta \end{pmatrix} \begin{pmatrix} \cosh\, t \\ \sinh\, t \\ 0 \end{pmatrix}$$

3. From a Lie Group to a Symmetric Pair

Now we are ready for the theorem that tell us if a Lie group and a closed subgroup can be made into a symmetric pair.

Lemma 3.6. *Let G be a Lie group and K a closed subgroup of G. Let $k \in K$. If we denote by $In\, g(g') = gg'g^{-1}$. Then*

$$\mu(k) \circ \pi = \pi \circ In\, k$$

and by differentiating at e

$$d\mu(k)_{eK} \circ d\pi_e = d\pi_e \circ Ad\, k.$$

PROOF. [**10**] Let $g \in G$ and $k \in K$ then $\mu(k)(gK) = kgK = kgk^{-1}K = \text{In}\,(k)gK = \pi(\text{In}\,(k)g)$, i.e. $\mu(k) \circ \pi = \pi \circ \text{In}\, k$. □

Theorem 3.7. [**10**] *Let G be a Lie group and let K be a closed subgroup, with identity components G_0, K_0. Form the quotient manifold G/K. Let μ denote the action $\mu(g')gK = g'gK$. Let $J \in Z(K) \subseteq K$ be such that $\mu(J)eK = eK$ and suppose*

$$(1) \quad d\mu(J)_{eK} = -id_{G/K}$$

Then for every compact subset K' in K where

$$K_0 \subseteq K' \subseteq K \cap G_0$$

(G_0, K') is a Riemannian symmetric pair with

$$s_{eK}(g) = JgJ^{-1} = In(J)g$$

The assumption (1) is equivalent to

$$(2) \quad X + Ad\,J(X) \in \mathfrak{k} \quad \textit{for all}\; X \in \mathfrak{g}.$$

condition (2) holds if $\mathfrak{k}$ *has a linear complement* $\mathfrak{m}$ *in* $\mathfrak{g}$, *i.e.* $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, *on which* $Ad\,J = -id$

PROOF. [**10**] By Lemma 3.6 $\mu(k) \circ \pi = \pi \circ \mathrm{In}\, k$ and

$$d\mu(k)_{eK} \circ d\pi_e = d\pi_e \circ \mathrm{Ad}\, k$$

If we put $k = J$ we get $-d\pi = d\pi \circ \mathrm{Ad}\, J$. So $d\pi(X + \mathrm{Ad}\,(J)X) = 0$ and thus $X + \mathrm{Ad}\,(J)X \in \mathfrak{k}$ for all $X \in \mathfrak{g}$. Thus (1) implies (2). By working backwards we get $-d\pi_e = d\pi \circ \mathrm{Ad}\, J = d\mu(J)_{eK} \circ d\pi_e$ so $d\mu(J)_{eK} = -id_{G/K}$ so also (2) implies (1).

Now we prove that we have a symmetric pair. Since $J \in Z(K)$,

$$\mathrm{In}\, J_{|K} = id_K \text{ and } \mathrm{Ad}\, J_{|\mathfrak{k}} = id_{\mathfrak{k}}.$$

If $X \in \mathfrak{m}$ where $\mathfrak{m}$ is the linear complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then

$$X + \mathrm{Ad}\,(X) = Y$$

where $Y \in \mathfrak{k}$ but Y depends linearly on X so we write $Y = T(X)$ where $T : \mathfrak{m} \to \mathfrak{k}$ is a linear map. Then

$$(\mathrm{Ad}\, J)^2(X) = \mathrm{Ad}\, J(-X + T(X)) = -(-X + T(X)) + T(X) = X.$$

Now since $(\mathrm{Ad}\, J_{|\mathfrak{k}})^2 = id_{\mathfrak{k}}$ we have $\mathrm{Ad}\,(J)^2 = id_{\mathfrak{g}}$. So if $X \in \mathfrak{g}$ then

$$\mathrm{In}\,(J)^2(\exp(x)) = \exp(d\mathrm{In}\,(J)^2 X) = \exp(\mathrm{Ad}\,(J)^2 X) = \exp(X).$$

So In is an involutive automorphism on G_0.

Now we'll show $((G_0)_{\mathrm{In}\, J})_0 \subseteq K' \subseteq (G_0)_{\mathrm{In}\, J}$. First $((G_0)_{\mathrm{In}\, J})_0 = (G_{\mathrm{In}\, J})_0$ since $(G_{\mathrm{In}\, J})_0 = (G_0)_{\mathrm{In}\, J}$. Also $(G_0)_{\mathrm{In}\, J} = G_0 \cap G_{\mathrm{In}\, J}$. So

$$K' \subseteq (G_0)_{\mathrm{In}\, J}$$

since $K' \subseteq G_0, K' \subseteq G_{\mathrm{In}\, J}$ because $J \in Z(K)$ and $K' \subseteq K$. Secondly if $\mathfrak{g}_{\mathrm{In}\, J} \subseteq \mathfrak{k}' = \mathfrak{k}$ then $(G_{\mathrm{In}\, J})_0 \subseteq K'$. So let $X \in g_{\mathrm{In}\, J}$ then

$$X = d\mathrm{In}\,(J)X = \mathrm{Ad}\, J(X)$$

and

$$d\pi(X) = d\pi(\mathrm{Ad}\,(J)X) = d\mu(J)d\pi(X) = -d\pi(X).$$

Thus $d(X) = 0$ and so $X \in \mathfrak{k}$.

Finally Ad K' is compact since Ad is a homeomorphism. □

Example 3.8. Let $G = \mathbf{O}(n+1)$ and $K = \mathbf{O}(n)$ in Theorem 3.7 with

$$J = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix}$$

with $J \in Z(K)$. If we form $G/K = \mathbf{O}(n+1)/\mathbf{O}(n)$ and since for $X \in \mathfrak{o}(n+1)$

$$X = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}, \quad \mathrm{Ad}\, J \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} = \begin{pmatrix} A & -B \\ B^T & D \end{pmatrix}$$

we have

$$X + \mathrm{Ad}\, J(X) = 2 \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0^T} & D \end{pmatrix} \in \mathfrak{k}$$

so J satisfies 2) in Theorem 3.7 so we have the symmetric pair

$$(\mathbf{SO}(n+1), \mathbf{SO}(n)) \text{ with } s = \mathrm{In}\, J$$

Example 3.9. Let $G = \mathbf{O}(n)$ and $K = \mathbf{O}(k) \times \mathbf{O}(n-k)$ in Theorem 3.7 with

$$J = \begin{pmatrix} I_k & \mathbf{0} \\ \mathbf{0^T} & -I_{n-k} \end{pmatrix}$$

with $J \in Z(K)$. If we form $G/K = \mathbf{O}(n)/\mathbf{O}(k) \times \mathbf{O}(n-k)$ and since for $X \in \mathfrak{o}(n+1)$

$$X = \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix}, \quad \mathrm{Ad}\, J \begin{pmatrix} A & B \\ -B^T & D \end{pmatrix} = \begin{pmatrix} A & -B \\ B^T & D \end{pmatrix}$$

we have

$$X + \mathrm{Ad}\, J(X) = 2 \begin{pmatrix} A & \mathbf{0} \\ \mathbf{0^T} & D \end{pmatrix} \in \mathfrak{k}$$

so J satisfies 2) in Theorem 3.7 so we have the symmetric pair

$$\big(\mathbf{SO}(n), \mathbf{S}(\mathbf{O}(\mathbf{k}) \times \mathbf{O}(\mathbf{n}-\mathbf{k}))\big) \text{ with } \mathbf{s} = \mathrm{In}\, \mathbf{J}$$

4. From a Symmetric Pair to a Symmetric Space

We will soon prove a theorem which shows that symmetric pairs induce symmetric spaces. With this it is possible to show that several familiar group quotients produce symmetric spaces.

First we need to produce some nice metrics for the intended symmetric space.

Lemma 3.10. [**10**] *Let G be a Lie group and K be a closed subgroup of G. If $Ad\, K$ is compact then there exists a G-invariant inner product on $\mathfrak{p} \cong T_{eK}G/K$, such that the action of G on G/K is an isometry. Here $\mathfrak{p}$ is the linear complement of $\mathfrak{k}$ in $\mathfrak{g}$, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

PROOF. [**10**] If we use Lemma C.2 with $H = \operatorname{Ad} K$, $V = \mathfrak{p}$ and $\varphi = id$, we get an $\operatorname{Ad} K$-invariant inner product $\{\cdot,\cdot\}$ on $\mathfrak{p}$. We transfer this to $T_{eK}G/K$ by $\mathrm{d}\pi(eK)$ by Corollary 1.5 and get an $\operatorname{Ad} K$-invariant inner product $\langle\cdot,\cdot\rangle_{eK}$ on $T_{eK}G/K$. Denote the action μ of G on G/K by

$$\mu(g')gK = g'(gK) = g'gK.$$

Define

$$(\cdot,\cdot)_{gK} = \langle \mathrm{d}\mu(g^{-1})\ \cdot, \mathrm{d}\mu(g^{-1})\ \cdot\rangle_{eK}$$

This is well defined since if $aK = bK$ then $a = bk$ so we get

$$\begin{aligned}
(\cdot,\cdot)_{aK} &= \langle \mathrm{d}\mu(a^{-1})\ \cdot, \mathrm{d}\mu(a^{-1})\ \cdot\rangle_{eK} \\
&= \langle \mathrm{d}\mu(k^{-1}b^{-1})\ \cdot, \mathrm{d}\mu(k^{-1}b^{-1})\ \cdot\rangle_{eK} \\
&= \langle \mathrm{d}\mu(k^{-1})\mathrm{d}\mu(b^{-1})\ \cdot, \mathrm{d}\mu(k^{-1})\mathrm{d}\mu(b^{-1})\ \cdot\rangle_{eK} \\
&= \langle Ad(k^{-1})\mathrm{d}\mu(b^{-1})\ \cdot, Ad(k^{-1})\mathrm{d}\mu(b^{-1})\ \cdot\rangle_{eK} \quad \text{by Lemma 3.6} \\
&= \langle \mathrm{d}\mu(b^{-1})\ \cdot, \mathrm{d}\mu(b^{-1})\ \cdot\rangle_{eK} \quad \text{since } \langle\cdot,\cdot\rangle \text{ is } Ad(K)\text{-invariant} \\
&= (\cdot,\cdot)_{bK}
\end{aligned}$$

□

Note that the submersion of Theorem 3.3 becomes a isometric submersion.

Finally we have the long awaited theorem.

Theorem 3.11. [**10**] *Let (G, K) be a symmetric pair with involutive automorphism s. Denote the action of G on G/K by $\mu(g')gK = g'gK$. Then there is a G-invariant metric g on $M = G/K$ which makes (M, g) a symmetric space with involution σ_{eK} such that*

$$\sigma_{eK} \circ \mu = \mu \circ s \quad \textit{i.e. } \sigma_{eK}(\mu(g)eK) = s(g)eK$$

For $X \in \mathfrak{p} \equiv \{x \in \mathfrak{g} : \mathrm{d}s(X) = -X\}$ we have

$$\mu(exp(X)) = \tau_{\mathrm{d}\mu(X)eK} \textit{ and } \mu(exp(X))eK = Exp_{eK}(\mathrm{d}\mu(X))$$

PROOF. For a symmetric pair $\operatorname{Ad} K$ is compact so by Lemma 3.10 we can pick a G-invariant inner product on the tangent space of G/K, as metric for G/K. Define the involution by

$$\sigma_{eK}(\mu(g)eK) = \mu(s(g))eK$$

this is well defined since

$$\sigma_{eK}(\mu(gk)eK) = \mu(s(g))\mu(s(k))eK = \mu(s(g))eK = \sigma_{eK}(\mu(g)eK).$$

It is smooth since $\sigma \circ \mu = \mu \circ s$ which is smooth.

Further

$$\sigma_{eK}^2(\mu(g)eK) \quad = \quad \sigma_{eK}(\mu(s(g))eK)$$

$$
\begin{aligned}
&= \mu(s(s(g)))eK \\
&= \mu(s^2(g))eK \\
&= \mu(e)eK = eK
\end{aligned}
$$

so $\sigma_{eK}^2 = e$ so it is involutive. Therefore σ_{eK}^{-1} is also smooth, so σ_{eK} is a diffeomorphism. To show that it is an isometry we note that by differentiating

$$(d\sigma_{eK})_{eK} \circ (d\mu) = (d\mu)_e \circ (ds)_e$$

so on $\mathfrak{p}$ we have $d\sigma_{eK} \circ d\mu = -d\mu$. But at $p_0 = eK$

$$\pi : g \to geK \quad \mu(g)eK = geK$$

so since $d\pi_{|\mathfrak{p}}$ is an isomorphism so $d\mu_{|\mathfrak{p}}$ is also that. Therefore

$$(d\sigma_{eK})_{eK} = -id.$$

So since $dpi_{|\mathfrak{p}}$ is an isomorphism by Corollary 1.5 $d\mu_{|\mathfrak{p}}$ is also that. Moreover

$$
\begin{aligned}
\mu(s(g))\sigma_{eK}(\mu(g')eK) &= \mu(s(g))\mu(s(g'))eK \\
&= \mu(s(gg'))eK \\
&= \sigma_{ek}(\mu(gg')eK) \\
&= \sigma_{eK}(\mu(g)\mu(g')eK)
\end{aligned}
$$

So

$$\mu(s(g)) \circ \sigma_{eK} = \sigma_{eK}\mu(g) \tag{3.1}$$

Let $u, v \in T_{gK}M = T_{\mu(g)eK}M$ then

$$
\begin{aligned}
& \langle d\sigma_{eK}(u), d\sigma_{eK}(v)\rangle_{\sigma_{eK}(\mu(g))eK} \\
= & \langle d\sigma_{eK}(u), d\sigma_{eK}(v)\rangle_{\mu(s(g))eK} \\
= & \langle d\mu(s(g)^{-1})d\sigma_{eK}(u), d\mu(s(g)^{-1})d\sigma_{eK}(v)\rangle_{eK} \quad \text{by } G\text{-invariance} \\
= & \langle d\sigma_{eK} \circ d\mu(g^{-1})u, d\sigma_{eK} \circ d\mu(g^{-1})v\rangle_{eK} \quad \text{by eq (3.1)} \\
= & \langle -d\mu(g^{-1})u, -d\mu(g^{-1})v\rangle_{eK} \quad \text{since } d\sigma_{ek} = -id \\
= & \langle -u, -v\rangle_{gK} \quad \text{by } G\text{-invariance} \\
= & \langle u, v\rangle_{gK}
\end{aligned}
$$

So σ_{ek} is an isometry and

$$
\begin{aligned}
(d\sigma_{eK})_{eK} &= id \\
\sigma_{eK}(eK) &= s_{eK}(\mu(e)eK) \\
&= \mu(s(e))eK \\
&= eK.
\end{aligned}
$$

So we have shown that (M, g, σ_{eK}) is a symmetric space.

The last claims follow from Theorem 3.3. □

Example 3.12. By applying Theorem 3.11 to Examples 3.8-3.9 we get the following symmetric spaces from the corresponding symmetric pairs

$$\begin{aligned} \big(\mathbf{SO}(n+1), \mathbf{SO}(n)\big) &\rightarrow \mathbf{SO}(n+1)/\mathbf{SO}(n) = S^n \\ \big(\mathbf{SO}(n), \mathbf{S}(\mathbf{O}(\mathbf{k}) \times \mathbf{O}(\mathbf{n}-\mathbf{k}))\big) &\rightarrow \mathbf{SO}(n)/\mathbf{S}(\mathbf{O}(\mathbf{k}) \times \mathbf{O}(\mathbf{n}-\mathbf{k})) \\ &\quad = G_k(\mathbb{R}^n). \end{aligned}$$

In a similar fashion one can get the symmetric spaces

$$\begin{aligned} \mathbf{Lor}(1,n)/\mathbf{SO}(n) &= \mathcal{H}^+_{1,n} \\ \mathbf{SU}(n+1)/\mathbf{SU}(1) \times \mathbf{SU}(n) &= \mathbb{C}\mathrm{P}^n \end{aligned}$$

CHAPTER 4

Curvature of a Symmetric Space

The aim of this chapter is to show how to transfer the curvature calculations from the symmetric space to the Lie group which is "behind" the symmetric space. So we need something like an inner product on the Lie algebra.

1. The Killing Form

The Killing form will provide us with a metric on the Lie algebra $\mathfrak{g}$ of the isometry group G on the symmetric space.

Definition 4.1. Let $\mathfrak{g}$ be a Lie algebra and let ϑ be an involutive automorphism on $\mathfrak{g}$ with fixed point set $\mathfrak{k}$, which is a Lie subalgebra or $\mathfrak{g}$. If Int $\mathfrak{k}$(see Appendix B) with Lie algebra ad $(\mathfrak{k})$ is compact, then $(\mathfrak{g}, \vartheta)$ is called an *orthogonal symmetric algebra*. The orthogonal symmetric algebra is called *effective* if $z(\mathfrak{g}) \cap \mathfrak{k} = \{0\}$.

Example 4.2. Let

$$\mathfrak{g} = \mathbb{R}, \vartheta = -id_{\mathbb{R}}, \mathfrak{k} = \{0\}$$

then we have an orthogonal symmetric algebra which corresponds to

$$G = \mathbb{R}, K = \{0\}, s = -id_{\mathbb{R}}$$

and also

$$G = T = \mathbb{R}/\mathbb{Z}, K = \{0, 1/2\}, s = -id_T$$

Definition 4.3. Let $\mathfrak{g}$ be a Lie algebra, then the Killing form B of $\mathfrak{g}$ over a field $\mathbb{F}$ is the bilinear form

$$B: \quad \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}, \quad (X, Y) \mapsto tr(\text{ad}\, X \circ \text{ad}\, Y).$$

The Lie group G and its Lie algebra $\mathfrak{g}$ are called *semisimple* if B is nondegenerate, i.e. if $B(X, Y) = 0$ for all $Y \in \mathfrak{g}$ then $X = 0$.

Example 4.4. To calculate[1] the trace of linear maps from a vector space to itself we calculate the trace of the corresponding matrix representations. Then we will have an expression that can be used for an arbitrary vector. It

[1] To save space we use the convention to sum over repeated indices without explicit summation expressions, i.e. $x_i y_i = \sum_{i=1}^{n} x_i y_i$.

becomes especially easy if we have an inner product $\langle \cdot, \cdot \rangle$, so that we have an orthonormal basis $\{e_i\}_{i\in I}$ where I is some finite set. Then the trace is given by

$$\text{trace}\,\varphi = \langle e_i, \varphi(e_i)\rangle$$

since φ has the matrix representation $\varphi_{ij} = \langle e_i, \varphi(e_j)\rangle$. We do an explicit calculation for $\mathfrak{so}(n)$ and display some others afterwards.

For $\mathfrak{so}(n)$ we chose the basis to be $\frac{1}{\sqrt{2}}(e^{ij} - e^{ji})$ for $1 \leq i < j \leq n$, where e^{ij} is the matrix with a 1 at position (i, j) and zeros elsewhere.

$$\text{trace}\,(\text{ad}\,X \circ \text{ad}\,Y) = \langle f_k, [X, [Y, f_k]]\rangle \quad \text{by Lemma } B.4$$

where f_k are the basis elements. This gives

$$\begin{aligned} \text{trace}\,(\text{ad}\,X \circ \text{ad}\,Y) &= \frac{1}{2}\langle e^{ij} - e^{ji}, [X, [Y, e^{ij} - e^{ji}]]\rangle \\ &= \frac{1}{2}\langle e^{ij}, [X, [Y, e^{ij}]]\rangle - \frac{1}{2}\langle e^{ij}, [X, [Y, e^{ji}]]\rangle \\ &\quad -\frac{1}{2}\langle e^{ji}, [X, [Y, e^{ij}]]\rangle + \frac{1}{2}\langle e^{ji}, [X, [Y, e^{ji}]]\rangle \end{aligned} \tag{4.1}$$

we only need to calculate the first two on the right hand side of Equation (4.1) since the last two are the first with ij switched. The inner product we use is

$$\langle A, B\rangle = \text{trace}\,(A^T B)$$

and the basis elements are orthogonal for this choice. The first expression in Equation (4.1) is

$$\frac{1}{2}\Big\{ (e^{ij})^T_{\alpha\beta} X_{\beta\gamma} Y_{\gamma\delta} e^{ij}_{\delta\alpha} \quad -(e^{ij})^T_{\alpha\beta} X_{\beta\gamma} e^{ij}_{\gamma\delta} Y_{\delta\alpha} - (e^{ij})^T_{\alpha\beta} Y_{\beta\gamma} e^{ij}_{\gamma\delta} X_{\delta\alpha} \\ +(e^{ij})^T_{\alpha\beta} e^{ij}_{\beta\gamma} Y_{\gamma\delta} X_{\delta\alpha} \Big\}$$

Now $(e^{ij})^T_{\alpha\beta} = e^{ji}_{\alpha\beta}$ so we get

$$= \frac{1}{2}\Big\{ e^{ji}_{\alpha\beta} X_{\beta\gamma} Y_{\gamma\delta} e^{ij}_{\delta\alpha} \quad -e^{ji}_{\alpha\beta} X_{\beta\gamma} e^{ij}_{\gamma\delta} Y_{\delta\alpha} - e^{ji}_{\alpha\beta} Y_{\beta\gamma} e^{ij}_{\gamma\delta} X_{\delta\alpha} \\ +e^{ji}_{\alpha\beta} e^{ij}_{\beta\gamma} Y_{\gamma\delta} X_{\delta\alpha} \Big\}$$

Also $e^{ij}_{\alpha\beta} X_{\beta\gamma} = \delta^i_\alpha X_{j\gamma}$ and $X_{\alpha\beta} e^{ij}_{\beta\gamma} = X_{\alpha i}\delta^j_\gamma$, $e^{ij}_{\alpha\beta} e^{kl}_{\beta\gamma} = \delta^i_\alpha \delta^{jk} \delta^l_\gamma$ so we get

$$\begin{aligned} &= \frac{1}{2}\Big\{ \delta^j_\alpha X_{i\gamma} Y_{\gamma i} \delta^j_\alpha - \delta^j_\alpha X_{i\gamma} \delta^i_\gamma Y_{j\alpha} - \delta^j_\alpha Y_{i\gamma} \delta^i_\gamma X_{j\alpha} + \delta^j_\alpha \delta^{ii} \delta^j_\gamma Y_{\gamma\delta} X_{\delta\alpha} \Big\} \\ &= \frac{1}{2}\Big\{ \delta^{jj} X_{i\gamma} Y_{\gamma i} - X_{ii} Y_{jj} - Y_{ii} X_{jj} + \delta^{ii} Y_{j\delta} X_{\delta j} \Big\} \end{aligned}$$

The fourth term of Equation (4.1) gives

$$= \quad \frac{1}{2}\left\{\delta^{ii}X_{j\gamma}Y_{\gamma j} - X_{jj}Y_{ii} - Y_{jj}X_{ii} + \delta^{jj}Y_{i\delta}X_{\delta i}\right\}$$

adding these gives

$$\frac{1}{2}(\delta^{jj}X_{i\gamma}Y_{\gamma i} + \delta^{ii}X_{j\gamma}Y_{\gamma j}) - X_{jj}Y_{ii} - \\ Y_{jj}X_{ii} + \frac{1}{2}(\delta^{ii}Y_{j\delta}X_{\delta j} + \delta^{jj}Y_{i\delta}X_{\delta i}) \tag{4.2}$$

Since $\mathfrak{so}(n)$ is skew symmetric the two middle terms in Equation (4.2) vanish and summing the other term over $1 \leq i < j \leq n$ gives

$$\frac{1}{2}(n\text{trace }(XY) + n\text{trace }(YX)) = n\text{trace }(XY)$$

The second term in Equation (4.1) becomes

$$\begin{aligned}
& -\frac{1}{2}\left\{(e^{ij})^T_{\alpha\beta}X_{\beta\gamma}Y_{\gamma\delta}e^{ji}_{\delta\alpha} - (e^{ij})^T_{\alpha\beta}X_{\beta\gamma}e^{ji}_{\gamma\delta}Y_{\delta\alpha} - (e^{ij})^T_{\alpha\beta}Y_{\beta\gamma}e^{ji}_{\gamma\delta}X_{\delta\alpha}\right. \\
& \left. +(e^{ij})^T_{\alpha\beta}e^{ji}_{\beta\gamma}Y_{\gamma\delta}X_{\delta\alpha}\right\} \\
= & -\frac{1}{2}\left\{e^{ji}_{\alpha\beta}X_{\beta\gamma}Y_{\gamma\delta}e^{ji}_{\delta\alpha} - e^{ji}_{\alpha\beta}X_{\beta\gamma}e^{ji}_{\gamma\delta}Y_{\delta\alpha} - e^{ji}_{\alpha\beta}Y_{\beta\gamma}e^{ji}_{\gamma\delta}X_{\delta\alpha}\right. \\
& \left. +e^{ji}_{\alpha\beta}e^{ji}_{\beta\gamma}Y_{\gamma\delta}X_{\delta\alpha}\right\} \\
= & -\frac{1}{2}\left\{\delta^j_\alpha X_{i\gamma}Y_{\gamma j}\delta^i_\alpha - \delta^j_\alpha X_{i\gamma}\delta^j_\gamma Y_{i\alpha} - \delta^j_\alpha Y_{i\gamma}\delta^j_\gamma X_{i\alpha} + \delta^j_\alpha\delta^{ij}\delta^i_\gamma Y_{\gamma\delta}X_{\delta\alpha}\right\} \\
= & -\frac{1}{2}\left\{0 - X_{ij}Y_{ij} - Y_{ij}X_{ij} + 0\right\} \quad \text{since } i \neq j \text{ we get the zeros} \\
= & -\text{trace }(XY) \quad \text{after summing over } i, j
\end{aligned}$$

the last equality follows since $X_{ij} = -X_{ji}$ for $X \in \mathfrak{so}(n)$. The third term of equation (4.1) similarly gives

$$-\text{trace }(XY)$$

So adding it all up gives the expression for the Killing form on $\mathfrak{so}(n)$ as

$$B(X, Y) = (n-2)\text{trace }(XY)$$

Some other explicit expressions of Killing forms are[**1**]

$$\begin{aligned}
B(X, Y) &= (n-2)\text{trace }(XY) && \text{on } \mathfrak{o}(n) \\
B(X, Y) &= 2n\text{trace }(XY) && \text{on } \mathfrak{su}(n) \\
B(X, Y) &= 2n\text{trace }(XY) - 2\text{trace } X\text{trace } Y && \text{on } \mathfrak{u}(u)
\end{aligned}$$

The Killing form has several nice symmetry properties.

Lemma 4.5. [**6**] *The Killing form B of $\mathfrak{g}$ is symmetric. Also B is invariant under automorphisms of $\mathfrak{g}$. In particular*

$$B((Ad\,g)X,(Ad\,g)Y) = B(X,Y) \quad \textit{for all } X,Y\in\mathfrak{g} \textit{ and } g\in G$$

also

$$B((ad\,X)Y,Z) + B(Y,(ad\,X)Z) = 0 \quad \textit{for all } X,Y,Z\in\mathfrak{g}$$

PROOF. [**6**] The symmetry follows from $trAB = trBA$.
If σ is an automorphism of $\mathfrak{g}$ then

$$\begin{aligned}(\mathrm{ad}\,\sigma X)Y &= [\sigma X, Y] \quad \text{by Lemma } B.4\\ &= [\sigma X, \sigma\sigma^{-1}Y]\\ &= \sigma[X,\sigma^{-1}Y]\\ &= \sigma\circ \mathrm{ad}\,X\circ\sigma^{-1}Y\end{aligned}$$

So

$$tr(\mathrm{ad}\,\sigma X\circ \mathrm{ad}\,\sigma Y) = tr(\sigma \mathrm{ad}\,X\sigma^{-1}\circ\sigma \mathrm{ad}\,Y\sigma^{-1}) = tr(\mathrm{ad}\,X\circ \mathrm{ad}\,Y)$$

by the cyclic property of the trace.
Now if $\sigma = \mathrm{Ad}\,\exp(tX)$ then

$$\begin{aligned}B((\mathrm{ad}\,X)Y,Z) &= \frac{\mathrm{d}}{\mathrm{d}t}B(\mathrm{Ad}\,(\exp(tX))Y,Z)_{|t=0}\\ &= \frac{\mathrm{d}}{\mathrm{d}t}B(Y,\mathrm{Ad}\,(\exp(-tX))Z)_{|t=0}\\ &= B(Y,(\mathrm{ad}\,X)Z)\end{aligned}$$

□

Theorem 4.6. [**10**] *Let $(\mathfrak{g},\vartheta)$ be an effective orthogonal symmetric algebra, then the Killing form B is negative definite on $\mathfrak{k}$.*

PROOF. [**10**] Since $(\mathfrak{g},\vartheta)$ is an orthogonal symmetric algebra Int $\mathfrak{g}$ is compact. By the proof of Lemma 3.10 there is an Ad G_0-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ so

$$\langle \mathrm{Ad}\,(\exp(tX))Y,Z\rangle = \langle Y,\mathrm{Ad}\,(\exp(-tX))Z\rangle$$

therefore by differentiating at $t=0$ we get

$$\langle(\mathrm{ad}\,X)Y,Z\rangle = -\langle Y,(\mathrm{ad}\,X)Z\rangle.$$

If we choose an orthonormal basis with respect to $\langle\cdot,\cdot\rangle$ for $\mathfrak{g}$, then the matrix representation of ad X is skew symmetric i.e. $(\mathrm{ad}\,X)_{ij} = a_{ij} = -a_{ji}$. Thus

$$B(X,X) = tr(\mathrm{ad}\,X\circ \mathrm{ad}\,X) = a_{ij}a_{ji} = \sum_{i,j} -a_{ij}^2 \le 0$$

Now $a_{ij} = \langle e_i, (\mathrm{ad}\, X)e_j\rangle$ so if $\sum_{i,j} -a_{ij}^2 = 0$ then $\mathrm{ad}\, X = 0$ so $X \in z(\mathfrak{g})$ but $\mathfrak{k} \cap z(\mathfrak{g}) = \{0\}$ so $X = 0$. Thus $B_{|\mathfrak{k}}$ is negative definite. □

The sign of the Killing form on $\mathfrak{p}$ leads to the following classifications,

Definition 4.7. An orthogonal symmetric algebra $(\mathfrak{g}, \vartheta)$ is said to be of

(1) *compact type* if B is negative definite on $\mathfrak{p}$,
(2) *noncompact type* if B is positive definite on $\mathfrak{p}$,
(3) *Euclidean type* if B is identically zero on $\mathfrak{p}$.

Example 4.8. We make show that $\mathfrak{so}(n)$ is a compact Lie algebra. Let $X^{ij} = \frac{1}{\sqrt{2}}(e^{ij} - e^{ji})$ be an arbitrary basis element in $\mathfrak{p}$ of $\mathfrak{so}(n)$ then

$$\begin{aligned} B(X^{ij}, X^{ij}) &= \frac{(n-2)}{2}\text{trace}\,(e^{ij} - e^{ji})(e^{ij} - e^{ji}) \\ &= -\frac{(n-2)}{2}\text{trace}\; 2e^{ij}e^{ji} \\ &= -(n-2). \end{aligned}$$

Hence it is negative definite.

It is also true that $\mathfrak{su}(n), \mathfrak{u}(n)$ and $\mathfrak{o}(n)$ are compact, while $\mathfrak{o}(p,q)$[2] and $\mathfrak{u}(p,q)$ are noncompact Lie algebras.

We now want to prove that the Lie algebra of a compact Lie group is compact i.e. agrees with the above classifications. To do so we need a lemma.

Lemma 4.9. [**10**] *Let $\mathfrak{g}$ be a Lie algebra, Aut $\mathfrak{g}$ be the Lie algebra automorphisms of $\mathfrak{g}$, $\partial\mathfrak{g}$ be the Lie algebra of Aut $\mathfrak{g}$ and let Int $\mathfrak{g}$ be the identity component of Aut $\mathfrak{g}$. If $\mathfrak{g}$ is semisimple then*

$$\textit{ad}\,\mathfrak{g} = \partial\mathfrak{g} \quad \textit{and therefore} \quad \textit{Int}\,\mathfrak{g} = (\textit{Aut}\,\mathfrak{g})_0$$

For the reader unfamiliar with this terminology we refer to Appendix B.

PROOF. [**10**] Let $D \in \partial\mathfrak{g}$ then $(D \circ \mathrm{ad}\, X)Y = \mathrm{ad}\, DX(Y) + \mathrm{ad}\, X \circ DY$ i.e

$$[D, \mathrm{ad}\, X] = \mathrm{ad}\, DX$$

so $[\partial\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}] \subseteq \mathrm{ad}\,\mathfrak{g}$.

The bi-linear form $C(F_1, F_2) = tr(F_1F_2)$ is nondegenerate on $\mathrm{ad}\,\mathfrak{g}$ by the assumptions of the lemma. Let $\mathfrak{a}$ be the orthogonal complement of $\mathrm{ad}\,\mathfrak{g}$ in $\partial\mathfrak{g}$ with respect to C. We will show that $\mathfrak{a} = 0$, then the claims follows. By construction $\partial\mathfrak{g} \subseteq \mathrm{ad}\,\mathfrak{g} + \mathfrak{a}$ and $(\mathrm{ad}\,\mathfrak{g}) \cap \mathfrak{a} = \{0\}$. Now

$$\begin{aligned} tr([F_1, F_2]F_3) &= tr(F_1F_2F_3 - F_2F_1F_3) \\ &= tr(F_1F_2F_3 - F_1F_3F_2) \end{aligned}$$

[2] see Example 4.21 for the definition

$$= \quad tr(F_1[F_2, F_3])$$

so for $A \in \mathfrak{a}, D \in \partial\mathfrak{g}$ and $X \in \mathfrak{g}$

$$tr([A, D]\text{ad } X) = tr(A[D, \text{ad } X]) = tr(A\text{ad } DX) = 0$$

so $[A, D] \in \mathfrak{a}$, thus $\mathfrak{a}$ is a Lie ideal.

Let $A \in \mathfrak{a}, X \in \mathfrak{g}$ then $\text{ad } AX = [A, \text{ad } X] \subseteq \mathfrak{a} \cap \text{ad } \mathfrak{g}$ so $\text{ad } AX = 0$ therefore $AX = 0$ since C is semisimple on $\text{ad } \mathfrak{g}$, so $A = 0$ and thus $\mathfrak{a} = \{0\}$. □

Theorem 4.10. [**10**] *Let B be the Killing form on the Lie algebra $\mathfrak{g}$ then then following are equivalent*

(1) *$\mathfrak{g}$ is the Lie algebra of a compact Lie group,*
(2) *$\mathfrak{g} = z(\mathfrak{g}) \oplus \mathfrak{g}'$ where B is negative definite on the ideal $\mathfrak{g}'$.*

PROOF. [**10**] Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We can assume that G is connected otherwise we consider G_0. Then $\text{Int } \mathfrak{g} = \text{Ad } G$ since G is connected and so $\text{Int } \mathfrak{g}$ is compact.

By Lemma C.2 we can pick an $\text{Int } \mathfrak{g}$-invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$. Let $\mathfrak{g}'$ be the orthogonal complement of $z(\mathfrak{g})$ with respect to $\langle\cdot,\cdot\rangle$. Then for $X \in \mathfrak{g}, Y \in \mathfrak{g}', Z \in z(\mathfrak{g})$ we have

$$\langle [X, Y], Z\rangle = \langle Y, [Z, X]\rangle = \langle Y, 0\rangle = 0$$

so $[\mathfrak{g}, \mathfrak{g}'] \subseteq \mathfrak{g}'$, and $\mathfrak{g}'$ is an ideal.

With an orthonormal basis of $\mathfrak{g}$ w.r.t. $\langle\cdot,\cdot\rangle$, the elements of $\text{Int } \mathfrak{g}$ are orthogonal matrices i.e $\exp(t \text{ ad } X)$ for $X \in \mathfrak{g}$ is an orthogonal matrix so $\text{ad } X$ is a skew symmetric i.e. $a_{ij} = -a_{ji}$ so for $X \in \mathfrak{g}'$

$$B(X, X) = tr(\text{ad } X)^2 = a_{ij}a_{ji} = \sum_{i,j} -a_{ij}^2 \leq 0$$

which is zero if and only if $\text{ad } X = 0$ is $X \in z(\mathfrak{g})$ i.e. $X = 0$. Thus $\mathfrak{g} = z(\mathfrak{g}) \oplus \mathfrak{g}'$ with $B_{|\mathfrak{g}' \times \mathfrak{g}'}$ negative definite.

Conversely if $\mathfrak{g} = z(\mathfrak{g}) \oplus \mathfrak{g}'$ with B negative definite on $\mathfrak{g}'$. Note that $\mathfrak{g}'$ is semisimple, since if $B(X, Y) = 0$ for all $Y \in \mathfrak{g}'$ then $B(X, X) = 0$ so $X = 0$.

We will show that $\mathfrak{g}'$ is the Lie algebra of a compact Lie group H, then $\mathfrak{g}' \oplus z(\mathfrak{g})$ correspond to $H \times T^{\dim z(\mathfrak{g})}$, where T^n is the n dimensional torus, which is compact.

Since B is $\text{Ad } G$ invariant, B is invariant under $\text{Int } \mathfrak{g}'$, so $\text{Int } \mathfrak{g}'$ is represented by orthogonal matrices, i.e. $\text{Int } \mathfrak{g}' \subseteq \mathbf{O}(\dim \text{Int } \mathfrak{g}')$. Also by Lemma 4.9 $\text{Int } \mathfrak{g}' = (\text{Aut } \mathfrak{g})_0$ so $\text{Int } \mathfrak{g}'$ is closed and thus compact. So $\mathfrak{g}'$ is the Lie algebra of a compact Lie group. □

It turns out that the Killing form decomposes the orhogonal symmetric algebra of a symmetric pair into a very convenient form.

Theorem 4.11. [**6**] *Let* $(\mathfrak{g}, \vartheta)$ *be an effective orthogonal symmetric algebra of a symmetric pair* (G, K)*. Then we can decompose* $\mathfrak{g}$ *as*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_m$$

where $p_i \perp p_j$, $i \neq j$ *with respect to* B*. We can also find an* $\operatorname{Ad} K$*-invariant inner product on* $\mathfrak{g}$ *given by*

$$g(\cdot,\cdot) = -B_{|\mathfrak{k}} + \frac{1}{\lambda_1} B_{|\mathfrak{p}_1} + \ldots + \frac{1}{\lambda_m} B_{|\mathfrak{p}_m}$$

We also have $[p_i, p_j] = 0$ *for* $i \neq j$*.*

PROOF. [**6**] By the proof of Lemma 3.10 we get a $\operatorname{Ad} K$ invariant inner product $(\cdot,\cdot)$ on $\mathfrak{p}$. Define

$$g(X,Y) = \begin{cases} -B(X,Y) & X, Y \in \mathfrak{k} \\ (X,Y) & X, Y \in \mathfrak{p} \\ 0 & X \in \mathfrak{k}, Y \in \mathfrak{p} \text{ or vice versa.} \end{cases}$$

Then $g(\cdot,\cdot)$ is positive definite and $\operatorname{Ad} K$ invariant.

Now for a fixed $X \in \mathfrak{p}$ consider the functional $f(Y) = B(X,Y)$ for all $Y \in \mathfrak{p}$. Then the Riesz-theorem implies

$$B(X,Y) = g(Y, T(X)) \tag{4.3}$$

Since B is symmetric T is self adjoint with respect to $g(\cdot,\cdot)$, so we can find an orthonormal basis of eigenvectors $\{X_j\}$ of T such that $T(X_j) = \lambda_j X_j$. Note that all $\lambda_j \neq 0$ since B is nondegenerate. Also since B is symmetric, the eigenspaces corresponding to different eigenvalues are orthogonal. In fact since $\{X_k\}$ are orthonormal with respect to $g(\cdot,\cdot)$ we have that $\lambda_k = B(X_k, X_k)$. So

$$\mathfrak{p} = \mathfrak{p}_1 \oplus \ldots \oplus \mathfrak{p}_m$$

The expression for $g(\cdot,\cdot)$ follows by Equation (4.3).

For the last claim let $Y_i \in \mathfrak{p}_i$, $Y_j \in \mathfrak{p}_j$ then

$$B([Y_i, Y_j], [Y_i, Y_j]) = B(Y_i, [Y_j, [Y_i, Y_j]]) = \lambda_i g(Y_i, [Y_j, [Y_i, Y_j]])$$

but also

$$\begin{aligned} B([Y_i, Y_j], [Y_i, Y_j]) &= -B([Y_j, Y_i], [Y_i, Y_j]) \\ &= -B(Y_j, [Y_i, [Y_i, Y_j]]) \\ &= -\lambda_j g(Y_j, [Y_i, [Y_i, Y_j]]) \\ &= \lambda_j g(Y_i, [Y_j, [Y_i, Y_j]]) \end{aligned}$$

where the last equality follows from Theorem 4.13 and the symmetries of the curvature tensor R^i_{jkl}. So if $i \neq j$ then $\lambda_i \neq \lambda_j$ so $B([Y_i, Y_j], [Y_i, Y_j]) = 0$ but $\mathfrak{g}$ is semisimple hence $[Y_i, Y_j] = 0$. Thus $[\mathfrak{p}_i, \mathfrak{p}_j] = 0$ for $i \neq j$. □

With the metric g in Theorem 4.11 the symmetric pair (G,K) makes the quotient G/K into a nice Riemannian manifold $(G/K, g)$.

2. The Curvature Formula

To do calculations on the symmetric space we turn our attention from G to the symmetric space $M = G/K$. By Theorem 3.3 we get the vector fields on M by parallel translation and the map $\mathrm{d}\nu_{\mathfrak{p}} p_0 \to T_pM$. The images $X^*(p_0) = \mathrm{d}\nu_X(p_0)$ are Killing fields (see Appendix D) i.e. their local flows are isometries.

We start with some properties of these Killing fields.

Lemma 4.12. [**6**] *Let G be the isometry group on the symmetric space (M,g), $K \subseteq G$ be the isotropy group at $p_0 \in M$. If $\mathfrak{g}$ is the Lie algebra of G with the standard decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, then*

$$\begin{aligned} X^*(p_0) &= 0 \qquad \textit{for all } X \in \mathfrak{k} \\ \nabla_v X^*(p_0) &= 0 \qquad \textit{for all } X \in \mathfrak{p}, v \in T_pM \end{aligned}$$

PROOF. [**6**] Let $X \in \mathfrak{k}$ then

$$X^*(p_0) = \mathrm{d}\nu_X(p_0) = \frac{\mathrm{d}}{\mathrm{d}t}(\exp(tX)p_0)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}(p_0)_{|t=0} = 0.$$

Next let $\gamma : t \to M$ be a curve such that $\dot\gamma(0) = v \in T_pM$ then

$$X^*(p_0) = \frac{\mathrm{d}}{\mathrm{d}t}\tau_{tX^*(p_0)}(p_0)_{|t=0} \quad \text{by Theorem3.3}$$

so

$$\begin{aligned} \nabla_v X^*(p_0) &= \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t}\tau_{tX^*(p_0)}(\gamma(s))_{|s=t=0} \\ &= \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\tau_{tX^*(p_0)}(\gamma(s))_{|s=t=0} \quad \text{since } \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0 \\ &= \nabla_{\frac{\partial}{\partial t}} \mathrm{d}\tau_{v|s=t=0} \\ &\qquad \text{but by Lemma 2.3 } \tau_v \text{ is a parallel} \\ &\qquad \text{transport of v along } \gamma(t)\text{, so} \\ &= 0 \end{aligned}$$

□

Since we can parallel translate all vector fields on M to the origin it is enough to be able to do the curvature calculations for vector fields at the origin.

Theorem 4.13. [**6**] *Let (M, g) be a symmetric space. With the identification*

$$\mathfrak{p} \cong T_pM \quad X \mapsto \mathrm{d}\nu_X(p)$$

in Theorem 3.3, the curvature tensor of M satisfies

$$R(X^*, Y^*)Z^* = -[[X^*, Y^*]Z^*](p)$$

for all $X^*, Y^*, Z^* \in \mathrm{d}\nu_{\mathfrak{p}}(p) = T_pM$

PROOF. [**6**] We denote the image $\mathrm{d}\nu_{\mathfrak{p}}(p)$ by $\mathfrak{p}^*$. Let $X^* \in \mathfrak{p}^*$, $Y^* \in \mathfrak{p}^*$. Then $X^*(p) = \mathrm{d}\nu_X(p)$. The geodesic $c(t) = \mathrm{Exp}_p tY^*(p)$ satisfies $Y^*(c(t)) = \dot{c}(t)$, since by theorem 3.3

$$\begin{aligned} Y^*(c(t)) &= \frac{\mathrm{d}}{\mathrm{d}s}\exp(sY(c(t)))c(t)_{|s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\tau_{sY(c(t))^*}(c(t))_{|s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}c(t+s)_{|s=0} = \dot{c}(t) \end{aligned}$$

By Lemma D.10 the Killing field X^* is a Jacobi field along c so we have the Jacobi equation

$$\nabla_{Y^*}\nabla_{Y^*}X^* + R(X^*, Y^*)Y^* = 0.$$

But $\mathfrak{p}^*$ is a subspace so for $Y^*, Z^* \in \mathfrak{p}^*$ we have $Y^* + Z^* \in \mathfrak{p}^*$. Inserting this into the above equation we obtain

$$\begin{aligned} 0 &= \nabla_{(Y^*+Z^*)}\nabla_{(Y^*+Z^*)}X^* + R(X^*, Y^*+Z^*)(Y^*+Z^*) \\ &= \nabla_{Y^*}\nabla_{Y^*}X^* + \nabla_{Y^*}\nabla_{Z^*}X^* + \nabla_{Z^*}\nabla_{Y^*}X^* + \nabla_{Z^*}\nabla_{Z^*}X^* + \\ &\quad R(X^*, Y^*)Y^* + R(X^*, Z^*)Y^* + R(X^*, Y^*)Z^* + R(X^*, Z^*)Z^* \\ &= \nabla_{Y^*}\nabla_{Z^*}X^* + \nabla_{Z^*}\nabla_{Y^*}X^* + R(X^*, Z^*)Y^* + R(X^*, Y^*)Z^* \\ &\qquad \text{(But } R(X^*, Z^*)Y^* = -R(Z^*, X^*)Y^*\text{)} \\ &= \nabla_{Y^*}\nabla_{Z^*}X^* + \nabla_{Z^*}\nabla_{Y^*}X^* - R(Z^*, X^*)Y^* + R(X^*, Y^*)Z^* \\ &\qquad \text{(By the Bianchy identity} \\ &\qquad -R(Z^*, X^*)Y^* = R(X^*, Y^*)Z^* + R(Y^*, Z^*)X^*\text{)} \\ &= \nabla_{Y^*}\nabla_{Z^*}X^* + \nabla_{Z^*}\nabla_{Y^*}X^* + 2R(X^*, Y^*)Z^* + R(Y^*, Z^*)X^* \\ &\quad \text{(Now } R(Y^*, Z^*)X^* = \nabla_{Y^*}\nabla_{Z^*}X^* - \nabla_{Z^*}\nabla_{Y^*}X^* - \nabla_{[Y^*, Z^*]}X^* \\ &\qquad \text{But for } Y^*, Z^* \in \mathfrak{p}^* \quad [Y^*, Z^*] \in \mathfrak{k}^* \text{ since} \\ &\qquad [Y^*, Z^*] = -[Y, Z]^* \text{by Appendix D so } [Y^*, Z^*](p) = 0 \\ &\qquad \text{Thus } R(Y^*, Z^*)X^* = \nabla_{Y^*}\nabla_{Z^*}X^* - \nabla_{Z^*}\nabla_{Y^*}X^*\text{)} \\ &= 2\nabla_{Y^*}\nabla_{Z^*}X^* + 2R(X^*, Y^*)Z^*. \end{aligned}$$

Hence

$$\nabla_{Y^*}\nabla_{Z^*}X^* + R(X^*, Y^*)Z^* = 0. \tag{4.4}$$

Therefore starting with the Bianchy identity

$$\begin{aligned}
R(X^*, Y^*)Z^*(p) &= -R(Y^*, Z^*)X^*(p) + R(X^*, Z^*)Y^*(p) \\
&= \nabla_{Z^*}\nabla_{X^*}Y^*(p) - \nabla_{Z^*}\nabla_{Y^*}X^*(p) \\
&\qquad \text{by eq. (4.4)} \\
&= \nabla_{Z^*}[X^*, Y^*](p) \quad \text{since } \nabla \text{ is torison free} \\
&= \nabla_{[X^*, Y^*]}Z^*(p) - [[X^*, Y^*], Z^*](p) \\
&\qquad \text{again since } \nabla \text{ is torision free} \\
&= -[[X^*, Y^*], Z^*](p) \quad \text{since } [X^*, Y^*](p) = 0
\end{aligned}$$

□

For $M = G/K$ we give M the Riemann structure induced by Theorem 3.11. There are a few details about Lie algebra operations in $\mathfrak{p}^*$ versus $\mathfrak{p}$, for our purposes it is the fact that $[X^*, Y^*] = -[X, Y]^*$, see Appendix D.

Corollary 4.14. [**6**] *Let $(G/K, g)$ be a symmetric space. Then the sectional curvature of $X^*, Y* \in T_pG/K$ is*

$$K(X^*, Y^*)(p) = \frac{-g([[X, Y], Y]^*, X^*)}{g(X^*, X^*)g(Y^*, Y^*) - g(X^*, Y^*)^2}$$

PROOF. This follows from the definition of the sectional curvature and that $[X^*, Y^*] = -[X, Y]^*$. □

Now consider the curvature tensor $R(X^*, Y^*)Z^* = -[[X^*, Y^*], Z^*]$ which can be rewritten as $R(X^*, Y^*)Z^* = -[[X, Y], Z]^*$. Since we know that $R(X^*, Y^*)Z^* \in T_pM$ we formally denote its corresponding vector in $\mathfrak{p}$ by $R(X, Y)Z$ therefore we formally have

$$R(X, Y)Z = -[[X, Y], Z] \quad \text{for } X, Y, Z \in p$$

Since $g(\cdot, \cdot)$ on TM derives from $\langle \cdot, \cdot \rangle$ on $\mathfrak{p}$ we can always write

$$\frac{-g([[X, Y], Y]^*, X^*)}{g(X^*, X^*)g(Y^*, Y^*) - g(X^*, Y^*)^2} = \frac{-\langle [[X, Y], Y], X)}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2}$$

in $\mathfrak{p}$. So in terms of this Corollary 4.14 becomes

$$K(X, Y) = \frac{-\langle [[X, Y], Y], X)}{\langle X, X\rangle\langle Y, Y\rangle - \langle X, Y\rangle^2} \quad \text{for } X, Y \in \mathfrak{p}$$

Example 4.15. By Corollary 4.14 and an Ad g invariant metric g the formula for the sectional curvature can be expressed as

$$K(X^*, Y^*) = \frac{g([X, Y]^*, [X, Y]^*)}{g(X^*, X^*)g(Y^*, Y^*) - g(X^*, Y^*)^2}$$

If we define a metric in the same way as in Lemma 3.10 we can write the formula in $\mathfrak{p}$ as

$$K(X, Y) = \frac{\langle [X, Y], [X, Y] \rangle}{\langle X, X \rangle \langle Y, Y \rangle - \langle X, Y \rangle^2}$$

where we use the same notation for the metrics. If we work with the real Grassmann manifolds we can choose a basis $e^{ij} - e^{ji}$. Before we start the calculations we produce some intermediate results.

$$[e^{ij}, e^{kl}]_{\alpha\gamma} = \delta^i_\alpha \delta^{jk} \delta^l_\gamma - \delta^k_\alpha \delta^{li} \delta^j_\gamma = \delta^{jk} e^{il} - \delta^{li} e^{kj}$$

this gives

$$\begin{aligned}
[e^{ij} - e^{ji}, e^{kj} - e^{lk}] &= [e^{ij}, e^{kl}] - [e^{ij}, e^{lk}] - [e^{ji}, e^{kl}] + [e^{ji}, e^{lk}] \\
&= \delta^{jk} e^{il} - \delta^{li} e^{kj} - \delta^{jl} e^{ik} + \delta^{ki} e^{lj} - \delta^{ik} e^{jl} + \delta^{lj} e^{ki} \\
&\quad + \delta^{il} e^{jk} - \delta^{jk} e^{il} - \delta^{kj} e^{li} \\
&= \delta^{jk}(e^{il} - e^{li}) + \delta^{il}(e^{jk} - e^{kj}) + \delta^{ik}(e^{lj} - e^{jl}) \\
&\quad + \delta^{jl}(e^{ki} - e^{ik})
\end{aligned}$$

Now if we choose the metric

$$\langle X, Y \rangle = -\frac{1}{2} \text{trace } (XY)$$

which is proportional to the Killing form on $\mathfrak{so}(n)$. Therefore

$$\begin{aligned}
\langle e^{ij} - e^{ji}, e^{kl} - e^{lk} \rangle &= -\frac{1}{2} \text{trace } (\delta^{jk} e^{il} - \delta^{jl} e^{ik} - \delta^{ik} e^{jl} + \delta^{il} e^{jk}) \\
&= -\frac{1}{2} (\delta^{jk}\delta^{il} - \delta^{jl}\delta^{ik} - \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \\
&= -\delta^{jk}\delta^{il} + \delta^{ik}\delta^{jl} \text{ but } i < j, k < l \text{ so} \\
&= \delta^{ik}\delta^{jl}
\end{aligned}$$

So it is indeed an orthogonal basis. Therefore

$$g\big([e^{ij} - e^{jk}, e^{kl} - e^{lk}], [e^{ij} - e^{jk}, e^{kl} - e^{lk}]\big) = \delta^{jk} + \delta^{il} + \delta^{ik} + \delta^{jl}$$

Note that only none or one of these can be satisfied, so

$$K(e^{ij} - e^{ji}, e^{kl} - e^{lk}) = \begin{cases} 1 & \text{if } (i = k, j \neq l), (i = l, j \neq k), \\ & , (i \neq k, j = l) \text{ or } (i \neq l, j = k) \\ 0 & \text{else} \end{cases}$$

So the sphere S^n has constant sectional curvature 1, while the real Grassmannian manifolds have sectional curvature 0 or 1.

3. The Dual Space

Definition 4.16. Let $(G/K, g, \sigma)$ be a symmetric space such that the inner product g on $\mathfrak{g}$ is both Ad G and $d\sigma$ invariant. Such that we can decompose the Lie algebra as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

The symmetric space is then called a *normal symmetric space*.

Note that when $\mathfrak{g}$ is semisimple the Killing form B makes

$$(G/K, B, \sigma)$$

a normal symmetric space.

The concept of dual spaces makes it possible to pair up symmetric spaces to that if one knows properties of one in the pair, one also knows the same properties of the dual and vice versa.

Definition 4.17. Two normal symmetric spaces $M = G/K$ and $M^\dagger = G^\dagger/K^\dagger$ are said to be *dual* provided

(1) There is a Lie algebra isomorphism $\varphi : \mathfrak{k} \to \mathfrak{k}^\dagger$ such that

$$g^\dagger(\varphi(X), \varphi(Y)) = -g(X, Y)$$

(2) A linear isometry $\hat{\varphi} : \mathfrak{p} \to \mathfrak{p}^\dagger$ such that

$$[\hat{\varphi}(X), \hat{\varphi}(Y)]^\dagger = -\varphi[X, Y],$$

where all the corresponding entities in $M^\dagger$ have been denoted with an $\dagger$.

Remark 4.18. Because φ is a Lie algebra isomorphism we have

$$[\varphi(X), \varphi(Y)]^\dagger = [X, Y]$$

and since $\hat{\varphi}$ is an isometry

$$g^\dagger(\hat{\varphi}(X), \hat{\varphi}(Y)) = g(X, Y).$$

The above isometry $\hat{\varphi}$ induce an isometry

$$T_{eK}M \to T_{eK^\dagger}M^\dagger$$

via the following commutative diagram

$$\begin{array}{ccc} \mathfrak{p} & \xrightarrow{\hat{\varphi}} & \mathfrak{p}^\dagger \\ {\scriptstyle \nu(p)}\downarrow & & \downarrow{\scriptstyle \nu(p^\dagger)} \\ T_{eK}M & \longrightarrow & T_{eK^\dagger}M^\dagger \end{array}$$

For compact spaces the following choice of metric is common

Definition 4.19. Let G/K be a homogeneous space with G *compact* and $\mathfrak{g}$ semisimple then the *standard homogeneous Riemannian metric* on G/K is chosen as the negative Killing form i.e. $g = -B$.

As promised duality provides important information about the dual spaces.

Theorem 4.20. [9] *Let* (M, g) *and* $(M^\dagger, g^\dagger)$ *be dual spaces. Then* $(M, g$ *and* $(M^\dagger, g^\dagger)$ *have opposite sectional curvature i.e.*

$$K^\dagger_{M^\dagger}(\hat{\varphi}X, \hat{\varphi}Y) = -K_M(X, Y) \textit{ for all } X, Y \in \mathfrak{p}$$

PROOF.

$$\begin{aligned} K_{M^\dagger}(\hat{\varphi}X, \hat{\varphi}Y) &= \frac{g^\dagger([\hat{\varphi}X, \hat{\varphi}Y], [\hat{\varphi}X, \hat{\varphi}Y])}{g(\hat{\varphi}X, \hat{\varphi}X)g(\hat{\varphi}Y, \hat{\varphi}Y) - g(\hat{\varphi}X, \hat{\varphi}Y)^2} \\ &= \frac{g^\dagger(-\varphi[X, Y], -\varphi[X, Y])}{g(X, X)g(Y, Y) - g(X, Y)^2} \\ &= \frac{-g([X, Y], [X, Y])}{g(X, X)g(Y, Y) - g(X, Y)^2} \\ &= -K_M(X, Y) \end{aligned}$$

□

To illustrate duality we find the the dual space of the real Grassmannian manifold.

Example 4.21. If we now generalize Example 1.7 by considering

$$Q = \begin{pmatrix} I_p & \mathbf{0^T} \\ \mathbf{0} & -I_q \end{pmatrix} \tag{4.5}$$

Then we define

$$\mathbf{O}(p, q) = \{A \in \mathbf{GL}_{p+q}(\mathbb{R}) : A^T Q A = Q\}$$

Then $\mathbf{O}(p, q)$ represents orthogonal matrices with respect to the inner product defined by Equation (4.5). We find the Lie algebra $\mathfrak{g} = \mathfrak{o}(p, q)$ of $\mathbf{O}(p, q)$ by taking the derivative of a curve at the origin in the defining expression for $\mathbf{O}(p, q)$. Then as in Example 1.7

$$\mathfrak{o}(p, q) = \left\{ \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} : A \in \mathfrak{o}(p), \;\; B \in M_{(p,q)}, \;\; C \in \mathfrak{o}(q) \right\}$$

If we let $\mathbf{O}(p, q)$ act on the real Grassmann manifold $G_p(\mathbb{R}^{p+q})$ as in Example 1.8, then the isotropy group consists of matrices of the form

$$\begin{pmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{pmatrix} \quad \text{where } B \in \mathbf{O}(p), C \in \mathbf{O}(q)$$

so we have that

$$\mathbf{O}(p,q)/(\mathbf{O}(p)\times\mathbf{O}(q))$$

is a manifold. Since the Lie algebra of $\mathbf{O}(k)$ consists of the skew symmetric matrices we get

$$\mathfrak{k}=\left\{\begin{pmatrix} A & \mathbf{0}\\ \mathbf{0} & D\end{pmatrix} : A\in\mathfrak{o}(p),\ \ D\in\mathfrak{o}(q)\right\}$$

and

$$\mathfrak{m}=\left\{\begin{pmatrix} \mathbf{0} & B\\ B^T & \mathbf{0}\end{pmatrix} : B\in M_{(p,q)}\right\}$$

Example 4.22. Let $G=\mathbf{O}(p,q)$ and $K=\mathbf{O}(p)\times\mathbf{O}(q)$ be as in Theorem 3.7 with

$$J=\begin{pmatrix} I_p & \mathbf{0}\\ \mathbf{0^T} & -I_q\end{pmatrix}$$

with $J\in Z(K)$. If we form $G/K=\mathbf{O}(p,q)/\mathbf{O}(p)\times\mathbf{O}(q)$ and observe that for $X\in\mathfrak{o}(p,q)$

$$X=\begin{pmatrix} A & B\\ B^T & D\end{pmatrix},\quad \operatorname{Ad}J\begin{pmatrix} A & B\\ B^T & D\end{pmatrix}=\begin{pmatrix} A & -B\\ -B^T & D\end{pmatrix}$$

we have

$$X+\operatorname{Ad}J(X)=2\begin{pmatrix} A & \mathbf{0}\\ \mathbf{0^T} & D\end{pmatrix}\in\mathfrak{k}$$

so J satisifies 2) in Theorem 3.7 so we have the symmetric pair

$$\big(\mathbf{SO_0(p,q)},\mathbf{S(O(p)\times O(q))}\big)\text{ with }\mathbf{s}=\operatorname{In}\mathbf{J}$$

Example 4.23. Let (G,K) be the symmetric pair

$$(G,K)=(\mathbf{SO}(p+q),(\mathbf{SO}(p)\times\mathbf{SO}(q))$$

with involutive automorphism s given by Example 3.9

$$\sigma(A)=\operatorname{In}S(A)=SAS^{-1}\text{ where }S=\begin{pmatrix} I_p & 0\\ 0 & -I_p\end{pmatrix}$$

on the manifold G/K and the Ad G-invariant metric

$$B(X,Y)=-\frac{1}{2}\text{trace }(XY)$$

at the origin. Then by extending the inner product B to all of M as in Theorem 3.11 σ becomes an isometry, so

$$M=\mathbf{SO}(p+q)/(\mathbf{SO}(p)\times\mathbf{SO}(q))$$

becomes a normal symmetric space. Note that B is positive definite on $\mathfrak{p}$.

On the symmetric pair $(G^\dagger, K^\dagger) = (\mathbf{SO_0(p,q)}, \mathbf{SO(p)} \times \mathbf{SO(q)})$ we pick the same involutive automorphism s as in Example 4.22. And on the manifold $G^\dagger/K^\dagger$ we take the metric

$$B^\dagger(X,Y) = \frac{1}{2}\text{trace}\,(XY)$$

at the origin, which again is Ad $G^\dagger$ invariant and σ is an isometry after extending the metric. So we have the normal symmetric space

$$M^\dagger = \mathbf{SO_0(p,q)}/(\mathbf{SO(p)} \times \mathbf{SO(q)})$$

Again $B^\dagger$ is positive definite on $p^\dagger$.

Let

$$\varphi : \mathfrak{k} \to \mathfrak{k}^\dagger = id_{\mathfrak{k}}$$

which certainly is a Lie algebra isomorphism such that

$$B(X,Y) = -\frac{1}{2}\text{trace}\,XY = -(-\frac{1}{2}\text{trace}\,XY) = -B^\dagger(X,Y)$$

and

$$\hat{\varphi} : \mathfrak{p} \to \mathfrak{p}^\dagger, \quad \begin{pmatrix} \mathbf{0} & x \\ -x^T & \mathbf{0} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{0} & x \\ x^T & \mathbf{0} \end{pmatrix}$$

which is an isometry such that

$$\begin{aligned}
[\hat{\varphi}(X), \hat{\varphi}(Y)]^\dagger &= \begin{pmatrix} \mathbf{0} & x \\ x^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & y \\ y^T & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & y \\ y^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & x \\ x^T & \mathbf{0} \end{pmatrix} \\
&= \begin{pmatrix} xy^T & \mathbf{0} \\ \mathbf{0} & x^Ty \end{pmatrix} - \begin{pmatrix} yx^T & \mathbf{0} \\ \mathbf{0} & y^Tx \end{pmatrix} \\
&= -\begin{pmatrix} \mathbf{0} & x \\ -x^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & y \\ -y^T & \mathbf{0} \end{pmatrix} + \\
&\quad \begin{pmatrix} \mathbf{0} & x \\ -x^T & \mathbf{0} \end{pmatrix}\begin{pmatrix} \mathbf{0} & y \\ -y^T & \mathbf{0} \end{pmatrix} \\
&= -[X,Y] \\
&= -\varphi([X,Y])
\end{aligned}$$

Therefore $M, M^\dagger$ are dual.

Example 4.24. Further examples of dual pairs of symmetric spaces can be found in the following table,

Noncompact	compact	*Dimension*
$\mathbf{SL(n,\mathbb{R})/SO(n)}$	$\mathbf{SU(n)/SO(n)}$	$\frac{1}{2}(n-1)(n+2)$
$\mathbf{SU(p,q)/S(U(p)\times U(q))}$	$\mathbf{SU(p+q)/S(U(p)\times U(q))}$	$2pq$
$\mathbf{SO_0(p,q)/SO(p)\times SO(q)}$	$\mathbf{SO(p+q)/SO(p)\times SO(q)}$	pq
$\mathbf{Sp(n,\mathbb{R})/U(n)}$	$\mathbf{Sp(n)/U(n)}$	$n(n+1)$
$\mathbf{Sp(p,q)/Sp(p)\times Sp(q)}$	$\mathbf{Sp(p+q)/Sp(p)+Sp(q)}$	$4pq$

see also [**5**].

The Hopf-Rinow Theorem

Definition A.1. A Riemannian manifold (M, g) is said to be *geodesically complete* if for all $p \in M$ the exponential map Exp_p is defined on all of T_pM, i.e. all geodesics $\gamma \in M$ can be defined for all $t \in \mathbb{R}$.

Definition A.2. A *normal ball* $B_\delta(p)$ at $p \in M$ on a Riemanninan manifold (M, g) is the image of an open ball $\tilde{B}_\delta(p)$ in the tangent space T_pM of the exponential map Exp_p.

Definition A.3. Let $p, q \in M$ then define the *distance* $d(p, q)$ as the infimum of the lengths of all curves on M from p to q.

Remark A.4. It can be shown that the distance function is continuous and that geodesics locally minimize the distance between two points, also that if a curve is a minimum then it must be a geodesic.

Theorem A.5. [**3**] ***Hopf-Rinow***
Let (M, g) be a Riamannian manifold if (M, g) is geodesically complete then any two points $p, q \in M$ can be connected by a geodesic.

PROOF. [**3**] If M is geodesically complete then the exponential map Exp_p is defined on all of T_pM.

Let $p, q \in M$ and $r = d(p, q)$. Let $B_\delta(p)$ be a normal ball in M and let $S_\delta(p)$ be the boundary of $B_\delta(p)$. Let x_0 be a point where the continuous function $d(q, x), \quad x \in S_\delta(p)$ takes on its infimum(It does so since $S_\delta(p)$ is compact). Then $x_0 = \mathrm{Exp}_p \delta v$ where $v \in T_pM$ and $|v| = 1$. Let γ be the geodesic given by $\gamma(s) = \mathrm{Exp}_p sv$. Now let

$$A = \{s \in [0, r] : d(\gamma(s), q) = r - s\}.$$

We will show that $r \in A$ and $\gamma(r) = q$.

Now A is not empty since $0 \in A$ and A is closed by continuity of the distance. We are going to show that if $s_0 \in A$ and $s_0 < r$ then there exists a $\delta > 0$ such that $s_0 + \delta \in A$. This will imply that $\sup A = r$ and $r \in A$ since A is closed. Thus $d(\gamma(r), q) = r - r = 0$ so $\gamma(r) = q$.

Let $B_{\delta'}(\gamma(s_0))$ be a normal ball at $\gamma(s_0)$ and S' be the boundary of $B_{\delta'}(\gamma(s_0))$. Let x_0' be the point where $d(x, q), \quad x \in S'$ takes on its minimum. So

$$d(\gamma(s_0), q) = \delta' + \min\{d(x, q) : x \in S'\} = \delta' + d(x_0', q)$$

But $d(\gamma(s_0), q) = r - s_0$ so $r - s_0 = \delta' + d(x_0', q)$. Now if $x_0' = \gamma(s_0 + \delta')$ we get

$$r - s_0 = \delta' + d(\gamma(s_0 + \delta', q) \quad \text{so } s_0 + \delta' \in A.$$

We only have to show that $x_0' = \gamma(s_0 + \delta')$. But by the triangle inequality

$$d(p, x_0') \geq d(p, q) - d(q, x_0') = r - (r - s_0 - \delta') = s_0 + \delta'$$

which is no surprise. But more importantly the broken curve joining p and x_0' by going along γ from p to $\gamma(s_0)$ and along the exponential ray from $\gamma(s_0)$ to x_0' in $B_{\delta'}(\gamma(s_0))$, has length $s_0 + \delta'$ so $d(p, x_0') = s_0 + \delta'$. By minimality of the distance that broken curve must be a geodesic. But at it is starting out as γ it must remain γ, so $x_0' = \gamma(s_0 + \delta')$. □

APPENDIX B

The Adjoint Representation

Definition B.1. Let G be a Lie group. For each $h \in G$ we define the inner automorphism of G by

$$\text{In}\, h : G \to G \quad g \mapsto hgh^{-1}$$

Denote by $\text{GL}(\mathfrak{g})$ the group of vector space automorphisms of $\mathfrak{g}$.

Definition B.2. Let G be a Lie group, then the *adjoint representation of G* is given by

$$\text{Ad} : G \to \text{GL}(\mathfrak{g}) \quad h \mapsto \text{d}_e In(h)$$

The map $\text{In}\, g$ is a Lie group automorphism so for fixed $g \in G$ $\text{Ad}\, g$ is a Lie algebra automorphism, thus it satisfies

$$\text{Ad}\, g([X, Y]) = [\text{Ad}\, g(X), \text{Ad}\, g(Y)] \quad \text{for all } X, Y \in \mathfrak{g}$$

and

$$\exp(\text{Ad}\, g(X)) = \text{In}\, g(\exp(X))$$

But we can also view Ad as a map $\text{Ad} : G \to \text{GL}(\mathfrak{g})$. Then

$$\text{Ad}\, g_1 \circ \text{Ad}\, g_2 = \text{Ad}\, g_1 g_2$$

so it is a Lie group homomorphism. Therefore it induces a Lie algebra automorphism.

Definition B.3. Let $\mathfrak{g}$ be a Lie algebra, then *the adjoint representation of* $\mathfrak{g}$ is defined as

$$\text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}), \quad X \mapsto (\text{d}_e\text{Ad})(X)$$

where $\text{End}(\mathfrak{g})$ is the space of linear maps $\mathfrak{g} \to \mathfrak{g}$

So we have

$$\text{ad}\, [X, Y] = [\text{ad}\, X, \text{ad}\, Y]$$

Lemma B.4. [**10**] *Let* $\mathfrak{g}$ *be the Lie algebra of the Lie group G then*

$$(adX)Y = [X, Y]$$

for $X, Y \in \mathfrak{g}$.

PROOF. For $g \in G, Y \in \mathfrak{g}$ we have

$$\operatorname{Ad} gY_e = \mathrm{d}(\operatorname{In} g)(Y_e) = \mathrm{d}(R_{g^{-1}}L_g)(Y_e) = \mathrm{d}R_{g^{-1}}Y_g$$

Let $\exp(tX)$ be an integral curve of X then

$$\begin{aligned}
\operatorname{ad} X(Y) &= \lim_{t\to 0}\frac{1}{t}(\operatorname{Ad}\exp(tX)(Y_e) - \operatorname{Ad} id_G(Y_e)) \\
&= \lim_{t\to 0}\frac{1}{t}(\mathrm{d}R_{\exp(-tX)}Y_{\exp tX} - Y_e) \\
&= \lim_{t\to 0}\frac{1}{t}(\mathrm{d}R_{\exp(-tX)}Y_{R_{\exp(tX)}e} - Y_e) \\
&= L_X Y = [X, Y]
\end{aligned}$$

since $R_{\exp tX}$ is the flow of X in G by Lemma D.15. □

Proposition B.5. [**6**] *Let $\mathfrak{g}$ be a Lie algebra, then for all $X \in \mathfrak{g}$*

$$exp(ad\, X) = Ad\, exp(X)$$

PROOF. [**6**]

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(t\operatorname{ad} X)_{|t=0} &= \operatorname{ad} X \\
&= (\mathrm{d}\operatorname{Ad})X \\
&= = \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Ad}\exp(tX)_{|t=0}.
\end{aligned}$$

So the claim follows by the uniqueness of ordinary differential equations with initial data. □

Let $\operatorname{Aut}\mathfrak{g}$ be the set of Lie algebra automorphisms on $\mathfrak{g}$. Then $\operatorname{Aut}\mathfrak{g}$ is a closed subgroup of the Lie group $\mathrm{GL}(\mathfrak{g})$. Let $\operatorname{Int}\mathfrak{g}$ be the identity component of $\operatorname{Aut}\mathfrak{g}$, with Lie algebra $\operatorname{ad}\mathfrak{g}$. By exponentiating $\operatorname{ad}\mathfrak{g}$ we get the entire $\operatorname{Int}\mathfrak{g}$ which is contained in $\operatorname{Ad}\mathfrak{g} \subseteq \operatorname{Aut}\mathfrak{g}$.

Definition B.6. Let $\partial\mathfrak{g}$ be the Lie algebra of $\operatorname{Aut}\mathfrak{g}$. Then

$$\operatorname{Aut}\mathfrak{g} \subseteq \mathrm{GL}(\mathfrak{g}) \text{ and } \partial\mathfrak{g} \subseteq \operatorname{End}(\mathfrak{g}).$$

Definition B.7. And endomorphism $D \in \operatorname{End}(\mathfrak{g})$ is called a derivation if

$$D[X, Y] = [DX, Y] + [X, DY]$$

Theorem B.8. [**10**] *Let $\mathfrak{g}$ be a Lie algebra and $\partial\mathfrak{g}$ be the Lie algebra of Aut g then $\partial\mathfrak{g}$ is the set of derivations on $\mathfrak{g}$.*

PROOF. [**10**] Let $D \in \partial\mathfrak{g}$ then for $t \in \mathbb{R}$, $\exp(tD) \in \operatorname{Aut}\mathfrak{g}$ so

$$\exp(tD)[X, Y] = [\exp(tD)X, \exp(tD)Y].$$

Therefore

$$\begin{aligned} D[X,Y] &= \frac{\mathrm{d}}{\mathrm{d}t}\exp(tD)[X,Y]_{|t=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}[\exp(tD)X, \exp(tD)Y]_{|t=0} \\ &= [DX,Y]+[X,DY], \end{aligned}$$

so D is a derivation.

Conversely let D be a derivation. Then

$$D[X,Y] = [DX,Y]+[X,DY]$$

so inductively

$$D^k[X,Y] = \sum_{i=0}^{k} \binom{k}{i} [D^iX, D^{k-i}Y].$$

So we get

$$\begin{aligned} \exp(tD)[X,Y] &= \sum_{l=0}^{\infty} \frac{t^l D^l}{l!}[X,Y] \\ &= \sum_{l=0}^{\infty} \frac{t^l}{l!} \sum_{i=0}^{l} \frac{l!}{(l-i)!i!}[D^iX, D^{l-i}Y] \\ &= \sum_{l=0}^{\infty} \sum_{i=0}^{l} \left[\frac{t^iD^iX}{i!}, \frac{t^{l-i}D^{l-i}Y}{(l-i)!}\right] \\ &\qquad \text{if we switch summation order we get} \\ &= \sum_{i=0}^{\infty} \sum_{l=i}^{\infty} \left[\frac{t^iD^iX}{i!}, \frac{t^{l-i}D^{l-i}Y}{(l-i)!}\right] \quad \text{let } k = l-i \\ &= \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \left[\frac{t^iD^iX}{i!}, \frac{t^kD^kY}{k!}\right] \\ &= [\exp(tD)X, \exp(tD)Y]. \end{aligned}$$

The above manipulations of the summations are legitimate since $\exp(tD)$ converges absolutely so can replace upper limit ∞ with N and do manipulations. Thus $\exp(tD) \in \operatorname{Aut}\mathfrak{g}$ so $D \in \partial\mathfrak{g}$. □

Theorem B.9. [**10**] *Let G be a Lie group and let $Z(G)$ be the center of G. If G is connected then $Z(G) = ker Ad\, \mathfrak{g}$.*

PROOF. [**10**] Obviously $Z(G) \subseteq ker\mathrm{Ad}\,\mathfrak{g}$. Now suppose $\mathrm{Ad}\, g = id$ then

$$\exp(X) = g\exp(X)g^{-1} \quad \text{by exponentiating,}$$

so $g \in Z(G_0) = Z(G)$ since G is connected. □

We define $z(\mathfrak{g}) = \{X \in \mathfrak{g} : \operatorname{ad} X = 0\}$

Theorem B.10. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. If G is connected then $z(\mathfrak{g})$ is the Lie algebra of $Z(G)$.*

PROOF. Let $X \in \mathfrak{g}$ then if X is belongs to the Lie algebra of $Z(G)$ then $\exp(tX) \in Z(G)$ so

$$\operatorname{Ad} \exp(tX) = id_{\mathrm{GL}(\mathfrak{g})}.$$

This is equivalent to that

$$\exp(\operatorname{ad} tX) = id_{\mathrm{GL}(\mathfrak{g})}$$

which happens if and only if $\operatorname{ad} X = 0$ so $X \in z(\mathfrak{g})$. □

APPENDIX C

Inner Products From the Haar Measure

Fact C.1. *Let G be a compact topological group and let $C(G)$ be the set of continuous real valued functions on G. Then there is a unique map*

$$I : C(G) \to \mathbb{R}, \quad I(f) \mapsto \int_G f(g)\mathrm{d}g$$

such that

(1) $I(e) = 1$,
(2) $I(f) \geq 0$ *for* $f \geq 0$,
(3) $I(\lambda f + \mu g) = \lambda I(f) + \mu I(g)$ *for* $\lambda, \mu \in \mathbb{R}$,
(4) I *is* $G -$ *invariant.*

The function I is called the Haar-integral.

Lemma C.2. [**1**] *Let H be a Lie group, V be a vector space and $\varphi : H \to Aut(V)$ be a representation of the group H on V. If H is compact there exists an H invariant inner product on V.*

PROOF. Since H is compact, we can define the Haar-integral

$$I(f) = \int_H f(h)\mathrm{d}h$$

The Haar-integral is H-invariant i.e.

$$I(f) = I(f \circ R_g) = I(L_g \circ f).$$

So if $\langle \cdot, \cdot \rangle$ is an inner product on V then we define the H invariant inner product on V

$$(u, v) = \int_H \langle \varphi(h)u, \varphi(h)v \rangle \mathrm{d}h.$$

□

Lie Derivatives and Killing Fields

We state the following result from the theory of ordinary differential equations

Lemma D.1. [**6**] *Let X be a vector field on the manifold M. Then for every point $p \in M$ there exists an open neighborhood U of p and an open open interval I of $\mathbb{R}$ such that $0 \in I$, with the property that for all $q \in U$ there is a curve $c_q : I \to M$ satisfying*

$$\dot{c}_q(t) = X(c_q(t)), \quad c_q(0) = q.$$

Then map $(t, q) \to c_q(t)$ from $I \times U$ to M is smooth.

Definition D.2. Let X, c be as in Lemma D.1, then the map $(t, q) \mapsto c_q(t)$ is called the *local flow* of the vector field X. The curve c_q is called the *integral curve* of X through q.

For a fixed t and varying q we write $\varphi_t(q) = c_q(t)$. Note that

$$\varphi_t \circ \varphi_s(q) = \varphi_{t+s}(q) \quad \text{if } s, t, s+t \in I_q \tag{D.1}$$

since

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_{s+t}(q)_{|t=0} = X(\varphi_s(q)) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(\varphi_s(q))_{|t=0}$$

and they have the same initial data for $t = 0$ at q.

It can be shown that if φ_t is defined on an open subset U of M, it maps U diffeomorphically onto its image.

Definition D.3. A family $(\varphi_t)_{t\in I}$ of diffeomorphisms from M to M satisfying equation (D.1) is called a local 1-parameter group of diffeomorphisms.

Definition D.4. Let $\psi : M \to N$ be a diffeomorphism between differentiable manifolds and let X be a vector field on M. Then we define the following vector field on N

$$(\psi_* X)(p) = \mathrm{d}\psi(X(\psi^{-1}(p))) \text{ for } p \in N$$

The following lemma follows from the definition of the differential map.

Lemma D.5. [**6**] *Let N be a differentiable manifold. For any differentiable function $f : N \to \mathbb{R}$ we have*

$$(\psi_* X)(f)(p) = X(f \circ \psi)(\psi^{-1}(p)).$$

Now consider the special case when $N = M$ so $\psi_{-t} = \psi_t^{-1}$.

Let f be a real valued function defined in the codomain of ψ_t, then the pull-back of f via ψ_t is defined by

$$\psi_t^* f \equiv f \circ \psi_t$$

For a vector field $X = a_k \frac{\partial}{\partial x^k}$ on the codomain of ψ_t

$$\psi_t^* X(p) \equiv (\psi_{-t})_* X(p) = a_k \frac{\partial \psi^i_{-t}}{\partial x^k} \frac{\partial}{\partial y^i} \text{ evaluated at } \psi_t(p)$$

where x,y are local coordinates in the codomain and domain and

$$\psi^i_{-t} = (y \circ \psi_{-t})^i.$$

For a 1-form $\omega = \omega_i \mathrm{d}y^i$ in the codomain of ψ_t we define the pull-back by

$$\psi_t^*(\omega)(p) = \omega_i(\psi_t(p)) \frac{\partial \psi_t^i}{\partial x^k} \mathrm{d}x^k$$

The action of ψ_t^* on higher order tensors is obtained by its action on the atomic tensors.

Definition D.6. Let X be a vector field on the differentiable manifold M with local 1-parameter group ψ_t of local diffeomorphisms and S a tensor field on M. Then the Lie-derivative of S in the direction of X is defined by

$$L_X S = \frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* S)_{|t=0}$$

Definition D.7. Let X be a vector field and S be a tensor of type $(0, r)$ then the *inner product* $\iota_X S$ is the contraction

$$\begin{aligned}(\iota_X S)(X_1, X_2, \ldots, X_{r-1})(p) &\equiv C_{1,1}(X \otimes S)(X_1, X_2, \ldots, X_{r-1})(p) \\ &= S(X, X_1, \ldots, X_{r-1})\end{aligned}$$

where $C_{1,1}$ is the contraction operator.

Theorem D.8. [6] *Let M be a differenitiable manifold and X a vector field on M.*

(1) *If $f : M \to \mathbb{R}$ be a differentiable function, then*

$$L_X(f) = \mathrm{d}f(X) = X(f)$$

(2) *If Y be a vector field on M, then*

$$L_X Y = [X, Y]$$

(3) *Let $\omega = \omega_j \mathrm{d}x^j$ be a 1-form on M. If $X = X^i \frac{\partial}{\partial x^i}$, then*

$$L_X \omega = \mathrm{d}(\iota_X \omega) + \iota_x(\mathrm{d}\omega).$$

For higher order tensors one simply uses the product rule for differentiations on the atomic tensors.

PROOF. [**6**] 1)

$$L_X(f) = \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^* f_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t} f \circ \psi_{t|t=0} = \frac{\partial f}{\partial x^i} X^i = X(f)$$

2) Let $Y = Y^i \frac{\partial}{\partial x^i}$

$$\begin{aligned}
L_X Y &= \frac{\mathrm{d}}{\mathrm{d}t}\psi_t^*\Big(Y^i \frac{\partial}{\partial x^i}\Big)_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\psi_{-t})_*\Big(Y^i \frac{\partial}{\partial x^i}\Big)_{|t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\Big(Y^i(\psi_t)\frac{\partial \psi^j_{-t}}{\partial x^i}\frac{\partial}{\partial x^j}\Big)_{|t=0} \\
&= \frac{\partial Y^i}{\partial x^k} X^k \delta^j_i \frac{\partial}{\partial x^j} + Y^i\Big(-\frac{\partial X^j}{\partial x^i}\Big)\frac{\partial}{\partial x^j} \quad \text{since} \\
&\qquad \Big\{\frac{\partial \psi^j_{-t}}{\partial x^i}{}_{|t=0} = \frac{\partial (x^j \circ \psi_{-t})}{\partial x^i}{}_{|t=0} = \frac{\partial x^j(\psi_0)}{\partial x^i} = \delta^j_i\Big\} \quad \text{and} \\
&\qquad \Big\{\frac{\mathrm{d}}{\mathrm{d}t}\psi_{-t|t=0} = -X\Big\} \\
&= \Big(X^k \frac{\partial Y^j}{\partial x^k} - Y^k \frac{\partial X^j}{\partial x^k}\Big)\frac{\partial}{\partial x^j} \\
&= [X, Y]
\end{aligned}$$

3)

$$\begin{aligned}
L_X \omega &= \frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^*\omega)_{|t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\omega_j(\psi_t)\frac{\partial \psi^j_t}{\partial x^k}\mathrm{d}x^k\Big)_{|t=0} \\
&= \frac{\partial \omega^j}{\partial x^i} X^i \delta^j_k \mathrm{d}x^k + \omega_j \frac{\partial X^j}{\partial x^k}\mathrm{d}x^k \quad \text{as in the previous} \\
&= \Big(\frac{\partial \omega_j}{\partial x^i} X^i + \omega_i \frac{\partial X^i}{\partial x^j}\Big)\mathrm{d}x^j.
\end{aligned}$$

Now if we expand

$$\begin{aligned}
\mathrm{d}(\iota_X\omega) + \iota_X(\mathrm{d}\omega) &= \mathrm{d}(\omega_j X^j) + \iota_X\Big(\frac{\partial \omega_j}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}x^j\Big) \\
&= \frac{\partial \omega_j}{\partial x^i} X^j \mathrm{d}x^i + \frac{\partial X^i}{\partial x^j}\omega_i \mathrm{d}x^j + \frac{\partial \omega_j}{\partial x^i} X^i \mathrm{d}x^j \\
&\quad - \frac{\partial \omega_j}{\partial x^i} X^j \mathrm{d}x^i \\
&= \frac{\partial X^i}{\partial x^j}\omega_i \mathrm{d}x^j + \frac{\partial \omega_j}{\partial x^i} X^i \mathrm{d}x^j
\end{aligned}$$

So indeed

$$L_X\omega = \mathrm{d}(\iota_X\omega) + \iota_X(\mathrm{d}\omega).$$

□

Definition D.9. Let (M, g) be a Riemannian manifold with a metric $g = g_{ij}\mathrm{d}x^i \otimes \mathrm{d}x^j$ then a vector field X on M is called a *Killing field* or *an infinitesimal isometry* if

$$L_X(g) = 0.$$

Lemma D.10. [**6**] *A vector field X on a Riemannian manifold (M, g) is a Killing field if and only if the local 1-parameter group generated by X consists of local isometries.*

PROOF. [**6**] By definition

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_t^* g)_{|t=0} = 0$$

holds at every point of M. Hence $\psi_t^* g = g$ for all $t \in I$, so the diffeomorphisms ψ_t are isometries.

Conversely if ψ_t are isometries then

$$g((\psi_t)_*(X), (\psi_t)_*(Y)) = g(X, Y)$$

so

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^* g = 0.$$

□

Lemma D.11. [**8**] *Let M be a differentiable manifold. Two derivations D_1, D_2 agree if they agree on vector fields and functions on M.*

PROOF. [**8**] Since the product rule applies for derivations on tensors, we only have to show the claim for one forms, since a general tensor can be written as a tensor product of vector fields, functions and one forms.

Let $\omega = \omega_i dx^i$ be an arbitrary one form on M and let $V = V^i \frac{\partial}{\partial x^i}$ be an arbitrary vector field on M. $\omega(V)$ is a function on M so if D is a derivation on M then

$$D(\omega(V)) = D(C_1^1 \omega \otimes V) = C_1^1(D\omega \otimes V + \omega \otimes DV) = (D\omega)(V) + \omega(DV)$$

where C_1^1 is the contraction operating on ω and V. This defines the derivation on one forms if one knows how the derivation operates on vector fields and functions. □

Corollary D.12. [**8**] *Let M be a differentiable manifold. Then the Lie derivative satisfies*

$$L_{[U,V]} = [L_U, L_V]$$

for arbitrary vector fields U, V on M.

PROOF. [**8**] Since both $[L_U, L_V]$ and $L_{[U,V]}$ are derivations, Lemma D.11 implies that it is enough that they agree on functions and vector fields on M.

Let f be a function on M then

$$\begin{aligned} [L_U, L_V]f &= L_U L_V f - L_V L_U f \\ &= U \circ V(f) - V \circ U(f) \\ &= [U, V](f) \\ &= L_{[U,V]} f. \end{aligned}$$

Similarly if W is a vector field on M then

$$\begin{aligned} [L_U, L_V] W &= L_U L_V W - L_V L_U W \\ &= [U, [V, W]] - [V, [U, W]] \\ &= [[U, V], W] \\ &= L_{[U,V]} W, \end{aligned}$$

where we have used the Jacobi identity and part 2 of Theorem D.8. □

Lemma D.13. [**6**] *The Killing fields of a Riemannian manifold (M, g) is a Lie-algebra.*

PROOF. [**8**],[**6**] Since vector fields on a differentiable manifold constitute a Lie-algebra, we only have to show that if X, Y are Killing fields then $L_{[X,Y]}g = 0$. But by Corollary D.12 the equation $L_{[X,Y]} = [L_X, L_Y]$ and the fact that X, Y are Killing fields $L_X g = 0, L_Y g = 0$ so $L_{[X,Y]}g = 0$. □

Proposition D.14. [**6**] *Every Killing field X on a Riemannian manifold (M, g) is a Jacobi field along any geodesic $\gamma \in M$.*

PROOF. [**6**] By Lemma D.10 each Killing field X generates a local 1-parameter group of isometries. Isometries map geodesics to geodesics, so X generates a variation of geodesics $c(t, s) = \psi_t(\gamma(s))$, where ψ is the local 1-parameter group of X. Since every variation of geodesics generates a Jacobi field by $X = \frac{\partial}{\partial t} c(t, s)_{|t=0}$, the claim follows. □

Lemma D.15. [**9**] *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let ψ be the one-parameter subgroup of $X \in \mathfrak{g}$ then the flow of X is R_{ψ_t}*

PROOF. [**9**] Here L_g is the left translation, i.e it has nothing to do with the Lie derivative. If $g \in G$ then $L_g \circ \psi$ is the integral curve of $\mathrm{d}L_g X = X$ starting at g, so $L_g \psi_t = g\psi_t = R_{\psi_t} g$. Thus R_{ψ_t} is the flow of X. □

Proposition D.16. [**9**] *Let G be an isometry group that acts on a Riemannian manifold (M, g) by $\rho_p(g) = gp$. Denote the Lie algebra of G by $\mathfrak{g}$. Let $X, Y \in \mathfrak{g}$ and define the Killing field*

$$X^* = \mathrm{d}\rho_p(X) = \frac{\mathrm{d}}{\mathrm{d}t} exp(tX)p$$

and similarly for Y^* then

$$[X^*, Y^*] = -[X, Y]^*$$

PROOF. [**9**] Again L_g is the left translation. Let ψ_t be the local flow of X^*. Then for $Y \in G$ we have

$$\psi_{-t} \circ \exp(Y) \circ \psi_t(p) = (\rho_p R_{\psi_t} L_{\psi_{-t}})\exp(Y)$$

so by the left-invariance of Y we have

$$\mathrm{d}\psi_{-t}(Y^*_{\psi_t(p)}) = (\mathrm{d}\rho_p \mathrm{d}R_{\psi_t} \mathrm{d}L_{\psi_{-t}})Y_e = \mathrm{d}\rho_p \mathrm{d}R_{\psi_t} Y_{\psi_{-t}} = \mathrm{d}\rho_p \mathrm{d}R_{\psi_t} Y_{R_{\psi_{-t}}e}.$$

Furthermore

$$\begin{aligned}
[X^*, Y^*] &= L_{X^*} Y^* \\
&= \lim_{t\to 0} \frac{1}{t}(\mathrm{d}R_{\psi_{-t}} Y^*_{\psi_t} - Y^*_p) \\
&= \lim_{t\to 0} \frac{1}{t}(\mathrm{d}\rho_p \mathrm{d}R_{\psi_t} Y_{R_{\psi_{-t}}e} - \mathrm{d}\rho_p Y_e) \\
&= \mathrm{d}\rho_p \lim_{t\to 0} \frac{1}{t}(\mathrm{d}R_{\psi_t} Y_{R_{\psi_{-t}}e} - Y_e) \\
&= \mathrm{d}\rho_p L_{-X} Y \\
&= [-X, Y]^*
\end{aligned}$$

The second last equality follows from the formula that if φ_t is the flow of X then

$$L_X Y = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_{-t} Y_{\varphi_t}.$$

By Lemma D.15 if ψ_t is the integral curve of X then R_{ψ_t} is the flow of X in G. □

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