# The Geometry of Naturally Reductive Homogeneous Spaces

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# Abstract

In this Master's thesis we investigate the geometry of naturally reductive Riemannian homogeneous spaces.

In Chapter 1, we cover some necessary background material and present several general results on Riemannian homogeneous spaces. In particular, we prove an important formula for the curvature tensor of the special class of naturally reductive homogeneous spaces.

In Chaper 2, we give an important characterization of homogeneous spaces due to Ambrose and Singer. Their results are then applied to give a classification of the four dimensional naturally reductive spaces due to Tricerri and Vanhecke.

In Chapter 3, we show that under certain conditions, on a naturally reductive homogeneous space, the existence of a totally geodesic hypersurface implies that the the space has constant sectional curvature. These interesting results are due to Tojo and Tsukada.

*Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.*

# Acknowledgments

I want to thank my supervisor Sigmundur Gudmundsson. He suggested this interesting topic, helped me understand essential details, and has devoted many hours to correcting and proof reading my drafts. A special thanks goes to my parents. While they have not proved very useful in dealing with the mathematics of this thesis, they have supported and loved me throughout my life. This thesis is dedicated to them.

Rune Suhr

# Contents

# Chapter 1

# Homogeneous Spaces

## 1.1 Basic Definitions and Results

**Definition 1.1.** An $m$-dimensional smooth manifold is a pair $(M, \mathcal{A})$ consisting of a topological Hausdorff space $M$ with a countable basis, and a differentiable structure $\mathcal{A}$ of class $C^\infty$, meaning a collection of coordinate systems $\{(U_\alpha, \phi_\alpha) : \alpha \in I\}$, such that:

1. each point $p \in M$ has a connected open neigborhood $U_\alpha$, such that $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^m$, with $V_\alpha$ open, is a homeomorphism,
2. $\bigcup_{\alpha \in I} U_\alpha = M$,
3. $\phi_\alpha \circ \phi_\beta^{-1}$ is $C^\infty$ for all $\alpha, \beta \in I$, and
4. $\mathcal{A} = \{(U_\alpha, \phi_\alpha) : \alpha \in I\}$ is maximal in the sense that if $(U_\beta, \phi_\beta)$ is a local chart and $\phi_\alpha \circ \phi_\beta^{-1}$ and $\phi_\beta \circ \phi_\alpha^{-1}$ are $C^\infty$ for all $\alpha \in I$, then $(U_\beta, \phi_\beta) \in \mathcal{A}$.

**Definition 1.2.** A Riemannian metric $g$ on a smooth manifold $M$ is a smooth tensor field

$$g : C_2^\infty(TM) \to C_0^\infty(TM),$$

such that for each point $p \in M$ the restriction

$$g_p = g|_{T_pM \otimes T_pM} : T_pM \otimes T_pM \to \mathbb{R}$$
$$(X_p, Y_p) \mapsto g(X, Y)(p),$$

is an inner product product on $T_pM$. The pair $(M, g)$ is called a Riemannian manifold.

**Definition 1.3.** A diffeomorphism

$$\varphi : (M, g) \to (N, h)$$

between two Riemannian manifolds $M$ and $N$ is called an *isometry*, if

$$g(X, Y)(p) = h(d\varphi X, d\varphi Y)(\varphi(p)),$$

for all $p \in M$.

Every Riemannian manifold has at least one isometry on it, namely the identity map $I : x \mapsto x$, but in general there need not exist more. Certain manifolds have plenty:

**Definition 1.4.** A smooth manifold M is called *homogeneous* if for every $p, q \in M$ there exist a diffeomorphism $\varphi : M \to M$ such that $\varphi(p) = q$. A Riemannian manifold $(M, g)$ is called homogenous if in addition the diffemorphisms are isometries of $M$.

A Riemannian metric on a smooth manifold turns $M$ into a metric space, where the distance function $d$ is defined as follows:

**Definition 1.5.** Let $\gamma : [a,b] \to M$ be a smooth curve. The *length* of $\gamma$ is defined to be the following integral:

$$L(\gamma) := \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t))dt.$$

**Definition 1.6.** For a connected Riemannian manifold $M$, and any two points $p, q \in M$, the Riemannian distance function $d(p,q)$ is defined to be the infimum of the lengths of all piecewise smooth curves from $p$ to $q$.

**Theorem 1.7.** *With the distance function $d$ defined above, any connected Riemannian manifold $M$ becomes a metric space. Moreover, the topology induced by the metric coincides with the topology of $M$ as a manifold.*

*Proof.* See [23]. □

**Theorem 1.8.** *A homogenous Riemannian manifold is complete.*

*Proof.* See for example [2]. □

A class of spaces that are homogeneous is given by the connected symmetric spaces:

**Definition 1.9.** A Riemannian manifold $(M, g)$ is called *symmetric* if for every $p \in M$ there exists an isometry $\sigma_p$ of $M$ such that $\sigma_x(x) = x$ and $d\sigma_x|_{T_xM} = -Id_{T_xM}$. The isometry $\sigma_p$ is called the *symmetry* at $p$.

**Proposition 1.10.** *A connected symmetric space is homogeneous.*

*Proof.* [2] $M$ is complete, since if $\gamma$ is a geodesic segment with endpoints $x, y \in M$, then it may be extended beyond $x$ and $y$ via the symmetries $\sigma_x$ and $\sigma_y$. Therefore, by the theorem of Hopf-Rinow (see e.g. [23]) for any two points there exists a geodesic $\gamma_{x,y}$ between them. Let $z$ be the midpoint of $\gamma_{x,y}$. Then the symmetry $\sigma_z$ satisfies $\sigma_z(x) = y$ and $\sigma_z(y) = x$, so the isometries of $M$ act transitively. □

The composition of two diffeomorphisms is in turn a diffemorphism, and likewise for isometries. The inverse $\varphi^{-1}$ of a diffeomorphism (isometry) is a diffeomorphism (isometry), and compositions yield the identity map: $\varphi^{-1} \circ \varphi = I = \varphi \circ \varphi^{-1}$, and naturally $\varphi \circ I = \varphi = I \circ \varphi$ for any diffeomorphism (isometry) $\varphi$. The set of diffeomorphism and the set of isometries on a (Riemannian) manifold therefore carry natural group structures, and we denote these by $D(M)$ and $I(M, g)$ (or $I(M)$ in case the metric $g$ need not be specified further), respectively.

**Theorem 1.11.** [22] *Let $(M, g)$ be a Riemannian manifold, then there is a unique manifold structure on $I(M, g)$ such that it is*

1. *a Lie Group,*
2. *the natural action $I(M) \times M \to M, \quad (\varphi, p) \mapsto \varphi(p)$, is smooth, and*

*Furthermore, the topology of $I(M)$ as a Lie group is then the compact open topology.*

We can therefore reformulate the notion of homogeneity in terms of smooth group actions: A Riemannian manifold is homogenous if there exists a smooth transitive group action (i.e. a smooth map $G \times M \to M$, that respects the group structure of $G$ in the sense that $(gh)p = g(hp), ep = p$ for all $g, h \in G, p \in M$ and $e$ the identity in $G$), such that for each $g \in G$, $p \mapsto gp$, is an isometry of M.

**Definition 1.12.** The isotropy group of a point $p \in M$ for a group action $\eta : G \times M \to M$, is the subgroup

$$H = \{\sigma \in G \,|\, \eta_\sigma(p) = \sigma p = p\},$$

or equivalently the inverse image of $p$ of the map $G \to M$, with $g \mapsto gp$.

The action being smooth, in particular continuous, makes $H$ a closed subgroup of $I(M, g)$ $(D(M))$.

**Theorem 1.13.** [27] *Let $H$ be a closed subgroup of a Lie group $G$. Then there exists a unique manifold structure for the set $G/H := \{gH : g \in G\}$ of all left cosets such that:*

1. *the natural projection $\pi(g) = gH$, is $C^\infty$,*
2. *there exist local smooth sections of $G/H$ in $G$, i.e. for each $gH \in G/H$, there exists an neighborhood $U \subset G/H$ of $gH$ and a smooth map $\tau : U \to G$, such that $\pi \circ \tau = I_{G/H}$, where $I_{G/H}$ is the identity map of $G/H$.*

**Definition 1.14.** A differentiable map $\varphi : M \to N$ between manifolds $M$ and $N$ is a submersion if the tangent map $d\varphi : T_pM \to T_{\varphi(p)}N$ is of full rank, i.e. surjective, for each $p \in M$.

**Theorem 1.15.** [27] *Let $\eta : G \times M \to M$ be a transitive left action of a Lie group $G$ on a manifold $M$, $p \in M$ and $H$ be the isotropy group at $p$. Then the mapping*

$$\tau : G/H \to M, \quad \tau(\sigma H) = \eta_\sigma(p),$$

*is a diffeomorphism. In particular, the map $g \mapsto gp$ is a submersion.*

A submersion $\varphi$ induces an isomorphism between $T_pM/ker(d\varphi)$ and $T_{\varphi(p)}N$, but in general there is no canonical way of choosing a complement $\mathcal{H}$ of $ker(d\varphi)$ such that $T_pM = ker(d\varphi) \oplus \mathcal{H}$. If $M$ is a Riemannian manifold, however, such a choice is natural, namely we choose $\mathcal{H}$ to be the orthogonal complement of $ker(d\varphi)$ in $T_pM$. $\mathcal{H}$ is then called the horizontal subspace, and $ker(d\varphi)$ is called the vertical subspace.

**Definition 1.16.** A submersion $\varphi : M \to N$ between Riemannian manifolds is said to be a *Riemannian submersion* if the restriction $d\varphi|_{\mathcal{H}} : \mathcal{H} \to T_{\varphi(p)}N$ is an isometry.

In other words, a Riemannian submersion preserves the length of horizontal vectors.

## 1.2 Killing Fields

To introduce the notion of Killing vector fields, we will first need some technical machinery. This treatment follows [21], which we refer to for most of the proofs.

**Definition 1.17.** Let $\mathfrak{T}^r_s(M)$ denote the set of all tensor fields of type $(r, s)$ on $M$. A *tensor derivation* $\mathfrak{D}$ on a smooth manifold is a set of $\mathbb{R}$-linear functions

$$\mathfrak{D} = \mathfrak{D}^r_s : \mathfrak{T}^r_s(M) \to \mathfrak{T}^r_s(M), \quad (r \geq 0, s \geq 0),$$

such that for any tensor fields $A$, $B$, $\mathfrak{D}$ satisfies

1. $\mathfrak{D}(A \otimes B) = \mathfrak{D}A \otimes B + A \otimes \mathfrak{D}B$
2. $\mathfrak{D}(\mathbf{C}A) = \mathbf{C}(\mathfrak{D}A)$, where $\mathbf{C}$ is any contraction.

**Theorem 1.18.** [21] *Given a vector field $V \in C^\infty(TM)$, and an $\mathbb{R}$-linear function $\delta : C^\infty(TM) \to C^\infty(TM)$, such that*

$$\delta(fX) = V(f)X + f\delta(X), \quad \textit{for all } f \in C(M), X \in C^\infty(TM),$$

*there exists a unique tensor derivation $\mathfrak{D}$ on $M$ such that*

$$\mathfrak{D}^0_0 = V : C^\infty(TM) \to C^\infty(TM), \textit{ and } \mathfrak{D}^1_0 = \delta.$$

**Definition 1.19.** If $V \in C^\infty(TM)$, the tensor derivation $L_V$ such that

$$L_V(f) = V(f), \text{and } L_V(X) = [V, X],$$

with $f \in C(M)$, $X \in C^\infty(TM)$, is called the Lie derivative relative to $V$.

This is well defined since according to the theorem above

$$L_V(fX) = [V, fX] = V(f)X + f[V, X] = V(f)X + fL_V X,$$

and therefore satisfies the hypothesis on $\delta$ in the theorem.

**Definition 1.20.** A curve $\alpha : [0, 1] \to M$ is an *integral curve* of $X \in C^\infty(TM)$ if $\alpha'(t) = X_{\alpha(t)}$ for all $t \in I$. A vector field $X$ is *complete* if all integral curves can be extended to all of $\mathbb{R}$. The *flow* of a complete vector field $V$ on $M$ is the map given in the following way:

$$\psi : M \times \mathbb{R} \to M, \quad \psi(p, t) = \alpha_p(t),$$

where $\alpha_p$ is the maximal integral curve starting at $p$.

Thus for a fixed $p$ the function is merely the integral curve through $p$, whereas for fixed $t$, the function lets each $p$ flow until time $t$. This is a diffeomorphism of $M$, and therefore defines a tangent map $d\psi_t : T_pM \to T_{\psi_t(p)}M$. $\psi(p, t)$ will also be written $\psi_t(p)$.

**Theorem 1.21.** [21] *Let $X, Y \in C^\infty(TM)$, and $\psi$ be the local flow of $X$ in a neighborhood of $p \in M$. Then the Lie bracket of $X$ and $Y$ at $p$, $[X, Y]_p$ satisfies:*

$$[X, Y]_p = \lim_{t\to 0} \frac{1}{t}[d\psi_{-t}(Y_{\psi_t(p)} - Y_p)].$$

So the Lie derivative $L_X$ with respect to a vector field $X$, applied to a vector field $Y$, can be interpreted as the rate of change of $Y$ under the flow of $X$. This interpretation can be extended to arbitrary tensor fields. In particular we have the special case where the tensor field is covariant:

**Proposition 1.22.** [21] *If $X \in C^\infty(TM), A \in \mathfrak{T}^0_s$, and $\psi_t$ is the (local) flow of $X$, then*

$$L_X A = \lim_{t\to 0} \frac{1}{t}[d\psi_t(A) - A], \tag{1.1}$$

*where the equality holds locally if the flow is local.*

**Definition 1.23.** A Killing vector field $X$ on a Riemannian manifold $(M, g)$ is a vector field such that $L_X g = 0$.

In other words the metric tensor $g$ is invariant under the flow of a Killing vector field $X$. For this reason Killing fields are also refered to as infinitesimal isometries.

**Proposition 1.24.** *$X \in C^\infty(TM)$ is a Killing vector field if and only if for any fixed $t$ the (local) flows $\psi_t$ are isometries.*

*Proof.* [21] One direction is immediate: if $\psi_t$ is an isometry for all $t \in \mathbb{R}$, then $d\psi_t(g) = g$, and so $L_X g = 0$. Conversely, suppose $L_X g = 0$. Let $U$ be an open subset of $M$ such that the flow is defined on $U$ (the flow might only be defined locally), and let $v$ be a tangent vector at some point $p \in U$. Then for small enough $s \in \mathbb{R}$, $w = d\psi_s(v)$ is also a tangent vector in the domain of the flow, i.e. at some point $q \in U \subseteq M$. By Proposition 1.22 we obtain

$$\lim_{t\to 0} \frac{1}{t}[g(d\psi_{s+t}(v), d\psi_{s+t}(v)) - g(d\psi_s(v), d\psi_s(v))] = 0.$$

This is merely the derivative with respect to $t$ of the real valued function

$$s \mapsto g(d\psi_s(v), d\psi_s(v)),$$

which is therefore constant, i.e. $g$ is invariant under the flow of $X$. □

**Lemma 1.25.** [21] *Let $\mathfrak{D}$ be a tensor derivation on $M$. Then for $A \in \mathfrak{T}^r_s(M)$, the following equality holds*

$$\mathfrak{D}(A(\varphi^1, ..., \varphi^r, X_1, ..., X_s) = (\mathfrak{D}A)(\varphi^1, ..., \varphi^r, X_1, ..., X_s)$$
$$+ \sum_{i=1}^{r} A(\varphi^1, ..., \mathfrak{D}\varphi^i, ..., \varphi^r, X_1, ..., X_s) + \sum_{j=1}^{s} A(\varphi^1, ..., \varphi^r, X_1, ..., \mathfrak{D}X_j, ..., X_s). \tag{1.2}$$

We will need some properties that are equivalent to being a Killing field.

**Proposition 1.26.** [21] *The following properties for vector fields on a Riemannian manifold are equivalent:*

1. *$X$ is a Killing field, i.e. $L_X g = 0$,*
2. *for all $X, Y, W \in C^\infty(TM)$, $\quad X < V, W >=< [X, V], W > + < V, [X, W] >$,*
3. *$< \nabla_V X, W >= - < \nabla_W X, V >$, i.e. $\nabla X : Y \mapsto \nabla_Y X$, is skew symmetric with respect to $g$.*

*Proof.* [21] In view of the product rule of Lemma 1.25, and remembering that the Lie derivative $L_X$ of a real valued function (such as $g(X, Y) : M \to \mathbb{R}, X, Y \in C^\infty(TM)$) is defined as $L_X(f) = Xf$, and $L_X(V) = [X, V]$, where $V$ is a vector field, we see that

$$X < V, W >=< [X, V], W > + < V, [X, W] >, \tag{1.3}$$

for all $V, W \in C^\infty(TM)$, which is equivalent to $(L_X g)(V, W) = 0$, for all $V, W$. So $L_X g = 0$. Expanding the left side of (1.3) and subtracting we get

$$< \nabla_X V, W > + < V, \nabla_X W > - < [X, V], W > - < V, [X, W] >= 0,$$

and since $[X, V] = \nabla_X V - \nabla_V X$, this is equivalent to

$$< \nabla_V X, W > + < \nabla_W X, V >= 0,$$

which is the skew symmetry. □

The Lie derivative is of course $\mathbb{R}$-linear, so the space of all Killing vector fields on $M$, which we shall denote by $i(M)$, is a real vector space. Since $L_X V = [X, V]$ we get that $L_{[X,Y]} Z = [[X, Y], Z] = -[Z, [X, Y]]$, and by the Jacobi identity

$$-[Z, [X, Y]] = [X, [Y, Z]] + [Y, [Z, X]] = [X, [Y, Z]] - [Y, [X, Z]]. \tag{1.4}$$

Returning to the Lie derivative notation we therefore get

$$L_{[X,Y]} = L_X(L_Y(Z)) - L_Y(L_X(Z)) = [L_X, L_Y]Z. \tag{1.5}$$

So the Lie bracket of two Killing fields is a Killing field, and we have proved:

**Theorem 1.27.** [21] *Let $M$ be a Riemannian manifold, then the space $i(M)$ of Killing vector fields on $M$ is a Lie algebra.*

Now we noted earlier that the group $I(M)$ of isometries on $M$ is a Lie group. Consider therefore an element $X$ in its Lie algebra $\mathfrak{i}(M)$, and let $\psi_t$ be its one-parameter subgroup. Define a smooth vector field $X^+$ on $M$ by setting

$$X_p^+ = \frac{d}{dt}(\psi_t(p))|_{t=0}, \tag{1.6}$$

i.e. $X_p^+$ is the initial velocity vector of the curve given by $t \to \psi_t(p)$. One parameter subgroups are defined on the whole of $\mathbb{R}$, so $X^+$ is complete. By construction the flow of $X^+$ is just $\psi_t$ so $X^+$ is a complete Killing field. This correspondence is not surjective in general, since a Killing field on an incomplete Riemannian manifold need not be complete: non-trivial infinitesimal translations on the open unit disc in $\mathbb{R}^2$ are not complete. We do however have the following results regarding the correspondence:

**Theorem 1.28.** *The set $i_C(M)$ of all complete Killing fields on $M$, is a Lie subalgebra of $i(M)$, and the map $\mathfrak{i}(M) \to i_C(M) : X \mapsto X^+$, is a Lie anti-isomorphism, i.e.:*

$$[X^+, Y^+] = -[X, Y]^+,$$

*for all $X, Y \in \mathfrak{i}(M)$.*

*Proof.* See [21]. □

**Theorem 1.29.** *On a complete Riemannian manifold every Killing field is complete.*

*Proof.* See [21]. □

**Corollary 1.30.** *On a homogenous Riemannian space $M = G/H$, any Killing field is complete. As a consequence, $X \mapsto X^+$ is a Lie anti-isomorphism $\mathfrak{i}(M) \to i(M)$.*

*Proof.* Homogenous manifolds are complete, and the claim follows. □

## 1.3 The Theory of Submersions

Recall that a Riemannian submersion $\pi : N \to M$ is a submersion of Riemannian manifolds such that $d\pi$ preserves the scalar product of horizontal vectors, that is, vectors that are orthogonal to the fibers $\pi^{-1}(m)$, $m \in M$.

**Definition 1.31.** Let $\pi : N \to M$, be a Riemannian submersion. For each $n \in \pi^{-1}(m) \subset N$, $\mathcal{H}$ and $\mathcal{V}$ will denote the orthogonal projection of $T_nN$ onto the horizontal and vertical subspaces, $\mathcal{H}_n$ and $\mathcal{V}_n$ respectively. Explicitly:

$$\mathcal{H}(X_n) \in T_n(\pi^{-1}(m))^{\perp} =: \mathcal{H}_n, \quad \mathcal{V}(X_n) \in T_n(\pi^{-1}(m)) =: \mathcal{V}_n, \tag{1.7}$$

where $X_n \in T_nN$.

In this section we shall present various results relating to Riemannian submersions. Our main goal is to prove the following theorem:

**Theorem 1.32.** [20] *Let $\pi : N \to M$ be a Riemannian submersion. If horizontal vector fields $X, Y \in C^\infty(TN)$ span 2-planes, then the Gaussian curvature $K_N$ of $N$ is given in terms of the curvature $K_M$ of $M$ by*

$$K_M(d\pi X, d\pi Y) = K_N(X, Y) + \frac{3}{4}\frac{< \mathcal{V}([X, Y]), \mathcal{V}([X, Y]) >}{Q(X, Y)}, \tag{1.8}$$

*where $Q(X, Y) =< X, X >< Y, Y > - < X, Y >^2$, and $<,>$ is the metric on $N$.*

**Definition 1.33.** Let $\pi : N \to M$ be a Riemannian submersion.

1. A vector field $X$ on $M$ is *horizontal* if $X_p \in \mathcal{H}_p$, for all $p \in M$.
2. A *vertical* vector field $X$ satisfies $X_p \in \mathcal{V}_p$, for all $p \in M$.
3. Given a vector field $X$ on $M$, the *horizontal lift* $\tilde{X}$ of $X$ is the unique horizontal vector field $\tilde{X}$ on $N$ such that $d\pi_p(\tilde{X}_p) = X_{\pi(p)}$, for all $p \in N$.

**Lemma 1.34.** *Let $\pi : N \to M$ be a Riemannian submersion. A vector field $Y$ on $N$ is vertical if and only if $Y(f \circ \pi) = 0$, for any $f \in C^\infty(M)$.*

*Proof.* See [6]. □

**Proposition 1.35.** [20] *Let $\tilde{X}, \tilde{Y}$ be horizontal lifts of $X, Y \in C^\infty(TM)$, and let $U \in C^\infty(TN)$ be vertical. Then the vector fields $[\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}$ and $[\tilde{X}, U]$ are vertical.*

*Proof.* [6] Since $\pi$ is a submersion, $d\pi$ respects the bracket, so we have $d\pi[\tilde{X}, \tilde{Y}] = [d\pi\tilde{X}, d\pi\tilde{Y}] = [X, Y]$. But by definition of the horizontal lift we of course also have $d\pi\widetilde{[X, Y]} = [X, Y]$. Therefore

$$d\pi([\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}) = [X, Y] - [X, Y] = 0, \tag{1.9}$$

and $[\tilde{X}, \tilde{Y}] - \widetilde{[X, Y]}$ is vertical. For the second claim, we use the above lemma and note that $U \cdot (f \circ \pi) = 0$. We thus have that

$$[\tilde{X}, U](f \circ \pi) = -U \cdot \tilde{X} \cdot (f \circ \pi). \tag{1.10}$$

Since $\tilde{X}$ is the horizontal lift of $X$, $\tilde{X} \cdot (f \circ \pi)$ is equal to $(X \cdot f) \circ \pi$. This is constant on fibres, and therefore $U \cdot \tilde{X} \cdot (f \circ \pi) = 0$, and by the lemma, we conclude that $[\tilde{X}, U]$ is vertical. □

We shall also need the following result:

**Proposition 1.36.** [20] *Let $\pi : N \to M$ be a Riemannian submersion, and $\tilde{\nabla}, \nabla$ the Levi-Civita connections on $N$ and $M$ respectively. Suppose $X, Y \in C^\infty(TM)$ with horizontal lifts $\tilde{X}, \tilde{Y}$. For the lifts, $\tilde{\nabla}_{\tilde{X}}\tilde{Y}$ can be calculated in terms of $\nabla_X Y$ as follows:*

$$\tilde{\nabla}_{\tilde{X}}\tilde{Y} = \widetilde{\nabla_X Y} + \frac{1}{2}\mathcal{V}[\tilde{X}, \tilde{Y}], \tag{1.11}$$

*where $\widetilde{\nabla_X Y}$ denotes the horizontal lift of $\nabla_X Y$. In particular, for any $p \in N$, we have*

$$(\nabla_X Y)_{\pi(p)} = d\pi(\tilde{\nabla}_{\tilde{X}}\tilde{Y})_p. \tag{1.12}$$

*Proof.* [6] Let $\tilde{g}$ and $g$ be the metric tensors of $N$ and $M$ respectively. First, since $d\pi$ is an isometry on the horizontal space, we have

$$\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y) \circ \pi. \tag{1.13}$$

First assume $\tilde{Z}$ is the horizontal lift of the vector field $Z$ on $M$. Starting with the Koszul formula we have the following equalities:

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y}, \tilde{Z}) &= \tilde{X} \cdot \tilde{g}(\tilde{Y}, \tilde{Z}) + \tilde{Y} \cdot \tilde{g}(\tilde{Z}, \tilde{X}) - \tilde{Z} \cdot \tilde{g}(\tilde{X}, \tilde{Y}) \\
&\quad + \tilde{g}([\tilde{X}, \tilde{Y}], \tilde{Z}) - \tilde{g}([\tilde{X}, \tilde{Z}], \tilde{Y}) - \tilde{g}([\tilde{Y}, \tilde{Z}], \tilde{X}) \\
&= \tilde{X} \cdot (g(Y, Z) \circ \pi) + \tilde{Y} \cdot (g(Z, X) \circ \pi) - \tilde{Z} \cdot (g(X, Y) \circ \pi) \\
&\quad + g(d\pi[\tilde{X}, \tilde{Y}], d\pi\tilde{Z}) - g(d\pi[\tilde{X}, \tilde{Z}], d\pi\tilde{Y}) - g(d\pi[\tilde{Y}, \tilde{Z}], d\pi\tilde{X}) \\
&= \tilde{X} \cdot (g(Y, Z) \circ \pi) + \tilde{Y} \cdot (g(Z, X) \circ \pi) - \tilde{Z} \cdot (g(X, Y) \circ \pi) \\
&\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X) \\
&= (X \cdot g(Y, Z)) \circ \pi + (Y \cdot g(Z, X)) \circ \pi - (Z \cdot g(X, Y)) \circ \pi \\
&\quad + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X).
\end{aligned}$$

This is equivalent to

$$\tilde{g}(\nabla^N_{\tilde{X}}\tilde{Y}, \tilde{Z})_p = g(\nabla^M_X Y, Z)_{\pi(p)}. \tag{1.14}$$

A couple of remarks on the 4 equalities above:

1. The first equality is merely the Koszul formula for $\tilde{\nabla}$.
2. The second equality follows from the remark earlier in the proof and the fact that $d\pi$ is an isometry (note in the 4th term after the second equality, that though $[\tilde{X}, \tilde{Y}]$ need not be horizontal, $\tilde{Z}$ is, and so we really do have $\tilde{g}([\tilde{X}, \tilde{Y}], \tilde{Z}) = g(d\pi[\tilde{X}, \tilde{Y}], d\pi\tilde{Z})$, and so on).
3. The third equality is just

$$g(d\pi[\tilde{X}, \tilde{Y}], d\pi\tilde{Z}) = g([d\pi\tilde{X}, d\pi\tilde{Y}], d\pi\tilde{Z}) = g([X, Y], Z),$$

   and so on.
4. The fourth equality follows from the fact that $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ are horizontal.

In the case that $U$ is a vertical vector field, the Koszul formula reduces to

$$\tilde{g}(\nabla^N_{\tilde{X}}\tilde{Y}, \tilde{U}) = \frac{1}{2}\tilde{g}([\tilde{X}, \tilde{Y}], U). \tag{1.15}$$

The remaining terms vanish since:

1. the relations: $\tilde{Y} \perp U$ and $\tilde{X} \perp U$ ensure that the first two terms of the Koszul formula vanish.
2. As $d\pi\tilde{X} = X$ and $d\pi\tilde{Y} = Y$, $\tilde{X}$ and $\tilde{Y}$ are constant along fibres, and so $U \cdot \tilde{g}(\tilde{X}, \tilde{Y}) = 0$.
3. By Proposition 1.35, $[\tilde{X}, U]$ and $[\tilde{Y}, U]$ are vertical and thus perpendicular to $\tilde{X}$ and $\tilde{Y}$.

Finally we observe that these two cases together imply the claim of the theorem. $\square$

Before proving Theorem 1.32 we give a result which will be useful in a later section.

**Proposition 1.37.** [21] *Under a Riemannian submersion $\pi : N \to M$, horizontal geodesics in $N$ are mapped to geodesics in $M$.*

*Proof.* [21] If $\gamma$ is a horizontal geodesic, then $\pi \circ \gamma$ is a regular smooth curve in $M$ and hence (locally) the integral curve of a smooth vector field $X$ on $M$. Hence, since it is horizontal, $\gamma$ is an integral curve of the horizontal lift $\tilde{X}$ of $X$. Therefore, using Proposition 1.36 we get

$$\nabla_X X = d\pi(\tilde{\nabla}_{\tilde{X}} \tilde{X}) = d\pi(0) = 0,$$

since $\gamma$ is a geodesic. This shows that $\pi \circ \gamma$ is a geodesic. $\square$

*Proof.* (Theorem 1.32)
We start by showing that

$$\tilde{g}(\tilde{\nabla}_U \tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{X}, \tilde{Y}], U), \tag{1.16}$$

where $U$ is a vertical vector field. Since $U$ is a vertical vector field on $N$, and $\tilde{X}$ and $\tilde{Y}$ are horizontal lifts of vector fields $X, Y \in C^\infty(TM)$, by Proposition 1.35 we have

$$\tilde{g}(\tilde{\nabla}_U \tilde{X}, \tilde{Y}) = \tilde{g}(\tilde{\nabla}_{\tilde{X}} U, \tilde{Y}) + \tilde{g}([U, \tilde{X}], \tilde{Y}) = \tilde{g}(\tilde{\nabla}_{\tilde{X}} U, \tilde{Y}), \tag{1.17}$$

where $\tilde{g}$ is the metric on $N$ and $\tilde{\nabla}$ the Levi-Civita connection on $N$. $U$ being vertical and $\tilde{Y}$ horizontal gives that $\tilde{g}(U, \tilde{Y}) = 0$. Therefore

$$0 = \tilde{X} \cdot \tilde{g}(U, \tilde{Y}) = \tilde{g}(\tilde{\nabla}_{\tilde{X}} U, \tilde{Y}) + \tilde{g}(U, \tilde{\nabla}_{\tilde{X}} \tilde{Y}), \tag{1.18}$$

and so

$$\tilde{g}(\tilde{\nabla}_{\tilde{X}} U, \tilde{Y}) = -\tilde{g}(U, \tilde{\nabla}_{\tilde{X}} \tilde{Y}). \tag{1.19}$$

By Proposition 1.36 we get

$$\tilde{g}(U, \tilde{\nabla}_{\tilde{X}} \tilde{Y}) = \frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{X}, \tilde{Y}], U), \tag{1.20}$$

since $U$ is vertical. Tracing back the equalities we therefore have

$$\tilde{g}(\tilde{\nabla}_U \tilde{X}, \tilde{Y}) = -\frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{X}, \tilde{Y}], U), \tag{1.21}$$

as desired.

Now from the proof of Proposition 1.36 we have that

$$\tilde{g}(\tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W}) = g(\nabla_Y Z, W), \tag{1.22}$$

for horizontal lifts $\tilde{Y}, \tilde{Z}, \tilde{W}$. This clearly implies that

$$\tilde{X} \cdot \tilde{g}(\tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W}) = X \cdot g(\nabla_Y Z, W). \tag{1.23}$$

We therefore get the following equalities:

$$\begin{aligned}
\tilde{g}(\tilde{\nabla}_{\tilde{X}} \tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W}) &= \tilde{X} \cdot \tilde{g}(\tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{W}) - \tilde{g}(\tilde{\nabla}_{\tilde{Y}} \tilde{Z}, \tilde{\nabla}_{\tilde{X}} \tilde{W}) \\
&= X \cdot g(\nabla_Y Z, W) - (\tilde{g}(\widetilde{\nabla_Y Z}, \widetilde{\nabla_X W}) + \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{Y}, \tilde{Z}], \mathcal{V}[\tilde{X}, \tilde{W}])) \\
&= X \cdot g(\nabla_Y Z, W) - g(\nabla_Y Z, \nabla_X W) - \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{Y}, \tilde{Z}], \mathcal{V}[\tilde{X}, \tilde{W}]) \\
&= g(\nabla_X \nabla_Y Z, W) - \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{Y}, \tilde{Z}], \mathcal{V}[\tilde{X}, \tilde{W}]).
\end{aligned}$$

1. The first equality is elementary.

2. The second equality follows from Equation (1.23) and Proposition 1.36.
3. The third equality follows from the submersion being Riemannian.
4. The fourth equality is the reverse of the first equality, though taking place in $M$ rather than $N$.

We have already established the identities

$$\tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) = g(\nabla_X Y, Z),$$

and

$$\tilde{g}(\tilde{\nabla}_{\tilde{U}}\tilde{X}, Y) = -\frac{1}{2}\tilde{g}([\tilde{X},\tilde{Y}], U),$$

for $U$ vertical, and from these we calculate $\tilde{g}(\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W})$:

$$\begin{aligned}\tilde{g}(\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}) &= \tilde{g}(\tilde{\nabla}_{\mathcal{H}[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}) + \tilde{g}(\tilde{\nabla}_{\mathcal{V}[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}) \\ &= g(\nabla_{[X,Y]}Z, W) - \frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{Z},\tilde{W}],\mathcal{V}[\tilde{X},\tilde{Y}]),\end{aligned}$$

the first equality following from linearity and the second from the aforementioned identities.

We can now calculate the curvature tensor $R$:

$$\begin{aligned}\tilde{g}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Z},\tilde{W}) &= \tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z},\tilde{W}) - \tilde{g}(\tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z},\tilde{W}) - \tilde{g}(\tilde{\nabla}_{[\tilde{X},\tilde{Y}]}\tilde{Z},\tilde{W}) \\ &= g(R(X,Y)Z,W) + \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{X},\tilde{Z}],\mathcal{V}[\tilde{Y},\tilde{W}]) \\ &\quad -\frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{Y},\tilde{Z}],\mathcal{V}[\tilde{X},\tilde{W}]) + \frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{Z},\tilde{W}],\mathcal{V}[\tilde{X},\tilde{Y}]).\end{aligned}$$

Setting $\tilde{Z} = \tilde{Y}$ and $\tilde{W} = \tilde{X}$, we get

$$\begin{aligned}\tilde{g}(\tilde{R}(\tilde{X},\tilde{Y})\tilde{Y},\tilde{X}) &= g(R(X,Y)Y,X) + \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{X},\tilde{Y}],\mathcal{V}[\tilde{Y},\tilde{X}]) \\ &\quad -\frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{Y},\tilde{Y}],\mathcal{V}[\tilde{X},\tilde{X}]) + \frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{Y},\tilde{X}],\mathcal{V}[\tilde{X},\tilde{Y}]) \\ &= g(R(X,Y)Y,X) - \frac{1}{4}\tilde{g}(\mathcal{V}[\tilde{X},\tilde{Y}],\mathcal{V}[\tilde{X},\tilde{Y}]) \\ &\quad -\frac{1}{2}\tilde{g}(\mathcal{V}[\tilde{X},\tilde{Y}],\mathcal{V}[\tilde{X},\tilde{Y}]).\end{aligned}$$

This is equivalent to the desired expression. □

## 1.4 Reductive Homogeneous Spaces

**Definition 1.38.** A Lie group $G$ is said to act effectively on a space $M$, if $L_g = Id_M$, for $g \in G$, implies that $g = e$.

If a Lie group $G$ acts transitively and effectively on a Riemannian space $M$ it is isomorphic to some subgroup of the isometry group $I(M)$. We shall require that $G$ is a closed subgroup of $I(M)$:

**Definition 1.39.** Let $G$ be a Lie group. A $G$-homogeneous space is a manifold $M$ with a transitive action of $G$. If $(M, g)$ is a Riemannian manifold and $G$ is a closed subgroup of the isometry group $I(M, g)$, we say that $M$ is a Riemannian $G$-homogeneous space.

Equivalently, $M$ is a coset manifold given by $G/H$, where $H$ is a closed subgroup of $G$. In the Riemannian case the metric will be called left invariant (under $G$).

As an example of a proper Lie subgroup $G$ of the full isometry group $I(M)$ acting transitively, we have for instance the group of translations on Euclidean space.

In this section we will show that any Riemannian homogeneous space is reductive, a property defined as follows:

**Definition 1.40.** A homogeneous space $M = G/H$ is called *reductive* if $\mathfrak{g}$ admits a decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, such that $Ad_H(\mathfrak{m}) \subset \mathfrak{m}$. $\mathfrak{m}$ is then called a Lie subspace for $G/H$.

Note that $\mathfrak{m}$ might not be unique, and the definition does not require that $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{m}$.

Nomizu ([18]) defines an affine connection as a rule which assigns to each $X \in C^\infty(TM)$, an endomorphism $t(X)$ of $C^\infty(TM)$, satisfying

$$t(X_1 + X_2) = t(X_1) + t(X_2),$$
$$t(fX)(Y) = ft(X)(Y) + (Yf)X,$$

where $f \in C^\infty(M)$.

That $t(X)$ should be a $C^\infty(M)$ endomorphism means that $t(X)(fY) = ft(X)(Y)$. We see therefore that defining $t(X)$ by $Y \mapsto \nabla_Y X$ makes $t(X)$ an affine connection in the sense of Nomizu.

**Theorem 1.41.** [18] *Let $G/H$ be a reductive homogeneous space with a fixed decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$, such that $Ad(H)\mathfrak{m} \subset \mathfrak{m}$. There exists a one-to-one correspondence between the set of all invariant affine connections on $G/H$, and the set of all bilinear functions $\Lambda$ on $\mathfrak{m} \times \mathfrak{m}$, with values in $\mathfrak{m}$ which are invariant by $Ad(H)$. The correspondence is given by*

$$\Lambda(X, Y) = t((Y), (X))_p,$$

*where $X, Y \in TM$ and $p \in M$.*

*Proof.* See [18]. □

To any reductive homogeneous Riemannian manifold there is an associated connection:

**Proposition 1.42.** [18] *Let $M = G/H$ be a reductive homogeneous space. Let $x(s)$ be the 1-paramater subgroup of $G$ generated by $X \in \mathfrak{m}$, and let $x^*(s) = \pi(x(s)) \subset M$, the image of $x(s)$ under the natural projection $G \to G/H$. Let $Y \in \mathfrak{m}$. There exists a unique $G$-invariant connection $\tilde{\nabla}$ on $M$ with respect to the decomposition $\mathfrak{g} = \mathfrak{m} + \mathfrak{h}$ such that the parallel translation of $Y$ along $x^*(s)$ coincides with the translation of $Y$ induced by the subgroup $x(s)$. Moreover, it corresponds to the connection function $\Lambda_\mathfrak{m} : \mathfrak{m} \to \mathfrak{m}$ that vanishes identically $\Lambda_\mathfrak{m} \equiv 0$.*

*Proof.* See [18]. □

**Definition 1.43.** The unique connection $\tilde{\nabla}$ of Proposition 1.42 is called the canonical connection on $M$ with respect to $\mathfrak{m}$.

**Proposition 1.44.** [18] *Let $M = G/H$ be a reductive homogeneous space. The canonical connection $\tilde{\nabla}$ satisfies*

$$\tilde{T}(X, Y) = -[X, Y]_\mathfrak{m},$$

$$\tilde{R}(X, Y)Z = -[[X, Y]_\mathfrak{h}, Z],$$

*as well as*

$$\tilde{\nabla}\tilde{T} = \tilde{\nabla}\tilde{R} = 0,$$

*for any $X, Y, Z \in \mathfrak{m}$, where $\tilde{T}$ and $\tilde{R}$ are the torsion and curvature tensors of $\tilde{\nabla}$, defined in the usual way.*

*Proof.* See [18]. □

The requirement that $G$ is a closed subgroup of $I(M)$ is not a significant restriction:

**Proposition 1.45.** [2] *If $G$ is any Lie group acting effectively and transitively on $M$, and if $G$ leaves invariant some Riemannian metric on $M$, then there exists a unique subgroup $\bar{G}$ of $Diff(M)$, such that for any $G$-invariant Riemannian metric $g$ on $M$, $\bar{G}$ is the closure of $G$ in $I(M, g)$.*

**Proposition 1.46.** [3] *Let $G$ be a Lie group acting transitively on some manifold $M$, let $H$ be the isotropy group at $p \in M$, and let $H_0$ be the largest subgroup of $H$ which is normal in $G$. Set*

$$G^* = G/H_0, \quad H^* = H/H_0.$$

*Then $G^*/H^*$ is diffeomorphic to $G/H \simeq M$, and $G^*$ acts effectively on $M$.*

In the light of Proposition 1.46 we will from now on assume that any homogeneous space is given by a Lie group acting effectively. Let $M$ be a Riemannian $G$-homogeneous space. The isotropy group $H$ at $p \in M$ then acts by isometries on $T_pM$. Since isometries commute with the exponential map of $T_pM$, an isometry on a connected Riemannian manifold is determined uniquely by its differential at one point. Therefore $H$ can be identified (not necessarily through an embedding, though - see [3]), via the map $H \ni h \mapsto dh \in O(T_pM)$, with a closed subgroup of $O(T_pM)$, the orthogonal group of $T_pM$, which implies that $H$ is compact (see [2], [3] or [6]).

Since the action of $G$ on $G$ and $G/H$ commutes with the projection $\pi$, for $h \in H$ and $X \in \mathfrak{g}$, we have

$$he^{tX}H = he^{tX}h^{-1}H.$$

Since $Ad(H)(\mathfrak{h}) \subset \mathfrak{h}$, as well as $ad(\mathfrak{h})(\mathfrak{h}) \subset \mathfrak{h}$ holds, we get an action on the quotient space $\mathfrak{g}/\mathfrak{h}$. We differentiate and obtain

$$dL_h(d\pi X) = d\pi(Ad(h)(X)), \tag{1.24}$$

showing that the linear isotropy group acting on the tangent space at $p \in M$ is equal to $Ad|_H$ under the projection $\pi$.

**Theorem 1.47.** *The set of $G$-invariant metrics on $G/H$ is naturally isomorphic to the set of scalar products $<,>$ on $\mathfrak{g}/\mathfrak{h}$ which are invariant under the action $Ad_H$ on $\mathfrak{g}/\mathfrak{h}$.*

*Proof.* [3] Given a left invariant metric $\mu$ on $G/H$, the restriction to the tangent space at $[H]$ yields an inner product $<,>$ on $\mathfrak{g}/\mathfrak{h}$. But as $d\pi(Ad_h(X)) = Ad_h(X) + \mathfrak{h} = Ad_h(X + \mathfrak{h})$, equation (1.24) shows that the left invariance of $\mu$ means that $<,>$ is $Ad_H$ invariant. Conversely, given an $Ad_H$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$ we naturally have an inner product $<,>_H$ on the tangent space of $G/H$ at $H$. We extend this to a $G$ left invariant metric on $G/H$ by setting

$$< X, Y >_{[g]} = < dL_{g^{-1}}(X), dL_{g^{-1}}(Y) >_H,$$

for $X, Y \in T_{[g]}G/H$. To show that this does not depend of the choice of $g$, we observe that if $<,>$ is $Ad_H$-invariant, we have

$$\begin{aligned} < dL_{hg^{-1}}(X), dL_{hg^{-1}}(Y) >_H &= < dL_h \circ dL_{g^{-1}}(X), dL_h \circ dL_{g^{-1}}(Y) >_H \\ &= < dL_{g^{-1}}(X), dL_{g^{-1}}(Y) >_H . \end{aligned}$$

By construction, the metric is left invariant on $G/H$. □

**Theorem 1.48.** *If $G$ acts effectively on $G/H$, then $G/H$ admits a $G$-invariant metric if and only if the closure of $Ad_H$ is compact in $GL(\mathfrak{g})$.*

*Proof.* [3] Since $G$ is assumed to act effectively by isometries, there exists an injective homomorphism $G \hookrightarrow I(G/H)$, and an associated map $\mathfrak{g} \to \mathfrak{i}(G/H)$. From the discussion above the full isotropy group $H^* \subset I(G/H)$ is compact, and therefore so is the image under the adjoint representation $Ad_{H^*} \subset GL(\mathfrak{g})$. Let $\omega$ be some right invariant volume form on $Ad_{H^*}$ and $<,>$ be some inner product on $\mathfrak{g}$, we then define $\ll, \gg$ as the average

$$\ll X, Y \gg \; = \int_{Ad_{H^*}} < Ad_{h^*}(X), Ad_{h^*}(Y) > \omega(h^*).$$

Now with respect to $\ll, \gg$, $Ad_{H^*}$ acts by isometries, since

$$\ll Ad_{h_1}(X), Ad_{h_1}(Y) \gg \; = \int_{Ad_{H^*}} < Ad_{h^*}Ad_{h_1}(X), Ad_{h^*}Ad_{h_1}(Y) > \omega(h^*)$$

$$
\begin{aligned}
&= \int_{Ad_{H^*}} < Ad_{h^*h_1}(X), Ad_{h^*h_1}(Y) > dR_{h_1^{-1}}\omega(h^*h_1) \\
&= \int_{Ad_{H^*}} < Ad_{h^*}(X), Ad_{h^*}(Y) > dR_{h_1^{-1}}\omega(h^*) \\
&= \int_{Ad_{H^*}} < Ad_{h^*}(X), Ad_{h^*}(Y) > \omega(h^*) \\
&= \ll X, Y \gg .
\end{aligned}
$$

So $H^*$ acts by isometries, and therefore so does $H \subset H^*$. Therefore $H$ is contained in $O(\mathfrak{g}, \ll, \gg)$ and so its closure is compact. Conversely, if the closure of $Ad_H$ is compact, we may construct an inner product $\ll, \gg$, with an averaging procedure similar to the one above, such that $Ad_H$ acts by isometries. Letting $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to $\ll, \gg$, makes $\ll, \gg |_\mathfrak{m}$ an $Ad_H$-invariant inner product on $\mathfrak{g}/\mathfrak{h}$, when we identify it with $\mathfrak{m}$. □

**Corollary 1.49.** *Any Riemannian G-homogeneous space M is reductive.*

*Proof.* [3] $H$ is closed, so given any inner product $<, >$ on $\mathfrak{g}$ we can, as in the proof of Theorem 1.48, take the average of $<, >$ over $Ad_H$, to get an $Ad_H$-invariant inner product $\ll, \gg$ on $\mathfrak{g}$. Letting $\mathfrak{m} = \mathfrak{h}^\perp$ with respect to $\ll, \gg$, yields the desired decomposition. □

Specializing Theorem 1.47 to the case of reductive homogeneous manifolds we get the more common

**Corollary 1.50.** *Let $M = G/H$ be a reductive homogeneous manifold with Lie subspace $\mathfrak{m}$. If we require that $d\pi : \mathfrak{m} \to T_pM = \mathfrak{g}/\mathfrak{h}$, is an isometry, a one-to-one correspondence between $Ad_H$-invariant inner products on $\mathfrak{m}$ and $G$-invariant metrics on $M$ is established.*

*Proof.* The result is an immediate consequence of Theorem 1.47 and the definitions. □

Later we shall need the following results on the holonomy algebra of reductive homogeneous spaces:

**Theorem 1.51.** [17] *Let $G/H$ be a homogeneous Riemannian space. The Lie algebra $\mathfrak{h}^*$, of the holonomy group of $G/H$, is generated by the endomorphisms of $\mathfrak{m}$ of the form $R(X,Y)$, $(\nabla R)(X,Y,Z)$, $(\nabla^2 R)(X,Y,Z,W), \ldots$, where $X, Y, Z, W, ... \in \mathfrak{m}$.*

*Proof.* See [17]. □

**Theorem 1.52.** [19] *Let $M = G/H$ be a reductive homogeneous space with an $Ad_H$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Then the holonomy algebra is equal to the smallest Lie algebra of endomorphisms $\mathfrak{h}^*$ of $\mathfrak{m}$, such that $R(X,Y) \in \mathfrak{h}^*$, for all $X, Y \in \mathfrak{m}$, and $[\Lambda_\mathfrak{m}(X), \mathfrak{h}^*] \subset \mathfrak{h}^*$, for all $X \in \mathfrak{m}$.*

*Proof.* See [19]. □

We noted above that we can identify the Lie algebra $\mathfrak{g}$ of $G$ with the Killing fields of $(M, g)$ generated by one-parameter subgroups of $G$ ($G$ can be a proper subgroup of $Isom(M, g)$). Let $\mathfrak{h}$ be the Lie algebra of the isotropy group $H$ of $p \in M$. We then identify $\mathfrak{h}$ with the set of Killing fields on $M$ that vanish at $p$. These Killing fields form a subalgebra of $i(M)$. We can identify $\mathfrak{m}$ with $T_pM$ (see [2]) by evaluating the remaining (i.e. the at $p$ non-vanishing) Killing fields at $p$:

$$\mathfrak{m} \ni X \mapsto (X^+)_p \in T_pM. \tag{1.25}$$

As we noted earlier homogenous spaces are complete, and therefore by Corollary 1.30 there is a one-to-one correspondence between the Lie algebra $\mathfrak{g}/\mathfrak{h}$ and $i(M)$, counting dimensions we conclude that the identification $X \mapsto X_p^+$ is a vector space isomorphism.

With these identifications, we can determine the Levi-Civita connection, the curvature tensor and so on at the point $p$ by making use of properties of Killing fields. Since our Riemannian manifolds are homogeneous, we need only determine them at one point to know them completely.

**Lemma 1.53.** [2] *Let $X, Y, Z$ be Killing fields on a Riemannian manifold $(M, g)$. Then*

$$2g(\nabla_X Y, Z) = g([X, Y], Z) + g([X, Z], Y) + g(X, [Y, Z]). \tag{1.26}$$

*Proof.* [2] Since $[X, Z] = \nabla_X Z - \nabla_Z X$, we have

$$g([X, Z], X) = g(\nabla_X Z, X) - g(\nabla_Z X, X).$$

But by Proposition 1.26 $X$ being a Killing field is equivalent to

$$g(\nabla_V X, W) = -g(\nabla_W X, V), \tag{1.27}$$

and we obtain

$$g(\nabla_X Z, X) = 0,$$

and

$$-g(\nabla_Z X, X) = g(\nabla_X X, Z),$$

and so

$$g([X, Z], X) = g(\nabla_X X, Z), \tag{1.28}$$

and the claim is therefore satisfied in the case $Y = X$. Now using

$$[X, Z] = \nabla_X Z - \nabla_Z X$$

again, we get

$$\begin{aligned} g([X, Z], Y) + g(X, [Y, Z]) &= g(\nabla_X Z - \nabla_Z X, Y) + g(X, \nabla_Y Z - \nabla_Z Y) \\ &= g(\nabla_X Z, Y) - g(\nabla_Z X, Y) \\ &\quad + g(\nabla_Y Z, X) - g(\nabla_Z Y, X) \end{aligned}$$

Now we use equation (1.27) to get

$$\begin{aligned} g([X, Z], Y) + g(X, [Y, Z]) &= g(\nabla_X Z, Y) + g(\nabla_Y X, Z) \\ &\quad - g(\nabla_X Z, Y) + g(\nabla_X Y, Z) \\ &= g(\nabla_Y X, Z) + g(\nabla_X Y, Z). \end{aligned}$$

But $g(\nabla_X Y, Z) - g(\nabla_Y X, Z) = g([X, Y], Z)$, so adding this relation we get

$$\begin{aligned} 2g(\nabla_X Y, Z) &= g(\nabla_Y X, Z) + g(\nabla_X Y, Z) + g(\nabla_X Y, Z) - g(\nabla_Y X, Z) \\ &= g([X, Z], Y) + g(X, [Y, Z]) + g([X, Y], Z), \end{aligned}$$

which proves the claim. □

For the rest of this section we let $U : \mathfrak{m} \times \mathfrak{m} \to \mathfrak{m}$, denote the symmetric function satisfying:

$$2g(U(X, Y), Z) = g([Z, X]_{\mathfrak{m}}, Y) + g(X, [Z, Y]_{\mathfrak{m}}),$$

for all $Z \in \mathfrak{m}$, where $[\ ,\ ]_{\mathfrak{m}}$ is the $\mathfrak{m}$-component of $[\ ,\ ]$.

**Proposition 1.54.** [2] *Let $(M, g)$ be a $G$-homogeneous space, and $\mathfrak{m}$ its Lie subspace. Let $X, Y \in \mathfrak{m}$. Then at $p \in M$, we have:*

$$(\nabla_X Y)_p = -\frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y). \tag{1.29}$$

We note that $X, Y$ being Killing fields does not imply that $\nabla_X Y$ is a Killing field, and so (1.29) in general only holds at $p$.

*Proof.* This follows from the definition of $U$, Equation (1.26) and the fact that $X \mapsto X^+$ is a Lie anti-isomorphism, by Corollary 1.30. □

We are now ready to prove a formula for the curvature:

**Theorem 1.55.** [2] *Let* $(M, <,>)$ *be a homogeneous Riemannian manifold, then the curvature tensor at* $p = H \in M$ *satisfies:*

$$\begin{aligned} < R(X,Y)Y, X >_p &= \frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + \frac{1}{2} < [X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y > \\ &+\frac{1}{2} < [Y,[Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X > -|U(X,Y)|^2 \\ &+ < U(X,X), U(Y,Y) > . \end{aligned}$$

*Proof.* [2] We will use Lemma 1.53, Proposition 1.29 and Equation (1.27) repeatedly, to get the following series of equalities:

$$\begin{aligned} - < R(X,Y)Y, X >_p &= < R(X,Y)X, Y >_p \\ &= < \nabla_{[X,Y]}X, Y >_p - < \nabla_X \nabla_Y X, Y >_p \\ &\quad + < \nabla_Y \nabla_X X, Y >_p \\ &= - < \nabla_Y X, [X,Y] >_p - X < \nabla_Y X, Y >_p \\ &\quad + < \nabla_Y X, \nabla_X Y >_p + Y < \nabla_X X, Y >_p \\ &\quad - < \nabla_X X, \nabla_Y Y >_p \\ &= |\nabla_Y X|_p^2 - < \nabla_X X, \nabla_Y Y >_p + Y < [X,Y], X >_p \\ &= \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 + < [X,Y]_{\mathfrak{m}}, U(X,Y) >_p + |U(X,Y)|^2 \\ &\quad - < U(X,X), U(Y,Y) >_p + < [Y,[X,Y]], X >_p \\ &\quad + < [X,Y], [Y,X] >_p \\ &= \frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 + |U(X,Y)|^2 - < U(X,X), U(Y,Y) >_p \\ &\quad + \frac{1}{2} < [[X,Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y >_p + \frac{1}{2} < X, [[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}} >_p \\ &\quad + < [Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}}, X >_p + < [Y,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, X >_p \\ &\quad - |[X,Y]_{\mathfrak{m}}|^2 \\ &= -\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + |U(X,Y)|^2 - < U(X,X), U(Y,Y) >_p \\ &\quad + \frac{1}{2} < [[X,Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y >_p + \frac{1}{2} < X, [[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}} >_p \\ &\quad + \frac{1}{2} < [Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}}, X >_p + \frac{1}{2} < Y, [[X,Y]_{\mathfrak{h}}], X]_{\mathfrak{m}} >_p \\ &\quad - < [[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}}, X >_p \\ &= -\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + |U(X,Y)|^2 - < U(X,X), U(Y,Y) >_p \\ &\quad + \frac{1}{2} < [[X,Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y >_p - \frac{1}{2} < X, [[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}} >_p \\ &\quad + \frac{1}{2} < [[X,Y]_{\mathfrak{h}}], X]_{\mathfrak{m}}, Y >_p + \frac{1}{2} < X, [Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}} >_p \\ &= -\frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + |U(X,Y)|^2 - < U(X,X), U(Y,Y) >_p \\ &\quad - \frac{1}{2} < [X,[X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y >_p - \frac{1}{2} < X, [[X,Y]_{\mathfrak{g}}, Y]_{\mathfrak{m}} >_p \end{aligned}$$

A couple of points are in order:

1. The first equality is a standard symmetry of $< R(X,Y)Z, W >$.

2. The second equality is merely the definition of $< R(X,Y)X, Y >_p$.

3. The third equality is given by shifting the $[X,Y]$ and $Y$ factors in the first term - which is allowed since we have identified $\mathfrak{g}$ with the Killing fields of $M$ - and making use of the standard equality

$$<\nabla_X\nabla_Y X, Y>= X<\nabla_Y X, Y> - <\nabla_Y X, \nabla_X Y> .$$

4. The fourth equality is given by expanding the terms $[X,Y]$ and collecting.

5. The fifth equality is given by using Equation 1.29 to write $|D_Y X|^2$ as the first three terms. Since $[X,X]=0$, $<\nabla_X X, \nabla_Y Y>=<U(X,X), U(Y,Y)>$, by Proposition 1.54. Using Proposition 1.26 part 2 gives the last two terms from $Y([X,Y],X)$.

6. The sixth equality uses the defining equation of $U$ to derive

$$<[X,Y]_{\mathfrak{m}}, U(X,Y)>=\frac{1}{2}<[[X,Y]_{\mathfrak{m}}, X]_{\mathfrak{m}}, Y>+\frac{1}{2}<X, [[X,Y]_{\mathfrak{m}}, Y]_{\mathfrak{m}}> .$$

Furthermore, $<[Y,[X,Y]],X>$ is expanded as:

$$\begin{aligned} <[Y,[X,Y]],X> &= <[Y,[X,Y]_{\mathfrak{h}}+[X,Y]_{\mathfrak{m}}],X> \\ &= <[Y,[X,Y]_{\mathfrak{h}}+[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},X> \\ &= <[Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}},X> \\ &\quad +<[Y,[X,Y]_{\mathfrak{m}}]_{\mathfrak{m}},X>, \end{aligned}$$

with the second equality due to $X\in\mathfrak{m}$. From Proposition 1.54 we have that $(\nabla_X Y)_p = -\frac{1}{2}[X,Y]_{\mathfrak{m}}+U(X,Y)$, so since all vector fields here are evaluated at $p\in M$ we get

$$\begin{aligned} [X,Y]_p &= (\nabla_X Y - \nabla_Y X)_p \\ &= -\frac{1}{2}[X,Y]_{\mathfrak{m}}+U(X,Y)+\frac{1}{2}[Y,X]_{\mathfrak{m}}-U(Y,X) \\ &= -\frac{1}{2}[X,Y]_{\mathfrak{m}}+\frac{1}{2}[Y,X]_{\mathfrak{m}}, \end{aligned}$$

since $U$ is symmetric, and therefore

$$<[X,Y],[Y,X]>_p=-|[X,Y]_{\mathfrak{m}}|^2.$$

7. Equality seven comes from $<,>$ being $Ad_H$-invariant, since this implies that:

$$<[Y,[X,Y]_{\mathfrak{h}}]_{\mathfrak{m}},X>=<Y,[[X,Y]_{\mathfrak{h}},X]_{\mathfrak{m}}> .$$

8. Equalities eight and nine use standard symmetries and linearity. Finally, taking the negative of both sides results in the formula of the theorem.

□

## 1.5 Naturally Reductive Homogeneous Manifolds

**Definition 1.56.** A *naturally reductive* homogenous space, is a reductive homogeneous Riemannian manifold $M=G/H$ with a decomposition $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, that satisfies

$$<[X,Y]_{\mathfrak{m}},Z>=-<Y,[X,Z]_{\mathfrak{m}}>, \quad \text{for } X,Y,Z\in\mathfrak{m}, \tag{1.30}$$

or equivalently, $U\equiv 0$.

We remark that the above definition depends on the choice of subgroup $G$ in the group of isometries of $M$. Thus if $G_1\subset G_2$ are two transitive groups of isometries, then a metric that is naturally reductive with respect to $G_1$ might not be so when considering $M=G_2/H_2$, or vice versa. For further discussion see [5].

To furnish Definition 1.56 with some context, note that in the special case of $H=\{e\}$, $\mathfrak{p}=\mathfrak{g}$, the condition is equivalent to the 4th property in the following list of equivalent properties:

**Theorem 1.57.** *Let $G$ be a connected Lie group with a left-invariant Riemannian metric $<,>$, i.e. a metric satisfying $< dL_gX, dL_gY >_{gh} = < X, Y >_h$, for all $g, h \in G$, and $X, Y \in T_hG$. Then the following are equivalent:*

1. *$<,>$ is right-invariant, and therefore bi-invariant.*
2. *$<,>$ is $Ad(G)$-invariant.*
3. *$h \mapsto h^{-1}$, is an isometry of $(G, <,>)$*
4. *$< [X, Y], Z > = < X, [Y, Z] >$, for all $X, Y, Z \in \mathfrak{g}$.*
5. *$\nabla_X Y = \frac{1}{2}[X, Y]$, for all $X, Y \in \mathfrak{g}$.*
6. *The geodesics starting at $e \in G$ are exactly the one-paramater subgroups of $G$.*

*Proof.* See [21]. □

**Theorem 1.58.** [2] *In a naturally reductive $G$-homogeneous Riemannian manifold, the curvature is given by*

$$< R(X,Y)Y, X > = - < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2.$$

*Proof.* Being naturally reductive is equivalent to $U \equiv 0$. From Theorem 1.55 we have that

$$\begin{aligned} < R(X,Y)Y, X >_p &= \frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + \frac{1}{2} < [X, [X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y > \\ &\quad + \frac{1}{2} < [Y, [Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X > \\ &\quad - |U(X,Y)|^2 + < U(X,X), U(Y,Y) > \\ &= \frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + \frac{1}{2} < [X, [X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y > \\ &\quad + \frac{1}{2} < [Y, [Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X > . \end{aligned}$$

But

$$\begin{aligned} \frac{1}{2} < [X, [X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y > &= \frac{1}{2} < [X, [X,Y]_{\mathfrak{h}} + [X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, Y > \\ &= \frac{1}{2} < [X, [X,Y]_{\mathfrak{h}}]_{\mathfrak{m}}, Y > + \frac{1}{2} < [X, [X,Y]_{\mathfrak{m}}]_{\mathfrak{m}}, Y > \\ &= -\frac{1}{2} < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{2}|[X,Y]_{\mathfrak{m}}|^2. \end{aligned}$$

And calculating the other term gives

$$\begin{aligned} \frac{1}{2} < [Y, [Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X > &= \frac{1}{2} < [Y, [Y,X]_{\mathfrak{h}}]_{\mathfrak{m}}, X > + \frac{1}{2} < [Y, [Y,X]_{\mathfrak{m}}]_{\mathfrak{m}}, X > \\ &= \frac{1}{2} < Y, [[Y,X]_{\mathfrak{h}}, X]_{\mathfrak{m}} - \frac{1}{2} < [Y,X]_{\mathfrak{m}}, [Y,X]_{\mathfrak{m}} > \\ &= -\frac{1}{2} < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{2} < [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}} > \\ &= -\frac{1}{2} < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{2}|[X,Y]_{\mathfrak{m}}|^2. \end{aligned}$$

Returning to the original expression, we get

$$\begin{aligned} < R(X,Y)Y, X >_p &= \frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 + \frac{1}{2} < [X, [X,Y]_{\mathfrak{g}}]_{\mathfrak{m}}, Y > \\ &\quad + \frac{1}{2} < [Y, [Y,X]_{\mathfrak{g}}]_{\mathfrak{m}}, X > \\ &= \frac{3}{4}|[X,Y]_{\mathfrak{m}}|^2 - \frac{1}{2} < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{2}|[X,Y]_{\mathfrak{m}}|^2 \end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2} < [[X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > -\frac{1}{2}|[X,Y]_{\mathfrak{m}}|^2 \\
= \quad & -\frac{1}{4}|[X,Y]_{\mathfrak{m}}|^2 - < [X,Y]_{\mathfrak{h}}, X]_{\mathfrak{m}}, Y > .
\end{aligned}
$$

□

**Lemma 1.59.** *For a naturally reductive homogeneous space $M = G/H$, with Lie-subspace $\mathfrak{m}$, if $X, Y \in \mathfrak{m}$ then $\nabla_X Y = -[X,Y]/2$.*

*Proof.* This follows from Proposition 1.54, with $U = 0$ since $M$ is naturally reductive. □

**Theorem 1.60.** *If $M = G/H$ is naturally reductive, the geodesics starting at $p = H \in M$, with tangent vector $d\pi X \in T_pM$ are given by*

$$\lambda_{d\pi X}(t) = e^{tX}p = \pi e^{tX},$$

*for all $t \in \mathbb{R}$, and $X \in \mathfrak{m}$, i.e. geodesics are given as orbits of $p$ under the action of one parameter subgroups of vectors in $\mathfrak{m}$.*

*Proof.* [21] The one parameter subgroup $e^{tX}$ is horizontal since it is the integral curve of $X \in \mathfrak{m}$. Using Lemma 1.59 we get that $\alpha(t) = e^{tX}$ is a geodesic:

$$\nabla_{\dot{\alpha}(t)}\dot{\alpha}(t) = \nabla_X X = -\frac{1}{2}[X,X] = 0.$$

By Proposition 1.37 a Riemannian submersion carries horizontal geodesics to geodesics, which proves the claim. □

# Chapter 2

# Classification Theorems

## 2.1 The Ambrose-Singer Theorem

This section is dedicated to proving a characterization of homogeneous manifolds, first given by W. Ambrose and I.M Singer in 1958 [1]. The proof presented here follows the outline given in [25].

**Definition 2.1.** An *affine* transformation of a connection $\tilde{\nabla}$ is a diffeomorphism $\varphi : M \to M$ such that

$$d\varphi(\tilde{\nabla}_X Y) = \tilde{\nabla}_{(d\varphi X)}(d\varphi Y),$$

for all $X, Y \in C^\infty(M)$.

**Proposition 2.2.** [19] *Let $(M, g)$ be a connected Riemannian manifold and let $\varphi$ be an affine transformation with respect to the connection $\tilde{\nabla}$. Suppose there exists $p \in M$ such that $d\varphi : T_pM \to T_{\varphi(p)}M$ is an isometry, then $\varphi$ is an isometry.*

*Proof.* [19] We will show that $d\varphi$ is isometric at any $q \in M$. Let $\gamma$ be any smooth curve from $q$ to $p$ and let $P_\gamma$ be the parallel transport along $\gamma$. Then we have

$$\begin{aligned} < X, Y >_q &= < P_\gamma X, P_\gamma Y >_p \\ &= < d\varphi(P_\gamma X), d\varphi(P_\gamma Y) >_{\varphi(p)} \\ &= < P_{\varphi(\gamma)} d\varphi(X), P_{\varphi(\gamma)} d\varphi(Y) >_{\varphi(p)} \\ &= < d\varphi(X), d\varphi(Y) >_{\varphi(q)}, \end{aligned}$$

for any $X, Y \in T_qM$. This shows that $\varphi$ is isometric at $q$, and since $q$ was arbitrary, $\varphi$ is an isometry of $M$. □

**Proposition 2.3.** *Let $(M, g)$ be a complete Riemannian manifold. Then each metric connection $\tilde{\nabla}$ is complete, i.e. every $\tilde{\nabla}$-geodesic on $M$ is defined on all elements of $\mathbb{R}$.*

*Proof.* See [25]. □

**Definition 2.4.** Let $(M, g)$ be a connected Riemannian manifold. A tensor-field $D$ of type $(1, 2)$ (we write $D_X Y := D(X, Y)$) is called an Ambrose-Singer tensor field if it satisfies the following conditions:

$$(A.S) \quad \begin{cases} g(D_X Y, Z) + g(Y, D_X Z) = 0, \\ (\nabla_X R)(Y, Z) = [D_X, R(Y, Z)] - R(D_X Y, Z) - R(Y, D_X Z), \\ (\nabla_X D)_Y = [D_X, D_Y] - D_{D_X Y}, \end{cases}$$

for $X, Y, Z \in C^\infty(M)$, where $\nabla$ is the Levi-Civita connection on $M$, and $R$ the Riemannian curvature tensor on $M$. A homogeneous (Riemannian) structure on $(M, g)$ is the triple $(M, g, D)$, where $D$ is an Ambrose-Singer tensor field.

Let $\tilde{\nabla}$ denote the difference

$$\tilde{\nabla} = \nabla - D,$$

then the conditions of (A.S) are equivalent to

$$\begin{cases} \tilde{\nabla} g = 0, \\ \tilde{\nabla} R = 0, \\ \tilde{\nabla} D = 0, \end{cases} \tag{2.1}$$

This equivalence is due to the following three equations ([25]):

$$(\tilde{\nabla}_W g)(X, Y) = g(D_W X, Y) + g(X, D_W Y), \tag{2.2}$$

$$\begin{aligned} (\tilde{\nabla}_W R)(X, Y) &= (\nabla_W R)(X, Y) - [D_W, R(X, Y)] & (2.3) \\ &\quad + R(D_W X, Y) + R(X, D_W Y), & (2.4) \end{aligned}$$

$$(\tilde{\nabla}_W D)_X = (\nabla_W D)_X - [D_W, D_X] + D_{D_W X}, \tag{2.5}$$

for $X, Y, Z \in C^\infty(TM)$.

Equation (2.2) follows from $\nabla$ being metric. (2.5) and (2.3) are obtained by writing $\tilde{\nabla} = \nabla - D$ and expanding. The equivalence now follows immediately from (A.S).

$\tilde{\nabla}_X Y$ is linear in both $X, Y$, and tensorial in $X$ (over $C^\infty(M)$) since $\nabla$ and $D$ are. In the second argument we get

$$\begin{aligned} \tilde{\nabla}_X(fY) &= \nabla_X(fY) - D_X(fY) \\ &= f\nabla_X Y + (Xf)Y - fD_X Y \\ &= f\tilde{\nabla}_X Y + (Xf)Y, \end{aligned}$$

for $f \in C^\infty$, so $\tilde{\nabla}$ is a connection on $M$. For $\tilde{\nabla}$, the curvature $\tilde{R}$ and torsion $\tilde{T}$ are defined in the usual way:

$$\tilde{R}(X, Y) = [\tilde{\nabla}_X, \tilde{\nabla}_Y] - \tilde{\nabla}_{[X,Y]},$$
$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y].$$

We shall need the following identity, which we state as a lemma.

**Lemma 2.5.** *For $X, Y \in M$, we have*

$$R(X, Y) = \tilde{R}(X, Y) + [D_X, D_Y] + D_{\tilde{T}(X,Y)}.$$

*Proof.* First we note that since $\tilde{\nabla} D = 0$, we get

$$0 = (\tilde{\nabla}_X D_Y)(Z) = \tilde{\nabla}_X(D_Y Z) - D_{\tilde{\nabla}_X Y} Z - D_Y(\tilde{\nabla}_X Z),$$

so

$$\tilde{\nabla}_X D_Y - D_Y \tilde{\nabla}_X = D_{\tilde{\nabla}_X Y}. \tag{2.6}$$

Since $D = \nabla - \tilde{\nabla}$ we have $\nabla = D + \tilde{\nabla}$, and so we get (using (2.6) in the fifth equality below)

$$\begin{aligned} R(X, Y) &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \\ &= (D_X + \tilde{\nabla}_X)(D_Y + \tilde{\nabla}_Y) - (D_Y + \tilde{\nabla}_Y)(D_X + \tilde{\nabla}_X) - D_{[X,Y]} - \tilde{\nabla}_{[X,Y]} \\ &= (\tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}) + D_X \tilde{\nabla}_Y + D_X D_Y + \tilde{\nabla}_X D_Y - \tilde{\nabla}_Y D_X \\ &\quad - D_Y D_X - D_Y \tilde{\nabla}_X - D_{[X,Y]} \\ &= \tilde{R}(X, Y) + (D_X D_Y - D_Y D_X) + (D_X \tilde{\nabla}_Y - \tilde{\nabla}_Y D_X) \\ &\quad + (\tilde{\nabla}_X D_Y - D_Y \tilde{\nabla}_X) - D_{[X,Y]} \\ &= \tilde{R}(X, Y) + [D_X, D_Y] + D_{\tilde{\nabla}_X Y} - D_{\tilde{\nabla}_Y X} - D_{[X,Y]} \\ &= \tilde{R}(X, Y) + [D_X, D_Y] + D_{\tilde{T}(X,Y)}. \end{aligned}$$

□

Notice that the torsion $\tilde{T}(X,Y)$ of $\tilde{\nabla}$ is given by $D_Y X - D_X Y$:

$$\begin{aligned} D_Y X - D_X Y &= (\nabla_Y X - \tilde{\nabla}_Y X) - (\nabla_X Y - \tilde{\nabla}_X Y) \\ &= (\nabla_Y X - \nabla_X Y) + (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) \\ &= -[X,Y] + \tilde{\nabla}_X Y - \tilde{\nabla}_Y X = \tilde{T}(X,Y). \end{aligned}$$

We can therefore write $\tilde{R}$ as

$$\tilde{R}(X,Y) = R(X,Y) - [D_X, D_Y] - D_{D_Y X - D_X Y},$$

and the covariant derivative of $\tilde{R}$ is given by

$$\begin{aligned} (\tilde{\nabla}_W \tilde{R})(X,Y) &= (\tilde{\nabla}_W R)(X,Y) - [D_X, (\tilde{\nabla}_W D)_Y] + [D_Y, (\tilde{\nabla}_W D)_X] \\ &\quad + (\tilde{\nabla}_W D)_{D_X Y} - (\tilde{\nabla}_W D)_{D_Y X} + D_{(\tilde{\nabla}_W D)_X Y} - D_{(\tilde{\nabla}_W D)_Y X}. \end{aligned}$$

Since we already have $\tilde{\nabla} D = 0$, this means that $\tilde{\nabla} R = 0$ if and only if $\tilde{\nabla}\tilde{R} = 0$, and the (A.S) conditions are therefore equivalent to

$$\begin{cases} \tilde{\nabla} g = 0, \\ \tilde{\nabla}\tilde{R} = 0, \\ \tilde{\nabla} D = 0. \end{cases} \tag{2.7}$$

Now if $(M,g)$ is a naturally reductive homogeneous space then the associated canonical connection of Proposition 1.42, $\tilde{\nabla}$, is a metric connection with parallel curvature tensor and parallel torsion tensor. Therefore by Equation (2.7) $D = \nabla - \tilde{\nabla}$ satisfies (A.S) and thus defines a homogeneous structure on $(M,g)$.

For later use we sum up some of the above results in a proposition:

**Proposition 2.6.** [16] *For a naturally reductive homogeneous space $M = G/H$, with Levi-Civita connection $\nabla$ and cannonical connection $\tilde{\nabla}$, the following relations hold*

$$\tilde{R}(X,Y)\cdot g = \tilde{R}(X,Y)\cdot T = \tilde{R}(X,Y)\cdot \tilde{T} = \tilde{R}(X,Y)\cdot \tilde{R} = 0, \tag{2.8}$$

$$\tilde{R}(X,Y)\cdot R = \tilde{R}(X,Y)\cdot(\nabla^k R) = 0, \tag{2.9}$$

*for any $X,Y \in C^\infty(TM)$, where $\tilde{R}(X,Y)$ acts at any fixed point $p \in M$ as a derivation of the corresponding tensor algebra.*

*Proof.* This follows from the above identities involving $\tilde{\nabla}$ and from the definition of $\tilde{R}$:

$$\tilde{R}(X,Y) = \tilde{\nabla}_X \tilde{\nabla}_Y - \tilde{\nabla}_Y \tilde{\nabla}_X - \tilde{\nabla}_{[X,Y]}.$$ □

**Lemma 2.7.** [18] *A connection $\tilde{\nabla}$ is invariant under parallelism if and only if its curvature and torsion tensors $\tilde{R}$ and $\tilde{T}$, respectively, are parallel i.e. $\tilde{\nabla}\tilde{R} = \tilde{\nabla}\tilde{T} = 0$.*

*Proof.* See [18]. □

**Corollary 2.8.** $\tilde{\nabla} = \nabla - D$ *is invariant under parallelism.*

**Definition 2.9.** A Riemannian manifold $(M,g)$ is *locally homogeneous* if for all $p,q \in M$, there exist open sets $U,V \subset M$, such that $p \in U$, $q \in V$ and there exists an isometry $\varphi : U \to V$ such that $\varphi(p) = q$.

We note two technical lemmas from [13] which we need to prove one direction of the Ambrose-Singer theorem.

**Lemma 2.10.** [13] *Let $M$ and $M'$ be differentiable manifolds with linear connections. Let $T$, $R$ and $\nabla$ (resp. $T'$, $R'$ and $\nabla'$) be the torsion, the curvature and the covariant differentiation of $M$ (resp. $M'$). Assume $\nabla T = 0$, $\nabla R = 0$, $\nabla' T' = 0$ and $\nabla' R' = 0$. If $F$ is a linear isomorphism of $T_{x_0}(M)$ onto $T_{y_0}(M')$ and maps the tensors $T_{x_0}$ and $R_{x_0}$ at $x_0$ into the tensors $T'_{y_0}$ and $R'_{y_0}$ respectively, then there is an affine isomorphism $f$ of a neighborhood $U$ of $x_0$ onto a neighborhood $V$ of $y_0$ such that $f(x_0) = y_0$ and that the differential of $f$ at $x_0$ coincides with $F$.*

**Lemma 2.11.** [13] *In Lemma 2.10 if $M$ and $M'$ are, moreover, connected, simply connected and complete then there exists a unique affine isomorphism $f$ of $M$ onto $M'$ such that $f(x_0) = y_0$ and that the differential of $f$ at $x_0$ coincides with $F$.*

**Theorem 2.12.** [1], [25] *Let $(M, g)$ be a connected Riemannian manifold and assume that there exists a tensor field $D$ of type $(1, 2)$ satisfying the (A.S) conditions. Then $(M, g)$ is locally homogeneous.*

*Proof.* [25] Set $\tilde{\nabla} = \nabla - D$. Let $p, q \in M$, and $\gamma(t)$ be a piecewise $C^\infty$ curve with $\gamma(0) = p, \gamma(1) = q$, and let $\tau_{pq}$ be the parallel transport with respect to $\tilde{\nabla}$ along $\gamma(t)$. Since $\tilde{\nabla}$ is metric and invariant under parallelism by Corollary 2.8, $\tau_{pq}$ is an isometry $T_pM \simeq T_qM$, which preserves $\tilde{R}$, $\tilde{T}$. Lemma 2.10 implies that there exist open sets $U, V \subset M$ with $p \in U$, $q \in V$, and an affine transformation $\varphi$ of $\tilde{\nabla}$, such that

$$\varphi : U \to V, \quad p \mapsto q$$
$$d\varphi|_p = \tau_{pq}.$$

Since $\tau_{pq}$ is an isometry, Proposition 2.2 implies that $\varphi$ is an isometry $U \to V$, i.e. a local isometry. □

**Theorem 2.13.** [1], [25] *Let $(M, g)$ be a connected, simply connected and complete Riemannian manifold satisfying the conditions of Theorem 2.12. Then $(M, g)$ is homogeneous.*

*Proof.* [25] Proposition 2.3 implies that $\tilde{\nabla}$ is complete. Lemma 2.11 implies that the local isometry of Theorem 2.12 can be exteded to a global isometry. □

We shall now proceed to prove the converse of the Ambrose-Singer theorem, we shall need the following lemma:

**Lemma 2.14.** [14] *Let $M = G/H$ be a reductive homogeneous space. If a tensor field $D$ is invariant by $G$, then it is parallel with respect to the canonical connection $\tilde{\nabla}$.*

*Proof.* [14] By definition the canonical connection is the unique connnection on $M$ satisfying the conditions of Proposition 1.42, namely that parallel translation of vectors along $x^*(s)$ should coincide with the translation given by $x(s)$, where $x(s)$ is a one parameter subgroup of $G$ induced by $X \in \mathfrak{g}$, and $x^*(s)$ is the projection of $x(s)$ to $M = G/H$. Therefore $\tilde{\nabla} D = 0$ at $p = H \in M$, and therefore, again by $G$-invariance, $\tilde{\nabla} D = 0$ in all of $M$. □

**Theorem 2.15.** [1], [25] *Let $(M, g)$ be a homogeneous Riemannian manifold. Then there exists a tensor field $D$ of type $(1, 2)$ satisfying the (A.S) conditions.*

*Proof.* [25] Recall from Corollary 1.49 that any homogeneous Riemannian manifold is reductive. Let $\tilde{\nabla}$ be the associated canonical connection. By Proposition 1.42 $\tilde{\nabla}$ is $G$-invariant, and since $G$ acts by isometries, so is $\nabla$. This implies that the difference $D = \nabla - \tilde{\nabla}$ is a $G$-invariant tensor field, and so by Lemma 2.14 $D$ is parallel with respect to the canonical connection: $\tilde{\nabla} D = 0$. Therefore $\tilde{\nabla}$ and $D$ together satisfy equation (2.7), which implies that $D$ is an Ambrose-Singer tensor field on $M$. □

Note that the converse Theorem is actually stronger: we need only assume that $M$ is a homogeneous Riemannian manfold with no further topological conditions.

## 2.2 The Classification of The Four-Dimensional Naturally Reductive Homogeneous Spaces

In this section we present a classification of the four dimensional naturally reductive homogeneous spaces, following the work of O. Kowalski and L. Vanhecke in [16]. Throughout this section $\tilde{\nabla}$ shall denote the canonical connection of a given space, and $\tilde{T}$ and $\tilde{R}$ its torsion and curvature tensors, respectively.

**Theorem 2.16.** [14] *Let $T$, $R$ be the torsion and curvature tensors of a linear connection $\nabla$ of $M$. Then for $X, Y, Z \in T_pM$ we have*

$$\mathfrak{S}_{X,Y,Z}(R(X,Y)Z) = \mathfrak{S}_{X,Y,Z}(T(T(X,Y),Z) + (\nabla_X T)(Y,Z)),$$

$$\mathfrak{S}_{X,Y,Z}((\nabla_X R)(Y,Z) + R(T(X,Y),Z)) = 0,$$

*where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum over $X, Y, Z$. The first equation is called the first Bianchi identity, and the second is called the second Bianchi identity.*

With the canonical connection $\tilde{\nabla}$ the two Bianchi identities then reduce to

$$\mathfrak{S}_{X,Y,Z}(\tilde{R}(X,Y)Z) = \mathfrak{S}_{X,Y,Z}(\tilde{T}(\tilde{T}(X,Y),Z)) \tag{2.10}$$

$$\mathfrak{S}_{X,Y,Z}(\tilde{R}(\tilde{T}(X,Y),Z)) = 0. \tag{2.11}$$

**Lemma 2.17.** [16] *Let $M = G/H$ be a naturally reductive homogeneous space, and let $D$ be the associated Ambrose-Singer tensor field, and $\tilde{T}$ the torsion tensor of the canonical connection $\tilde{\nabla}$ on $M$. The following identity holds at any point $p \in M$ and for any vectors $X, Y \in T_pM$.*

$$D_X Y = -\frac{1}{2}\tilde{T}(X,Y). \tag{2.12}$$

*Proof.* [16] First, as calculated earlier, since the Levi-Civita connection is symmetric, we get

$$\begin{aligned} D_X Y - D_Y X &= (\nabla_X Y - \tilde{\nabla}_X Y) - (\nabla_Y X - \tilde{\nabla}_Y X) \\ &= [X,Y] - (\tilde{\nabla}_X Y - \tilde{\nabla}_Y X) \\ &= -(\tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]) \\ &= -\tilde{T}(X,Y). \end{aligned}$$

From Proposition 1.44 we know that $\tilde{T}$ satisfies

$$\tilde{T}(X,Y) = -[X,Y]_{\mathfrak{m}},$$

for $X, Y \in \mathfrak{m}$.

Hence, using the fact that $M$ is naturally reductive, we get:

$$\begin{aligned} 0 &= < [X,Y]_{\mathfrak{m}}, Z > + < Y, [X,Z]_{\mathfrak{m}} > \\ &= - < \tilde{T}(X,Y), Z) > - < Y, \tilde{T}(X,Z) > \\ &= < D_X Y - D_Y X, Z > + < D_X Z - D_Z X, Y > \\ &= < D_X Y, Z > + < D_X Z, Y > - < D_Y X, Z > - < D_Z X, Y > . \end{aligned}$$

But by (A.S)

$$< D_X Y, Z > + < D_X Z, Y >= 0,$$

so continuing (using that $- < D_Y X, Z >=< X, D_Y Z >$), we get

$$\begin{aligned} 0 &= - < D_Y X, Z > - < D_Z X, Y >=< X, D_Y Z > + < X, D_Z Y > \\ &= < X, D_Y Z + D_Z Y > . \end{aligned}$$

This holds for all $X \in \mathfrak{m}$, so $D_Y Z + D_Z Y = 0$. Adding the equations

$$D_Y Z + D_Z Y = 0,$$

$$D_Y Z - D_Z Y = -\tilde{T}(Y,Z),$$

we get

$$2D_Y Z = -\tilde{T}(Y,Z).$$

□

**Corollary 2.18.** [16] *Let $M$ be a naturally reductive homogeneous space. Then we have*

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\tilde{T}(X,Y).$$

*Proof.* [16] The claim is immediate from

$$\nabla_X Y - \tilde{\nabla}_X Y = D_X Y = -\frac{1}{2}\tilde{T}(X,Y).$$

□

We now wish to show that $(\nabla_W R)(X,Y)Z$ can be written solely in terms of $\tilde{T}$ and $\tilde{R}$. The first step is to transform the expression in Lemma 2.5 using Lemma 2.17. We get

$$\begin{aligned} R(X,Y)Z &= \tilde{R}(X,Y)Z + [D_X, D_Y]Z + D_{\tilde{T}(X,Y)}Z \\ &= \tilde{R}(X,Y)Z + \frac{1}{4}[\tilde{T}(X,\tilde{T}(Y,Z)) - \tilde{T}(Y,\tilde{T}(X,Z))] \\ &\quad -\frac{1}{2}\tilde{T}(\tilde{T}(X,Y),Z). \end{aligned}$$

Using the skew-symmetry of $\tilde{T}$ and the first Bianchi identity we get the following string of identities:

$$\begin{aligned} & -\frac{1}{4}\mathfrak{S}_{(X,Y,Z)}\tilde{R}(X,Y)Z + \frac{1}{4}\tilde{T}(Z,\tilde{T}(X,Y)) \\ =\ & -\frac{1}{4}\mathfrak{S}_{(X,Y,Z)}\tilde{T}(\tilde{T}(X,Y),Z) + \frac{1}{4}\tilde{T}(Z,\tilde{T}(X,Y)) \\ =\ & -\frac{1}{4}\tilde{T}(\tilde{T}(X,Y),Z) - \frac{1}{4}\tilde{T}(\tilde{T}(Y,Z),X) - \frac{1}{4}\tilde{T}(\tilde{T}(Z,X),Y) \\ & -\frac{1}{4}\tilde{T}(\tilde{T}(X,Y),Z) \\ =\ & -\frac{1}{2}\tilde{T}(\tilde{T}(X,Y),Z) - \frac{1}{4}[\tilde{T}(\tilde{T}(Y,Z),X) + \tilde{T}(\tilde{T}(Z,X),Y)] \\ =\ & -\frac{1}{2}\tilde{T}(\tilde{T}(X,Y),Z) + \frac{1}{4}[\tilde{T}(X,\tilde{T}(Y,Z)) + \tilde{T}(Y,\tilde{T}(Z,X))] \\ =\ & -\frac{1}{2}\tilde{T}(\tilde{T}(X,Y),Z) + \frac{1}{4}[\tilde{T}(X,\tilde{T}(Y,Z)) - \tilde{T}(Y,\tilde{T}(X,Z))]. \end{aligned}$$

Therefore, tracing back the equalities, and substituting with the new expression for $R(X,Y)Z$ we get

$$R(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{4}\mathfrak{S}_{(X,Y,Z)}\tilde{R}(X,Y)Z + \frac{1}{4}\tilde{T}(Z,\tilde{T}(X,Y)). \tag{2.13}$$

Since $\tilde{\nabla}R = ... = \tilde{\nabla}\nabla^k R = 0$ $(k = 0, 1...)$, we can get by induction (see [16]) that

$$D^k R = (\nabla - \tilde{\nabla})^k R = \nabla^k R, \quad (k = 0, 1...)$$

Then we can prove that $(\nabla_U R)(X,Y)Z$ has the desired form:

**Proposition 2.19.** [16] *$(\nabla_U R)(X,Y)Z$ can be written as a sum of compositions of $\tilde{T}$ and $\tilde{R}$.*

*Proof.* [16] We have

$$\begin{aligned} (\nabla_U R)(X,Y)Z &= \nabla R(U,(X,Y)Z) = (\nabla - \tilde{\nabla})R(U,(X,Y)Z) \\ &= ((\nabla_U - \tilde{\nabla}_U)R)(X,Y)Z \\ &= [\nabla_U(R(X,Y)Z) - R(\nabla_U X,Y)Z - R(X,\nabla_U Y)Z \\ &\quad -R(X,Y)\nabla_U Z] - [\tilde{\nabla}_U(R(X,Y)Z) - R(\tilde{\nabla}_U X,Y)Z \end{aligned}$$

$$
\begin{aligned}
&-R(X,\tilde{\nabla}_U Y)Z - R(X,Y)\tilde{\nabla}_U Z]\\
=\ & [\nabla_U(R(X,Y)Z) - \tilde{\nabla}_U(R(X,Y)Z)] + [-(R(\nabla_U X,Y)Z\\
&+R(\tilde{\nabla}_U X,Y)Z] + [-(R(X,\nabla_U Y)Z + R(X,\tilde{\nabla}_U Y)Z]\\
&+[-(R(X,Y)\nabla_U Z + R(X,Y)\tilde{\nabla}_U Z]\\
=\ & (\nabla_U - \tilde{\nabla}_U)(R(X,Y)Z) - R((\nabla_U - \tilde{\nabla}_U)X,Y)Z\\
&-R(X,(\nabla_U - \tilde{\nabla}_U)Y)Z - R(X,Y)(\nabla_U - \tilde{\nabla}_U)Z)\\
=\ & D_U(R(X,Y)Z) - R(D_U X,Y)Z - R(X,D_U Y)Z\\
&-R(X,Y)D_U Z\\
=\ & -\frac{1}{2}\tilde{T}(U,(R(X,Y)Z)) - R(-\frac{1}{2}\tilde{T}(U,X),Y)Z\\
&-R(X,-\frac{1}{2}\tilde{T}(U,Y))Z + \frac{1}{2}R(X,Y)\tilde{T}(U,Z).
\end{aligned}
$$

Using Equation (2.13) we can replace each occurrence of $R$ in the above expression with a sum of terms involving only $\tilde{R}$ and $\tilde{T}$. The proposition follows. □

**Corollary 2.20.** [16] *Any covariant derivative*

$$(\nabla^k_{U_1,\dots,U_k}R)(X,Y)Z, \tag{2.14}$$

*can be expressed as a sum of compositions of the tensors $\tilde{R}$ and $\tilde{T}$.*

*Proof.* [16] We've already established the cases $k = 0, 1$. The general case follows by induction using Proposition 2.19 and

$$\nabla^k R = D^k R, \quad k \in \mathbb{N}.$$

We leave the details to the reader. □

We note that an explicit formula for the $k = 1$ case is given in [16].

**Theorem 2.21.** *Let $(M, g)$ be a simply connected naturally reductive homogeneous space with the decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$ given. Suppose that $T_pM \simeq \mathfrak{m}$ admits an orthogonal decomposition*

$$T_pM = V_1 \oplus V_2,$$

*such that for the canonical projections $\pi_1, \pi_2$, the following conditions hold:*

$$\begin{cases} \pi_i\tilde{T}(X,Y) = \tilde{T}(\pi_i X, \pi_i Y), \\ \pi_i\tilde{R}(X,Y)Z = \tilde{R}(\pi_i X, \pi_i Y)(\pi_i Z), \end{cases} \tag{2.15}$$

*for $X, Y, Z \in T_pM$, $i = 1, 2$. Then $M$ is a Riemannian direct product:*

$$(M, g) = (M_1, g_1) \times (M_2, g_2),$$

*with $dim(M_i) = dim(V_i)$, $i = 1, 2$.*

*Proof.* If the tensors $\tilde{R}$ and $\tilde{T}$ satisfy conditions (2.15) with respect to such a decomposition $V_1, V_2$, then it follows that any sum of compositions of the tensors will also leave the subspaces $V_1, V_2$ invariant. From this and Corollary 2.20 it follows that any covariant derivative

$$(\nabla^k_{U_1,\dots,U_k}R)(X,Y)Z,$$

will leave $V_1, V_2$, invariant. But recall from Theorem 1.51 that the endomorphisms $\nabla^k R$, $k \in \mathbb{N}$ generate the Lie algebra of the holonomy group of $M$ at $p$ .This in turn implies that the subspaces $V_1, V_2$ are invariant under the action of the holonomy group, and $(M, g)$ therefore decomposes in accordance with the de Rham decomposition theorem. □

**Theorem 2.22.** [14] *Let $(M, g) = (M_1, g_1) \times (M_2, g_2)$ be a homogeneous manifold. Then $(M, g)$ is naturally reductive if and only if both factors $(M_i, g_i)$ are naturally reductive.*

*Proof.* See [14]. □

Now, having established the above results, we can proceed more directly towards obtaining the classification. Let $(M, g)$ be a 4-dimensional simply connected naturally reductive homogeneous space, with $p \in M$ its origin. Let $\{X_1, X_2, X_3, X_4\}$ be some orthonomal basis for $T_pM$. The operator $\tilde{T}$ is then determined at $p$ by $3^n = 3^4$ constants $t_{ij}^k$:

$$\tilde{T}(X_i, X_j) = \sum_{k=1}^{4} t_{ij}^k X_k, \quad i, j = 1, ..., 4.$$

Remember that for $X, Y \in T_pM$, $\tilde{T}$ satisfies

$$\tilde{T}(X, Y) = -[X, Y]_{\mathfrak{m}}.$$

Combining this identity with the criterion

$$< [X, Y]_{\mathfrak{m}}, Z > + < [X, Z]_{\mathfrak{m}}, Y >= 0, \quad X, Y, Z \in T_pM,$$

we conclude that $t_{ij}^k + t_{ik}^j = 0$, as well as $t_{ij}^k + t_{ji}^k = 0$, the last of which also follows from the skew symmetry of $\tilde{T}$. From these two identities it follows that

$$t_{ij}^i = t_{ij}^j = 0, \quad i, j = 1, ..., 4,$$

as $t_{ij}^i = -t_{ii}^j = t_{ii}^j = -t_{ij}^i$ and $t_{ij}^j = -t_{ij}^j$. Therefore, naming $a := t_{12}^3 = -t_{13}^2 = t_{23}^1$, and so on with $b, c, d$, we get the following table describing $\tilde{T}(X_i, X_j)$:

$$\begin{cases} \tilde{T}(X_1, X_2) = aX_3 + bX_4, & \tilde{T}(X_2, X_3) = aX_1 + dX_4, \\ \tilde{T}(X_1, X_3) = -aX_2 + cX_4, & \tilde{T}(X_2, X_4) = bX_1 - dX_3, \\ \tilde{T}(X_1, X_4) = -bX_2 - cX_3, & \tilde{T}(X_3, X_4) = cX_1 + dX_2, \end{cases} \tag{2.16}$$

**Lemma 2.23.** [16] *If there exists $X, Y \in T_pM$ such that $A = \tilde{R}(X, Y)$ is a curvature transformation such that*

$$\begin{cases} AX_1 = X_2, \\ AX_2 = -X_1, \\ AX_3 = AX_4 = 0, \end{cases}$$

*holds for some orthonormal basis $\{X_1, X_2, X_3, X_4\}$, then $(M, g)$ is either symmetric or a Riemannian product.*

*Proof.* [16] Recall that $A \cdot \tilde{T} = \tilde{R}(X, Y) \cdot \tilde{T} = 0$. The action is given by

$$A \cdot \tilde{T} = A(\tilde{T}(W, Z)) - \tilde{T}(AW, Z) - \tilde{T}(W, AZ). \tag{2.17}$$

Applying $A$ to the 6th equation in (2.16) we obtain

$$\begin{aligned} A \cdot \tilde{T}(X_3, X_4) &= A(cX_1 + dX_2) - \tilde{T}(AX_3, X_4) - \tilde{T}(X_3, AX_4) \\ &= cX_2 - dX_1 - \tilde{T}(0, X_4) - \tilde{T}(X_3, 0) = cX_2 - dX_1. \end{aligned}$$

Thus $A \cdot \tilde{T} = 0$ implies that $c = d = 0$.

We can now reduce Table 2.16 to

$$\begin{cases} \tilde{T}(X_1, X_2) = aX_3 + bX_4, & \tilde{T}(X_2, X_3) = aX_1, \\ \tilde{T}(X_1, X_3) = -aX_2, & \tilde{T}(X_2, X_4) = bX_1, \\ \tilde{T}(X_1, X_4) = -bX_2, & \tilde{T}(X_3, X_4) = 0. \end{cases} \tag{2.18}$$

If $a^2 + b^2 = 0$, then $\tilde{T} \equiv 0$, which implies that $\nabla = \tilde{\nabla}$, so $0 = \tilde{\nabla} R = \nabla R$, and since $M$ is simply connected it would be symmetric.

Assume therefore that $\rho = (a^2+b^2)^{\frac{1}{2}} > 0$. We can define a new othogonal - though in general not orthonormal - basis $\{X_1', X_2', X_3', X_4'\}$ by setting

$$X_1' = \frac{1}{\rho}X_1, \quad X_3' = \frac{1}{\rho^2}(aX_3 + bX_4),$$
$$X_2' = \frac{1}{\rho}X_2, \quad X_4' = \frac{1}{\rho^2}(-bX_3 + aX_4).$$

A case by case check shows that the table for $\tilde{T}$ in this basis becomes

$$\begin{cases} \tilde{T}(X_1', X_2') = X_3', & \tilde{T}(X_1', X_4') = 0, \\ \tilde{T}(X_1', X_3') = -X_2', & \tilde{T}(X_2', X_4') = 0, \\ \tilde{T}(X_2', X_3') = X_1', & \tilde{T}(X_3', X_4') = 0. \end{cases} \tag{2.19}$$

Now we apply the second Bianchi identity, $\mathfrak{S}_{(X,Y,Z)}\tilde{R}(\tilde{T}(X,Y),Z) = 0$, to the following three cases

$$\begin{cases} X = X_1', & Y = X_2', & Z = X_4', \\ X = X_2', & Y = X_3', & Z = X_4', \\ X = X_1', & Y = X_3', & Z = X_4'. \end{cases}$$

This yields:

$$\begin{aligned} 0 &= \mathfrak{S}_{(X_1',X_2',X_4')}\tilde{R}(\tilde{T}(X_1', X_2'), X_4') \\ &= \tilde{R}(\tilde{T}(X_1', X_2'), X_4') + \tilde{R}(\tilde{T}(X_2', X_4'), X_1') + \tilde{R}(\tilde{T}(X_4', X_1'), X_2') \\ &= \tilde{R}(X_3', X_4') + \tilde{R}(0, X_1') + \tilde{R}(0, X_2') = \tilde{R}(X_3', X_4'). \end{aligned}$$

$$\begin{aligned} 0 &= \mathfrak{S}_{(X_2',X_3',X_4')}\tilde{R}(\tilde{T}(X_2', X_3'), X_4') \\ &= \tilde{R}(\tilde{T}(X_2', X_3'), X_4') + \tilde{R}(\tilde{T}(X_3', X_4'), X_2') + \tilde{R}(\tilde{T}(X_4', X_2'), X_3') \\ &= \tilde{R}(X_1', X_4') + \tilde{R}(0, X_2') + \tilde{R}(0, X_3') = \tilde{R}(X_1', X_4'). \end{aligned}$$

$$\begin{aligned} 0 &= \mathfrak{S}_{(X_1',X_3',X_4')}\tilde{R}(\tilde{T}(X_1', X_3'), X_4') \\ &= \tilde{R}(\tilde{T}(X_1', X_3'), X_4') + \tilde{R}(\tilde{T}(X_3', X_4'), X_1') + \tilde{R}(\tilde{T}(X_4', X_1'), X_3') \\ &= \tilde{R}(-X_2', X_4') + \tilde{R}(0, X_1') + \tilde{R}(0, X_3') = \tilde{R}(-X_2', X_4'). \end{aligned}$$

So we have

$$\tilde{R}(X_3', X_4') = \tilde{R}(X_1', X_4') = \tilde{R}(X_2', X_4') = 0. \tag{2.20}$$

The first Bianchi identity, $\mathfrak{S}_{(X,Y,Z)}\tilde{R}(X,Y)Z = \mathfrak{S}_{(X,Y,Z)}\tilde{T}(\tilde{T}(X,Y),Z)$ applied to the same three cases then reduces to

$$\begin{cases} \mathfrak{S}_{(X_1',X_2',X_4')}\tilde{R}(X_1', X_2')X_4' = \tilde{R}(X_1', X_2')X_4' = \mathfrak{S}_{(X_1',X_2',X_4')}\tilde{T}(\tilde{T}(X_1', X_2'), X_4') = 0, \\ \mathfrak{S}_{(X_2',X_3',X_4')}\tilde{R}(X_2', X_3')X_4' = \tilde{R}(X_2', X_3')X_4' = \mathfrak{S}_{(X_2',X_3',X_4')}\tilde{T}(\tilde{T}(X_2', X_3'), X_4') = 0, \\ \mathfrak{S}_{(X_1',X_3',X_4')}\tilde{R}(X_1', X_3')X_4' = \tilde{R}(X_1', X_3')X_4' = \mathfrak{S}_{(X_1',X_3',X_4')}\tilde{T}(\tilde{T}(X_1', X_3'), X_4') = 0, \end{cases}$$

since $\tilde{T}(X_i', X_4') = 0$, for $i = 1, 2, 3$ by (2.19). Summing up we have

$$\tilde{R}(X_i', X_4') = 0, \quad \tilde{R}(X_i', X_j')X_4' = 0, \quad i, j = 1, 2, 3.$$

This, together with 2.19 implies that $\tilde{R}$ and $\tilde{T}$ satisfy (2.15), with the decomposition

$$T_pM = V_1 \oplus V_2 = span(\{X_1^{'}, X_2^{'}, X_3^{'}\}) \oplus span(\{X_4^{'}\}).$$

Therefore $(M, g) = (M_1, g_1) \times (M_2, g_2)$, with $dim(M_2) = dim(V_2) = 1$, and so since $\pi_1(X \times Y, (x_0, y_0)) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$, we must have $M_2 = \mathbb{R}$ for $M$ to be simply connected and homogeneous. By Theorem 2.22, $M = M_3 \times \mathbb{R}$, for some 3 dimensional naturally reductive simply connected manifold $M_3$. ☐

**Lemma 2.24.** [16] *Suppose there exists $X, Y \in T_pM$ such that $\tilde{R}(X,Y) = \lambda A + \mu B$, $\lambda\mu \neq 0$ with $A, B : T_pM \to T_pM$, such that*

$$AX_1 = X_2, \quad AX_2 = -X_1, \quad AX_3 = AX_4 = 0,$$
$$BX_1 = BX_2 = 0, \quad BX_3 = X_4, \quad BX_4 = -X_3,$$

*for some othonormal basis $\{X_1, X_2, X_3, X_4\}$. Then $(M, g)$ is symmetric.*

*Proof.* As in the proof of Lemma 2.23, we get $\lambda c = \lambda d = 0$, but similarly also $\mu a = \mu b = 0$, so $a = b = c = d = 0$, and therefore $\tilde{T} \equiv 0$. This gives, as before, that $M$ is symmetric. ☐

We can now prove the classification of the four dimensional simply connected naturally reductive homogeneous spaces:

**Theorem 2.25.** [16] *Let $(M, g)$ be a four dimensional, simply connected naturally reductive homogeneous space. Then either $M$ is symmetric or it is a Riemannian product $M = M_3 \times \mathbb{R}$, where $M_3$ is a naturally reductive homogeneous space, isometric to one of the following spaces:*

1. *$SU(2)$ with a special left-invariant metric,*
2. *the universal covering of $SL(2, \mathbb{R})$ with a special left-invariant metric,*
3. *the Heisenberg group with a left-invariant metric.*

*Proof.* [16] If $\tilde{R} \equiv 0$, then $(M, g)$ is symmetric since $\nabla R = 0$ by the explicit formula for $\nabla_U R(X,Y)Z$ given above.

Suppose therefore that $\tilde{R}(X, Y) \neq 0$ for some $X, Y \in T_pM$. Define

$$h(U, V) = g(\tilde{R}(X, Y)U, V).$$

Since we know that $\tilde{R}(X, Y) \cdot g(U, V) = 0$, where $\tilde{R}(X, Y)$ acts as $[X, Y]_{\mathfrak{m}} \in T_pM$, we get

$$0 = \tilde{R}(X, Y)g(U, V) = g(\tilde{R}(X, Y)U, V) + g(U, \tilde{R}(X, Y)V),$$

so

$$h(V, U) = g(U, \tilde{R}(X, Y)V) = -g(\tilde{R}(X, Y)U, V) = -h(U, V),$$

and $h$ is an alternating 2-form on $T_pM$. Then there exists an orthonormal basis $\{\psi^1, \psi^2, \psi^3, \psi^4\}$ of $T_pM^*$ such $h$ can be written as either

$$h = \lambda\psi^1 \wedge \psi^2, \quad \lambda \neq 0,$$

or

$$h = \lambda\psi^1 \wedge \psi^2 + \mu\psi^3 \wedge \psi^4, \quad \lambda\mu \neq 0,$$

depending on the rank of $h$. The first case implies that $\tilde{R}(X, Y)$ satisfies Lemma 2.23, and the second case implies that $\tilde{R}(X, Y)$ satisfies Lemma 2.24. Thus if $(M, g)$ is not symmetric, it must be a direct product $M_3 \times \mathbb{R}$ of naturally reductive spaces. The last claim now follows from Theorem 2.26. ☐

**Theorem 2.26.** [25] *Let $(M, g)$ be a three-dimensional connected and simply connected homogeneous naturally reductive space. Then $(M, g)$ is either one of*

$$\mathbb{R}^3, \quad S^3 \, or \, \mathbb{H}^3,$$

*or it is isometric to one of the following Lie groups with a suitable left invariant metric:*

1. *$SU(2)$ with a special left-invariant metric,*
2. *the universal covering of $SL(2, \mathbb{R})$ with a special left-invariant metric,*
3. *the Heisenberg group with a left-invariant metric.*

*Proof.* See [25]. □

# Chapter 3

# Totally Geodesic Hypersurfaces and Sectional Curvature

## 3.1 The Theorem of Tojo

In [24] it is proved that for dimensions 3, 4 and 5, if an irreducible naturally reductive space contains a totally geodesic hypersurface, then the ambient space has constant sectional curvature. This theorem was generalized by K. Tsukada in [26] to cover any dimension. Tsukada's proof, which we shall cover in the next section, relies on Theorem 3.1. Before we state the Theorem we first need a definition. Recall from Proposition 1.54 that

$$(\nabla_X Y)_p = -\frac{1}{2}[X,Y]_{\mathfrak{m}} + U(X,Y),$$

for $X, Y \in \mathfrak{m}$. So in a naturally reductive space, $\Lambda(X,Y) = -\frac{1}{2}[X,Y]_{\mathfrak{m}}$. Each $X \in \mathfrak{m}$ then defines the function:

$$\Lambda(X) : Y \mapsto \Lambda(X,Y) = \frac{1}{2}[X,Y]_{\mathfrak{m}},$$

and for notational convenience we write $\varphi_X = \Lambda(X)$.

**Theorem 3.1.** [24] *Let $(M,g) = G/H$, be a naturally reductive homogeneous space and $V$ be a linear subspace of $\mathfrak{m}$. Let $R_o$ be the curvature tensor at $o \in M$, and $\varphi_X$ the connection function associated with $\nabla$. Then there exists a totally geodesic submanifold tangent to $V$ at $o$, if and only if for any $X \in V$, the following condition is satisfied:*

$$R_p(X, e^{-\varphi_X}(V))(e^{-\varphi_X}(V)) \subset e^{-\varphi_X}(V), \tag{3.1}$$

*where*

$$e^{-\varphi_X} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!}(\varphi_X)^l.$$

We remark that the theorem can be seen as an analog of the Lie tripple systems for symmetric spaces (see [9]).

Since $(M, <,>)$ is a naturally reductive space, we have that $\varphi_X$ is skew symetric with respect to the metric:

$$< \varphi_X Y, Z >=< \frac{1}{2}[X,Y]_{\mathfrak{m}}, Z >= - < Y, \frac{1}{2}[X,Z]_{\mathfrak{m}} >= - < Y, \varphi_X Z > .$$

Therefore $\varphi_X \in \mathfrak{so}(\mathfrak{m})$ so $e^{-\varphi_X} : (\mathfrak{m}, <,>) \to (\mathfrak{m}, <,>)$ is a linear isometry.

**Lemma 3.2.** [24] *Let $\gamma_X$ denote the geodesic in $M$ satisfying $\gamma(0) = p$, and $\dot{\gamma}(0) = X$. The parallel vector field $V_Y(t)$, along $\gamma_X$, such that $V_Y(0) = Y \in \mathfrak{m}$, is given by*

$$V_Y(t) = d\exp(tX)(e^{-\varphi_{tX}}(Y)).$$

*Proof.* [24] First we have that

$$\nabla_{\dot\gamma_X(t)}(d\exp tX)(Z) = (d\exp tX)(\varphi_X(Z)),$$

for $Z \in \mathfrak{m}$. This holds since by Theorem 1.60 we have $\gamma_X(t) = \exp(tX)(p)$, so $\dot\gamma_X(t) = d\exp(tX)(X)$, and since $G$ acts by isometries, $\exp(tX)$ is an isometry, hence $d\exp(tX)$ is an affine map:

$$\nabla_{(d\exp(tX)(X))}(d\exp(tX)(Z)) = d\exp(tX)(\nabla_X Z).$$

Now $X, Z \in \mathfrak{m}$, and we replace $\nabla_X Z$ with the associated connection function to obtain the equality:

$$\nabla_{\dot\gamma_X(t)}(d\exp(tX)(Z)) = d\exp(tX)(\nabla_X Z) = (d\exp(tX))(\varphi_X(Z)).$$

Then we calculate

$$\begin{aligned}
\nabla_{\dot\gamma_X(t)} V_Y(t) &= d\exp(tX)(\frac{d}{dt}e^{-\varphi_{tX}}(Y)) + d\exp(tX)(\varphi_X \circ e^{-\varphi_{tX}}(Y)) \\
&= d\exp(tX)(\sum_{l=1}^{\infty}\frac{(-1)^l}{(l-1)!}t^{l-1}\varphi_X^l(Y) + \varphi_X \circ (\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}t^l\varphi_X^l(Y))) \\
&= 0,
\end{aligned}$$

where the first equality follows from the well know formula for the covariant differentiation of a vector field $Y(t) = \sum_{j=1}^{m}\alpha_j(t)(X_j)_{\gamma(t)}$ along a curve $\gamma(t)$, where $X_i = \partial/\partial x_i$ (see eg. [8]):

$$(\frac{DY}{dt})(t) = \sum_{k=1}^{m}\Big(\dot\alpha_k(t) + \sum_{i,j=1}^{m}\Gamma_{ij}^k(\gamma(t))\dot\gamma_i(t)\alpha_j(t)\Big)(X_k)_{\gamma(t)},$$

where $\Gamma_{ij}^k$ are the Christoffel symbols of $\nabla$. The second equality follows from the fact that $\varphi_{tX} = t\varphi_X$. This proves that $V_Y(t)$ is parallel. □

There is a theorem on the existence of totally geodesic submanifolds in general Riemannian manifolds by R. Hermann, the statement of which requires the following definitions:

**Definition 3.3.** Let $(M, g)$ be a Riemannian manifold with Levi-Civita connection $\nabla$, $p \in M$, and $U \in T_pM$. Then $P_U$ denotes the parallel transport, with respect to $\nabla$, along the geodesic $\gamma_{(p,U)}(t) = \exp_p(tU)$, from $p = \gamma_{(p,U)}(0)$ to $\gamma_{(p,U)}(1)$, where $\exp_p$ is the Riemannian exponential map at $p \in M$.

**Definition 3.4.** Let $R$ be the curvature tensor on $M$ with respect to $\nabla$. Then the $(1,3)$-tensor $R_U(t)$, $U \in T_pM$, on $T_pM$ is defined as follows:

$$R_U(t)(X,Y)Z = (P_{(tU)})^{-1} \circ R_{\gamma_{(tU)}(1)}(P_{(tU)}(X), P_{(tU)}(Y))P_{(tU)}(Z),$$

with $X, Y, Z \in T_pM$.

**Theorem 3.5.** [10] *Let $V$ be a subspace of $T_pM$. Then the following conditions are equivalent.*

1. *There exists a totally geodesic submanifold tangent to $V$ at $p \in M$.*
2. *There is a positive number $\epsilon$ such that for each $t \in (-\epsilon, \epsilon)$ and each $U \in V$, with $\| U \| = 1$, the following is satisfied:*

$$R_U(t)(V,V)V \subset V.$$

3. *There is a positive number $\epsilon$ such that for each $t \in (-\epsilon, \epsilon)$ and each $U \in V$, with $\| U \| = 1$, the following is satisfied:*

$$r_U(t)(V,V) \subset V,$$

*where $r_U(t)(X,Y) = R_U(t)(U,X)Y$.*

In particular, if Condition 1 is satisfied, then setting $t = 0$ we obtain

$$R(V, V)V \subset V.$$

Using Lemma 3.2 we can rewrite Theorem 3.5 in terms of the bracket operation when the space is naturally reductive, and thus prove the Theorem 3.1. We shall need the following well known fact:

**Theorem 3.6.** *If $f$ and $g$ are real analytic functions on an open interval $I$ and there is an open set $J \subset I$ such that*

$$f(x) = g(x),$$

*for all $x \in J$, then*

$$f(x) = g(x),$$

*for all $x \in I$.*

*Proof.* See [12]. □

We now proceed with the proof of Theorem 3.1:

*Proof.* Let $V \subset \mathfrak{m}$ be given, $X, Y, Z \in V$, with $\parallel X \parallel = 1$, and let $\xi \in V^{\perp}$ and $t \in (-\epsilon, \epsilon)$, where $\epsilon$ is given by Theorem 3.5. Assume that condition (3) in 3.5 is satisfied. Then we get the following equalities:

$$\begin{aligned} 0 &= < r_X(t)(Y, Z), \xi > \\ &= < P_{tX}^{-1} \circ R_{\gamma_{tX}(1)}(P_{tX}(X), P_{tX}(Y))P_{tX}(Z), \xi > \\ &= < d\exp(tX)(R_p(X, e^{-\varphi_{tX}}(Y))e^{-\varphi_{tX}}(Z)), d\exp(tX)(e^{-\varphi_{tX}}(\xi)) > \\ &= < R_o(X, e^{-\varphi_{tX}}(Y))e^{-\varphi_{tX}}(Z), e^{-\varphi_{tX}}(\xi) >, \end{aligned}$$

where the third equality comes from Lemma 3.2 and breaking out $d\exp(tX)$, and the fourth from the $G$-invariance of the metric. We now put

$$f(t) =< R_p(X, e^{-\varphi_{tX}}(Y))e^{-\varphi_{tX}}(Z), e^{-\varphi_{tX}}(\xi) > .$$

Then $f(t) = 0$ for $t \in (\epsilon, \epsilon)$, and therefore, since $f(t)$ is real analytic $f \equiv 0$, so

$$tf(t) =< R_o(tX, e^{-\varphi_{tX}}(Y))e^{-\varphi_{tX}}(Z), e^{-\varphi_{tX}}(\xi) >\equiv 0,$$

and since $e^{-\varphi_{tX}}$ is an isometry, Equation (3.1) follows.

Conversely, supposing equation (3.1) is satisfied we clearly have $tf(t) = 0$ for all $t \in \mathbb{R}$. Then $f(t) = 0$ for $t \neq 0$, and at $t = 0$ we have by continuity that $f(0) = 0$, so $f(t) \equiv 0$ on $\mathbb{R}$. Since $< r_X(t)(Y, Z), \xi >= f(t)$, condition (3) in Theorem 3.5 is satisfied. □

## 3.2 Totally Geodesic Hypersurfaces of Naturally Reductive Homogeneous Spaces

K. Tojo proved that a simply connected, irreducible (as a Riemannian manifold, i.e. if $T_pM$ is irreducible under the action of the holonomy group) naturally reductive homogeneous space $(M, g)$, of dimension 3, 4 or 5, admitting a totally geodesic hypersurface is either a sphere or a hyperbolic space, in particular $(M, g)$ has constant sectional curvature. K. Tsukada extended this result, proving that this holds true for any dimension $n \geq 3$ ([26]). In this section we will present the proof of this theorem.

**Definition 3.7.** Let $(M, g)$ be a reductive homogeneous space, with decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. A subspace $V$ of $\mathfrak{m}$ is said to be $\Lambda_{\mathfrak{m}}$-invariant if it satisfies $\Lambda_{\mathfrak{m}}(X)(V) \subset V$, for any $X \in \mathfrak{m}$. Moreover a $\Lambda_{\mathfrak{m}}$-invariant subspace $V$ is $\Lambda_{\mathfrak{m}}$-irreducible if $V$ has only trivial $\Lambda_{\mathfrak{m}}$-invariant subspaces.

**Definition 3.8.** A representation $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ of a real Lie algebra $\mathfrak{g}$ is said to be unitary if there is an inner product on $V$ which is $\mathfrak{g}$-invariant, i.e. satisfying:

$$< \rho(X)Y, W > + < (Y, \rho(X)W) >= 0,$$

for any $X \in \mathfrak{g}$ and $Y, W \in V$.

**Theorem 3.9.** *Each unitary representation is completely reducible i.e. it is isomorphic to a direct sum of irreducible representations:* $V \simeq \bigoplus V_i$ *with each* $V_i$ *irreducible.*

*Proof.* See [11]. □

Since $(M, <, >)$ is naturally reductive, it follows that the representation

$$\begin{aligned} \Lambda_{\mathfrak{m}}(X) : \mathfrak{m} &\to \mathfrak{m}, \\ Y &\mapsto \frac{1}{2}[X, Y]_{\mathfrak{m}}, \end{aligned}$$

is a unitary representation with respect to $g|_o$. We set

$$\mathfrak{m}_0 = \{V \in \mathfrak{m} \,|\, \Lambda_{\mathfrak{m}}(X)(V) = 0 \text{ for all } X \in V\}.$$

It follows that $\mathfrak{m}$ has an orthogonal decomposition into $\Lambda_{\mathfrak{m}}$-invariant, and $\Lambda_{\mathfrak{m}}$-irreducible subspaces $\mathfrak{m}_i$, $0 \leq i \leq r$,

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r, \tag{3.2}$$

where for $i \geq 1$, $\Lambda_{\mathfrak{m}}(X)|_{\mathfrak{m}_i} \neq 0$, for some $X \in \mathfrak{m}$.

**Theorem 3.10.** [15], [5] *Let* $H$ *be a closed subgroup of* $G$ *and suppose that* $G$ *acts almost effectively on* $M = G/H$*, i.e. the subset of* $g \in G$ *such that* $g$ *acts as the identity on* $M$ *is discrete. If* $<, >$ *is a Riemannian metric on* $M$ *which is naturally reductive with respect to* $G$ *and* $\mathfrak{m} \subset \mathfrak{g}$*, then* $\bar{\mathfrak{g}} := \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$*, as ideal in* $\mathfrak{g}$ *whose corresponding analytic subgroup* $\bar{G} \subset G$ *is transitive on* $M$ *and there exists a unique* $Ad(\bar{G})$*-invariant, symmetric, nondegenerate, bilinear form* $Q$ *on* $\bar{\mathfrak{g}}$ *(not necessarily positive definite) such that*

$$Q(\mathfrak{h} \cap \bar{\mathfrak{g}}, \mathfrak{m}) = 0 \quad \textit{and} \quad Q|_{\mathfrak{m}} =<, > |_p.$$

We may therefore assume that $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$ holds.

**Lemma 3.11.** [26] *Let* $M = G/H$ *be a homogeneous space with an* $Ad(H)$*-invariant decomposition* $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$*. Then for* $X, Y, Z \in \mathfrak{m}$*, we have*

$$[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{h}} + [[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{h}} + [[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{h}} = 0.$$

*Proof.* Using the Jacobi identity for $\mathfrak{g}$ and splitting the inner parts into $\mathfrak{m}$- and $\mathfrak{h}$-components, the linearity of the bracket yields

$$\begin{aligned} 0 &= [[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] \\ &= ([[X, Y]_{\mathfrak{h}}, Z] + [[Y, Z]_{\mathfrak{h}}, X] + [[Z, X]_{\mathfrak{h}}, Y]) \\ &\quad + ([[X, Y]_{\mathfrak{m}}, Z] + [[Y, Z]_{\mathfrak{m}}, X] + [[Z, X]_{\mathfrak{m}}, Y]). \end{aligned}$$

Taking the $\mathfrak{h}$-components of the expression the first parentheses vanishes, since $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, leaving us with the equality

$$0 = [[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{h}} + [[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{h}} + [[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{h}}.$$ □

**Lemma 3.12.** [26] *Let $M = G/H$ be a naturally reductive homogeneous space with an $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and let $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r$ be the $\Lambda_{\mathfrak{m}}$-invariant decomposition of equation* (3.2)*. Then the following relations hold:*

1. $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$, *for* $i \neq j$.
2. $[[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_j] = 0$, *for* $i \neq j$.
3. $[[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}, \mathfrak{m}_i] \subset \mathfrak{m}_i$.
4. $[[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i] \subset \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i]$.

*Proof.* From the $\Lambda_{\mathfrak{m}}$-invariance of each $\mathfrak{m}_i$ we get that $[\mathfrak{m}, \mathfrak{m}_i] \subset \mathfrak{m}_i$, for each $i$. This in particular implies that

$$[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{m}} = 0, \tag{3.3}$$

for $i \neq j$. Since each $\mathfrak{m}_i$ is $\Lambda_{\mathfrak{m}}$-irreducible, $[\mathfrak{m}_i, \mathfrak{m}_i]$ cannot be contained in a proper subspace of $\mathfrak{m}_i$ for $i \geq 1$, therefore we also have

$$[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{m}} = \mathfrak{m}_i. \tag{3.4}$$

(1) By equation (3.3) it is only necessary to prove that $[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{h}} = 0$, if $i \neq j$. The case $i = 0$ is trivial. Let $X, Y \in \mathfrak{m}_i$ and $Z \in \mathfrak{m}_j$. Lemma 3.11 gives that

$$[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{h}} = -[[Y, Z]_{\mathfrak{m}}, X]_{\mathfrak{h}} - [[Z, X]_{\mathfrak{m}}, Y]_{\mathfrak{h}} = 0,$$

where the last equality is a direct consequence of equation (3.3). Equation (3.4) tells us that any element of $\mathfrak{m}_i$ can be written as a bracket, for $i \geq 1$, hence the calculation above implies that $[\mathfrak{m}_i, \mathfrak{m}_j]_{\mathfrak{h}} = 0$, proving (1).

(2) Let $X, Y \in \mathfrak{m}_i$, $Z \in \mathfrak{m}_j$. From the Jacobi identity we obtain

$$[[X, Y], Z] = -[[Y, Z], X] - [[Z, X], Y].$$

Using $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$, from (1) we get

$$-[[Y, Z], X] - [[Z, X], Y] = 0,$$

which proves (2).

(3) Let $X, Y \in \mathfrak{m}_i$, $Z \in \mathfrak{m}_j$, $i \neq j$. Since

$$[[X, Y], Z] = [[X, Y]_{\mathfrak{m}}, Z] + [[X, Y]_{\mathfrak{h}}, Z],$$

(1) and (2) together imply that that $[[X, Y]_{\mathfrak{h}}, Z] = 0$. The $Ad(H)$-invariance of the metric then implies that

$$< [[X, Y]_{\mathfrak{h}}, V], Z >= - < V, [[X, Y]_{\mathfrak{h}}, Z] >= 0,$$

for $V \in \mathfrak{m}_i$. So $[[X, Y]_{\mathfrak{h}}, \mathfrak{m}_i] \subset \mathfrak{m}_i$, which is the claim.

(4) We calculate:

$$\begin{aligned}[][[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i] &= [[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}, \mathfrak{m}_i] + [[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{m}}, \mathfrak{m}_i] \\ &\subset \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i],\end{aligned}$$

where the inclusion is a consequence of (3) and equation (3.4). □

**Theorem 3.13.** [26] *Let $M = G/H$ be a naturally reductive homogeneous space with $Ad(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$. Let*

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r$$

*be the* $\Lambda_{\mathfrak{m}}$*-invariant decomposition of* $\mathfrak{m}$ *in equation* (3.2). *Let* $\mathfrak{g}_i$ *be defined by*

$$\mathfrak{g}_i = \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i],$$

*for* $i = 0, 1, ..., r$ *and further set*

$$\mathfrak{h}_i = \mathfrak{g}_i \cap \mathfrak{h}.$$

*Then* $\mathfrak{g}$ *and* $\mathfrak{h}$ *can be written as direct sums of Lie algebras:*

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_r, \\ \mathfrak{h} &= \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus ... \oplus \mathfrak{h}_r. \end{aligned}$$

*Proof.* We shall first need to establish that each $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$. Recall that $\mathfrak{g} = \mathfrak{m} + [\mathfrak{m}, \mathfrak{m}]$, by Theorem 3.10. Thus to check that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$, we need only show the following four inclusions:

$$[\mathfrak{m}, \mathfrak{m}_i] \subset \mathfrak{g}_i, \quad [\mathfrak{m}, [\mathfrak{m}_i, \mathfrak{m}_i]] \subset \mathfrak{g}_i$$
$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}_i] \subset \mathfrak{g}_i, \quad [[\mathfrak{m}, \mathfrak{m}], [\mathfrak{m}_i, \mathfrak{m}_i]] \subset \mathfrak{g}_i.$$

We show this using the relations of Lemma 3.12. First we have that

$$[\mathfrak{m}, \mathfrak{m}_i] \subset [\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{g}_i,$$

by (1) in Lemma 3.12, proving the first inclusion. By (2) and (4) in Lemma 3.12 we get

$$\begin{aligned} [\mathfrak{m}, [\mathfrak{m}_i, \mathfrak{m}_i]] &\subset [\mathfrak{m}_i, [\mathfrak{m}_i, \mathfrak{m}_i]] \\ &\subset \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i] = \mathfrak{g}_i, \end{aligned}$$

which proves the second inclusion. Since $[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{m}} = \mathfrak{m}_i$, $[[\mathfrak{m}_i, \mathfrak{m}_i]_{\mathfrak{h}}, \mathfrak{m}_i] \subset \mathfrak{m}_i$, and $[\mathfrak{m}_i, \mathfrak{m}_j] = 0$, for $i \neq j$, we can write $\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r$ and then use the linearity of the bracket to obtain

$$[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}_i] \subset [\sum_{j=0}^{r} [\mathfrak{m}_j, \mathfrak{m}_j], \mathfrak{m}_i].$$

Using (1) and (4) of Lemma 3.12 again we prove the inclusion

$$\begin{aligned} [[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}_i] &\subset [\sum_{j=0}^{r} [\mathfrak{m}_j, \mathfrak{m}_j], \mathfrak{m}_i] \\ &\subset [[\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i] \subset \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i] = \mathfrak{g}_i. \end{aligned}$$

By the Jacobi identity and the third inclusion just above, we get

$$\begin{aligned} [[\mathfrak{m}, \mathfrak{m}], [\mathfrak{m}_i, \mathfrak{m}_i]] &\subset [[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}_i], \mathfrak{m}_i] \\ &\subset [\mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i], \mathfrak{m}_i] \subset \mathfrak{m}_i + [\mathfrak{m}_i, \mathfrak{m}_i] = \mathfrak{g}_i. \end{aligned}$$

This proves that $\mathfrak{g}_i$ is an ideal of $\mathfrak{g}$.
For a proof of the direct sum properties, see [26]. □

We need a well known lemma before we can go on to prove that $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible if $\Lambda_{\mathfrak{m}} \neq 0$:

**Lemma 3.14.** *Let* $M = G/H$ *be a naturally reductive homogeneous space with an* $Ad(H)$*-invariant decomposition* $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ *and a* $G$*-invariant Riemannian metric* $<,>$*. Then the curvature tensor* $R$ *satisfies*

$$\begin{aligned} R_p(X, Y)Z &= -[[X, Y]_{\mathfrak{h}}, Z] + \frac{1}{4}[X, [Y, Z]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad -\frac{1}{4}[Y, [X, Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[X, Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}}. \end{aligned}$$

*Proof.* See [14]. □

**Corollary 3.15.** [26] *Let $M = G/H$ be a simply connected irreducible (as a Riemannian manifold) naturally reductive homogeneous space. If $\Lambda_{\mathfrak{m}} \neq 0$ then $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible.*

*Proof.* Let

$$\mathfrak{m} = \mathfrak{m}_0 \oplus \mathfrak{m}_1 \oplus ... \oplus \mathfrak{m}_r, \tag{3.5}$$

be the decomposition in (3.2). We will show that each $\mathfrak{m}_i$ is invariant under the action of the holonomy algebra $\mathfrak{h}^*$ of the Levi-Civita connection. This implies that the decomposition of equation (3.5) has only one factor, since $M$ is assumed to be irreducible. $\Lambda_{\mathfrak{m}} \neq 0$, implies that $\mathfrak{m} \neq \mathfrak{m}_0$ so $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible.

To show that $\mathfrak{m}_i$ is invariant under $\mathfrak{h}^*$, we observe that Theorem 1.52 implies that we only have to check invariance under $R(Y, Z)$ and $[\Lambda_{\mathfrak{m}}(X), R(Y, Z)]$, for all $X, Y, Z \in \mathfrak{m}$. Since

$$[\Lambda_{\mathfrak{m}}(X), R(Y, Z)] = \Lambda_{\mathfrak{m}}(X) \circ R(Y, Z) - R(Y, Z) \circ \Lambda_{\mathfrak{m}}(X),$$

and $\mathfrak{m}_i$ is already invariant under $\Lambda_{\mathfrak{m}}(X)$, for all $X \in \mathfrak{m}$, this reduces to showing that $\mathfrak{m}_i$ is invariant under $R(X, Y)$, for all $X, Y \in \mathfrak{m}$. Recall that $\Lambda_{\mathfrak{m}}(X)(Y) = \frac{1}{2}[X, Y]_{\mathfrak{m}}$. From Lemma 3.14 we then get

$$\begin{aligned} R_o(X, Y)Z &= -[[X, Y]_{\mathfrak{h}}, Z] + (\Lambda_{\mathfrak{m}}(X) \circ \Lambda_{\mathfrak{m}}(Y))(Z) \\ &\quad -(\Lambda_{\mathfrak{m}}(Y) \circ \Lambda_{\mathfrak{m}}(X))(Z) - 2\Lambda_{\mathfrak{m}}(\Lambda_{\mathfrak{m}}(X)(Y))(Z). \end{aligned}$$

Assume now that $Z \in \mathfrak{m}_i$, and $X, Y \in \mathfrak{m}$. Since $\mathfrak{m}_i$ is invariant under $\Lambda_{\mathfrak{m}}$, we only need to check that $[[X, Y]_{\mathfrak{h}}, Z] \in \mathfrak{m}_i$ as well. This follows from Lemma 3.12. □

We shall now start working more directly towards the proof of Tsukada's theorem. So let $M$ be a simply connected, irreducible and naturally reductive homogeneous manifold, admitting a totally geodesic hypersurface. Suppose $\Lambda_{\mathfrak{m}} \equiv 0$. Then $\nabla R = 0$, at the origin. Since $M$ is homogeneous it is then locally symmetric, and being simply connected, it is symmetric, by the following theorem:

**Theorem 3.16.** *A complete, simply connected, locally symmetric semi-Riemannian manifold is symmetric.*

*Proof.* See [21] or [9]. □

This case of the theorem of Tsukada is now taken care of by a theorem of Chen and Nagano:

**Theorem 3.17.** [4] *Spheres and hyperbolic spaces are the only simply connected, irreducible symmetric spaces admitting a totally geodesic hypersurface.*

We therefore assume that $\Lambda_{\mathfrak{m}} \neq 0$. By Corollary 3.15 $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible. So assume that $M$ admits a totally geodesic hypersurface, which we by homogeneity may assume goes through the origin. Call this hypersurface $S$ and let $V := T_oS \subset \mathfrak{m}$ be the corresponding hyperplane in $\mathfrak{m}$, which is tangent to $S$ at $o \in M$. Fix some unit vector $\xi \in \mathfrak{m}$ normal to $V$ so that $\mathfrak{m} = \mathbb{R}\xi \oplus V$. Define the subspace $V_1$ of $V$ by:

$$V_1 = \{\varphi_\xi X | X \in \mathfrak{m}\} = \{\varphi_\xi X | X \in V\},$$

and put $O_1 := \mathbb{R}\xi \oplus V_1$. That $V_1$ is a subspace of $\mathfrak{m}$ is clear. That it is contained in $V$ follows from $M$ being naturally reductive:

$$< \varphi_\xi X, \xi >= - < X, \varphi_\xi \xi >= 0,$$

since $\varphi_\xi \xi = [\xi, \xi]_{\mathfrak{m}} = 0$. As $\mathfrak{m}$ is $\Lambda_{\mathfrak{m}}$-irreducible, $V_1 \neq 0$: if $V_1 = 0$, then $\varphi_\xi X = 0$, for all $X \in V$, i.e. $[\xi, X]_{\mathfrak{m}} = 0$. But $[\xi, X]_{\mathfrak{m}} = -[X, \xi]_{\mathfrak{m}} = -\varphi_X \xi$, so $\varphi_X \xi = 0$, and $\varphi_\xi \xi = 0$, and $0 \in \mathbb{R}\xi$, contradicting the irreducibility.

**Lemma 3.18.** [26] *For any $X, Y, Z \in V$, $W \in \mathfrak{m}$ we have $< R(X, Y)Z, \xi >= 0$ and*

$$\begin{aligned} < R(Y, Z)W, \varphi_\xi X > &= < \varphi_\xi X, Y >< R(Z, \xi)\xi, W > \\ &\quad - < \varphi_\xi X, Z >< R(Y, \xi)\xi, W > . \end{aligned}$$

*Proof.* From Theorem 3.1 we have that

$$R(e^{t\varphi_X}(V), e^{t\varphi_X}(V))e^{t\varphi_X}(V) \subset e^{t\varphi_X}(V),$$

for any $X \in V$, $t \in \mathbb{R}$. In particular,

$$< R(e^{t\varphi_Q}(X), e^{t\varphi_Q}(Y))e^{t\varphi_Q}(Z), e^{t\varphi_Q}(\xi) >= 0, \tag{3.6}$$

for $Q, X, Y, Z, \in V$. At $t = 0$ this is $< R(Y, Z)W, \xi >= 0$, where we for now assume that $W \in V$. To prove the second statement of the lemma we differentiate equation (3.6) with respect to $t$ and evaluate at $t = 0$, and obtain

$$\begin{aligned} &< R(\varphi_X Y, Z)W, \xi > + < R(Y, \varphi_X Z)W, \xi > \\ &+ < R(Y, Z)\varphi_X W, \xi > + < R(Y, Z)W, \varphi_X \xi >= 0. \end{aligned} \tag{3.7}$$

We can write $\varphi_X Y =< \varphi_X Y, \xi > +Q$, for some $Q \in V$. Then by linearity and the first statement of the lemma, we get

$$\begin{aligned} < R(\varphi_X Y, Z)W, \xi > &= < R(< \varphi_X Y, \xi > \xi + Q, Z)W, \xi > \\ &= < \varphi_X Y, \xi >< R(\xi, Z)W, \xi > + < R(Q, Z)W, \xi > \\ &= - < \varphi_X \xi, Y >< R(\xi, Z)W, \xi > \\ &= < \varphi_\xi X, Y >< R(\xi, Z)W, \xi > \\ &= < \varphi_\xi X, Y >< R(Z, \xi)\xi, W > . \end{aligned}$$

Similarly, write $\varphi_X Z =< \varphi_X Z, \xi > \xi + Q'$, for $Q' \in V$. Then

$$\begin{aligned} < R(Y, \varphi_X Z)W, \xi > &= < \varphi_X Z, \xi >< R(Y, \xi)W, \xi > + < R(Y, Q')W, \xi > \\ &= < \varphi_\xi X, Z >< R(Y, \xi)W, \xi > \\ &= - < \varphi_\xi X, Z >< R(Y, \xi)\xi, W > . \end{aligned}$$

Finally, we write $\varphi_X W =< \varphi_X W, \xi > \xi + Q''$, with $Q'' \in V$. Then

$$\begin{aligned} < R(Y, Z)\varphi_X W, \xi > &= < \varphi_X W, \xi >< R(Y, Z)\xi, \xi > + < R(Y, Z)Q'', \xi > \\ &= < \varphi_\xi X, W >< R(Y, Z)\xi, \xi >= 0. \end{aligned}$$

Substituting these back into equation (3.7) we get

$$\begin{aligned} 0 &= < \varphi_\xi X, Y >< R(Z, \xi)\xi, W > - < \varphi_\xi X, Z >< R(Y, \xi)\xi, W > \\ &+ < R(Y, Z)W, \varphi_X \xi >, \end{aligned}$$

which is the second claim of the Lemma. Recalling that $\mathfrak{m} = \mathbb{R}\xi \oplus V$, we get that $W$ can be taken in all of $\mathfrak{m}$, since the equation is zero if $W = \lambda\xi$. □

Writing $\varphi_\xi X = Q \in V_1$, $Y, Z \in V$, $W \in \mathfrak{m}$, the equality then becomes

$$\begin{aligned} - < R(Y, Z)Q, W > &= < R(Y, Z)W, Q > \\ &= < Q, Y >< R(Z, \xi)\xi, W > \\ &\quad - < Q, Z >< R(Y, \xi)\xi, W > . \end{aligned}$$

For each $X \in \mathfrak{m}$ we define the map $R_X : \mathfrak{m} \to \mathfrak{m}$, by

$$R_X Y = R(Y, X)X.$$

Then $R_X$ is a symmetric endomorphism of $\mathfrak{m}$, since

$$\begin{aligned} < R(Y, X)X, Z > &= < R(X, Z)Y, X > \\ &= - < R(Z, X)Y, X > \\ &= < R(Z, X)X, Y > . \end{aligned}$$

**Lemma 3.19.** [26] *There exists a constant $c \in \mathbb{R}$, such that*

$$R_\xi X = cX,$$

*holds for any $X \in V_1$.*

*Proof.* Let $X \in V_1$, and set $Q = Z = W = X$. With $Y \in X^\perp$ the equality

$$-< R(Y,Z)Q, W >=< Q, Y >< R(Z,\xi)\xi, W > - < Q, Z >< R(Y,\xi)\xi, W >,$$

becomes (via standard symmetries of $< R(X,Y)Z, W >$)

$$\begin{aligned} < Q, Z >< R(X,\xi)\xi, Y > &= - < Q, Z >< R(X,\xi)Y, \xi > \\ &= - < X, X >< R(Y,\xi)\xi, X > \\ &= - < R(Y,X)X, X) = 0, \end{aligned}$$

so $< R(X,\xi)\xi, Y >= 0$, and by symmetry we clearly also have $< R(X,\xi)\xi, \xi >= 0$. This holds for any $Y \in X^\perp$, and since $X$ was arbitrary, $V_1$ is a subspace of an eigenspace of $R_\xi$, and we may let $c$ be the eigenvalue of $R_\xi$ with respect to this eigenspace. □

**Lemma 3.20.** [26] *Let $c \in \mathbb{R}$ be as in Lemma 3.19 and $Q \in O_1 = \mathbb{R}\xi \oplus V_1$. Then the following relations hold:*

$$R(Y,Z)Q = 0, \quad \textit{for any } Y, Z \in Q^\perp, \tag{3.8}$$

$$R_Q X = c(< Q, Q > X - < X, Q > Q), \quad \textit{for } X \in O_1, \textit{ and} \tag{3.9}$$

$$R_Q X =< Q, Q > R_\xi X, \quad \textit{for } X \in O_1^\perp, \tag{3.10}$$

*where the orthogonal complements are taken in $\mathfrak{m}$.*

*Proof.* As $O_1 = \mathbb{R}\xi \oplus V_1$, we only need to consider three cases, namely $Q = \xi$, $Q \in V_1$ with $|Q| = 1$, and $Q \in O_1$ with $|Q| = 1$. The lemma then follows from linearity.

Case 1. Assume that $Q = \xi$. Equation (3.8) follows from Lemma 3.18 gives that

$$-< R(X,Y)\xi, Z >=< R(X,Y)Z, \xi >= 0,$$

for all $X, Y, Z \in V$, and by symmetry, $< R(X,Y)\xi, \xi >= 0$, so $R(X,Y)\xi \equiv 0$, which is Equation (3.8). Lemma 3.19 which says that

$$R_Q X = R_\xi X = c(X - < X, \xi > \xi),$$

since $R_\xi \equiv 0$, on the $\xi$-component of $X \in O_1$. This establishes equation (3.9), and (3.10) is trivial when $Q = \xi$.

Case 2. Let $Q \in V_1 \subset O_1$, with $|Q| = 1$, and let $Z, Y \in Q^\perp \cap V$. From the previously noted equality we get

$$\begin{aligned} -< R(Y,Z)Q, W > &= < Q, Y >< R(Z,\xi)\xi, W > \\ &\quad - < Q, Z >< R(Y,\xi)\xi, W > \\ &= 0, \end{aligned}$$

with $W \in \mathfrak{m}$. Since $Q^\perp = Q^\perp \cap V \oplus Q^\perp \cap \xi$, the only remaining detail to check is that $R(Y,\xi)Q = 0$. But from Lemma 3.18 we have (via symmetry of $R$) that

$$< R(Y,\xi)Q, W >=< R(Q,W)Y, \xi >= 0,$$

for $W \in V$. And since $Q \in V_1$, Lemma 3.19 gives that

$$< R(Y,\xi)Q, \xi >= - < R(Q,\xi)\xi, Y >= - < R_\xi Q, Y >= -c < Q, Y >= 0,$$

which proves (3.8). Recalling that

$$< Q, Y >< R(Z,\xi)\xi, W > - < Q, Z >< R(Y,\xi)\xi, W >= - < R(Y,Z)Q, W >,$$

setting $Z = Q$, and letting $Y \in Q^{\perp} \cap V$ yields

$$- < Q, Q >< R(Y,\xi)\xi, W >= - < R(Y,Q)Q, W >,$$

i.e. $R_{\xi}Y = R_{Q}Y$. Therefore (3.9) and (3.10) hold.

Case 3. The last case is left for the reader, or see [26]. □

We note the 2nd Bianchi identity (see e.g. [13] and [14]):

**Lemma 3.21.** *Let $M$ be a Riemannian manifold with $p \in M$ and Levi-Civita connection $\nabla$. Let $X, Y, Z \in T_pM$, then the following identity holds*

$$\mathfrak{S}_{X,Y,Z}\{(\nabla_X R)(Y,Z)\} = 0, \tag{3.11}$$

*where $\mathfrak{S}_{X,Y,Z}$ denotes the cyclic sum over $X$, $Y$ and $Z$.*

**Corollary 3.22.** [26] *Let $X, Y, Z, W \in \mathfrak{m}$, then we have*

$$\mathfrak{S}_{X,Y,Z}\{\varphi_X(R(Y,Z)W) - R(\varphi_X Y, Z)W - R(Y, \varphi_X Z)W - R(Y,Z)\varphi_X W\} = 0.$$

*Proof.* By definition of $(\nabla_X R)(Y,Z)W$ and $\varphi_X$ we have

$$\begin{aligned}(\nabla_X R)(Y,Z)W &= (\varphi_X R)(Y,Z)W \\ &= \varphi_X(R(Y,Z)W) - R(\varphi_X Y, Z)W \\ &\quad -R(Y,\varphi_X Z)W - R(Y,Z)\varphi_X W.\end{aligned}$$

Applying the 2nd Bianchi identity we obtain the statement. □

As we noted earlier $R_{\xi}$ is a symmetric endomorphism of $\mathfrak{m}$. But since $R_{\xi}(V) = R(V,\xi)\xi$, we have $< R_{\xi}(V), \xi >=< R(V,\xi)\xi, \xi >= 0$, and therefore $R_{\xi}(V) \subset V$. Then, since $R_{\xi}$ is symmetric, $V$ is decomposed into orthogonal eigenspaces of $R_{\xi}$:

$$V = \mathfrak{p}_1 \oplus ... \oplus \mathfrak{p}_r,$$

where $\mathfrak{p}_i$ has eigenvalue $\lambda_i$, say, and we choose indices of the eigenspaces such that $\lambda_1 = c$. We note that Lemma 3.19 implies that $V_1 \subset \mathfrak{p}_1$.
We shall now proceed to investigate how $\varphi_X Y$ behaves with respect to this decomposition of $V$.

**Lemma 3.23.** [26] *If $X, Y \in \mathfrak{p}_1$, then $\varphi_X Y \in \mathbb{R}\xi \oplus \mathfrak{p}_1$. If $X \in \mathfrak{p}_i$, $1 \leq i \leq r$, and $Y \in \mathfrak{p}_j$, $j > 1$, then $\varphi_X Y$ is contained in the eigenspace of $R_{\xi}$ corresponding to the eigenvalue $\frac{\lambda_i + \lambda_j}{2}$.*

*Proof.* [26] By Lemma 3.22, for $X \in \mathfrak{p}_i$, $Y \in \mathfrak{p}_j$, when permuting the $\xi$ in $\varphi_{\xi}$, and not the $\xi$ in $R(X,Y)\xi$, we have

$$\begin{aligned}0 &= \mathfrak{S}_{X,Y,\xi}\{\varphi_{\xi}(R(X,Y)\xi) - R(\varphi_{\xi}X, Y)\xi - R(X, \varphi_{\xi}Y)\xi - R(X,Y)\varphi_{\xi}\xi\} \\ &= \varphi_{\xi}(R(X,Y)\xi) - R(\varphi_{\xi}X, Y)\xi - R(X,\varphi_{\xi}Y)\xi - R(X,Y)\varphi_{\xi}\xi \\ &\quad +\varphi_X(R(Y,\xi)\xi) - R(\varphi_X Y, \xi)\xi - R(Y, \varphi_X \xi)\xi - R(Y,\xi)\varphi_X \xi \\ &\quad +\varphi_Y(R(\xi,X)\xi) - R(\varphi_Y \xi, X)\xi - R(\xi, \varphi_Y X)\xi - R(\xi,X)\varphi_Y \xi.\end{aligned}$$

Now $Y \in \mathfrak{p}_j$, so

$$\varphi_X(R(Y,\xi)\xi) = \varphi_X(R_{\xi}(Y)) = \varphi_X(\lambda_j Y) = \lambda_j \varphi_X Y,$$

and similarly

$$\varphi_Y(R(\xi,X)\xi) = \varphi_Y(-R(X,\xi)\xi) = -\varphi_Y(R_{\xi}(X)) = \lambda_i \varphi_Y X.$$

The term $-R(\xi,\varphi_Y X)\xi$ becomes $-R(\varphi_X Y,\xi)\xi$, under symmetries of $R$ and $\varphi$, so we have a $-2R_\xi(\varphi_X Y)$ term left in the end. Recall that equation (3.8) says that $R(X,Y)Z=0$, for any $Z\in O_1=\mathbb{R}\xi\oplus V_1$ and $X,Y\in Z^\perp$. But $\xi\in O_1$, and $X,Y\in V$, so $X,Y\in\xi^\perp$ - by choice of $\xi$ - and $\varphi_\xi X,\varphi_\xi Y\in V_1\subset\xi^\perp$. Since $\varphi_\xi X=-\varphi_X\xi$, we get that the following terms vanish:

$$\begin{aligned} 0 &= \varphi_\xi(R(X,Y)\xi)=-R(\varphi_\xi X,Y)\xi=-R(X,\varphi_\xi Y)\xi \\ &= -R(X,Y)\varphi_\xi\xi=-R(Y,\varphi_X\xi)\xi=-R(\varphi_Y\xi,X)\xi. \end{aligned}$$

We are then left with

$$\begin{aligned} 0 &= \mathfrak{S}_{X,Y,\xi}\{\varphi_\xi(R(X,Y)\xi)-R(\varphi_\xi X,Y)\xi-R(X,\varphi_\xi Y)\xi-R(X,Y)\varphi_\xi\xi\} \\ &= \lambda_j\varphi_X Y-2R_\xi(\varphi_X Y)-R(Y,\xi)\varphi_X\xi-\lambda_i\varphi_Y X+R(X,\xi)\varphi_Y\xi \\ &= (\lambda_i+\lambda_j)\varphi_X Y-2R_\xi(\varphi_X Y)-R(Y,\xi)\varphi_X\xi+R(X,\xi)\varphi_Y\xi. \end{aligned}$$

The term $-R(Y,\xi)\varphi_X\xi$ can be written

$$\begin{aligned} -R(Y,\xi)\varphi_X\xi &= -R\big(<\frac{\varphi_X\xi}{|\varphi_X\xi|},Y>\frac{\varphi_X\xi}{|\varphi_X\xi|},\xi\big)\varphi_X\xi \\ &\quad -R\big(Y-<\frac{\varphi_X\xi}{|\varphi_X\xi|},Y>\frac{\varphi_X\xi}{|\varphi_X\xi|},\xi\big)\varphi_X\xi. \end{aligned}$$

Since $\Big(Y-<\frac{\varphi_X\xi}{|\varphi_X\xi|},Y>\frac{\varphi_X\xi}{|\varphi_X\xi|}\Big)$, and $\xi$ are orthogonal to $\varphi_X\xi\in V_1\subset O_1$, equation (3.8) implies that the second term here vanishes. We then calculate

$$\begin{aligned} -R(Y,\xi)\varphi_X\xi &= -R\big(<\frac{\varphi_X\xi}{|\varphi_X\xi|},Y>\frac{\varphi_X\xi}{|\varphi_X\xi|},\xi\big)\varphi_X\xi \\ &= -\frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}R(\varphi_X\xi,\xi)\varphi_X\xi \\ &= \frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}R(\xi,\varphi_X\xi)\varphi_X\xi \\ &= \frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}R_{(\varphi_X\xi)}\xi. \end{aligned}$$

From Lemma 3.20, Equation (3.9) and $\xi\perp\varphi_X\xi$, we then obtain

$$\begin{aligned} \frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}R_{(\varphi_X\xi)}\xi &= \frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}c(<\varphi_X\xi,\varphi_X\xi>\xi-<\xi,\varphi_X\xi>\varphi_X\xi) \\ &= \frac{<\varphi_X\xi,Y>}{|\varphi_X\xi|^2}c(<\varphi_X\xi,\varphi_X\xi>\xi) \\ &= c<\varphi_X\xi,Y>\xi. \end{aligned}$$

Similarly, we get

$$\begin{aligned} R(X,\xi)\varphi_Y\xi &= -c<\varphi_Y\xi,X>\xi \\ &= c<\varphi_\xi Y,X>\xi \\ &= -c<Y,\varphi_\xi X>\xi \\ &= c<Y,\varphi_X\xi>\xi. \end{aligned}$$

Thus the terms $-R(Y,\xi)\varphi_X\xi$, and $R(X,\xi)\varphi_Y\xi$, add to $2c<\varphi_X\xi,Y>\xi$. Collecting, we therefore get

$$0=(\lambda_i+\lambda_j)\varphi_X Y-2R_\xi(\varphi_X Y)+2c<\varphi_X\xi,Y>\xi,$$

which is equivalent to

$$2R_\xi(\varphi_X Y)=(\lambda_i+\lambda_j)\varphi_X Y+2c<\varphi_X\xi,Y>\xi. \tag{3.12}$$

If $i=j=1$, then (3.12) becomes

$$R_\xi(\varphi_X Y)=c(\varphi_X Y-<\varphi_X Y,\xi>\xi),$$

which implies the first statement of the Lemma. If $j\neq 1$, then (3.12) implies that $R_\xi(\varphi_X Y)=\frac{\lambda_i+\lambda_j}{2}\varphi_X Y$, which is the second statement of the lemma. □

**Lemma 3.24.** [26] *If $X \in \mathfrak{p}_i$, $Y \in \mathfrak{p}_j$ with $i \neq j$, then $\varphi_X Y = 0$.*

*Proof.* We may assume that $j \neq 1$. We argue indirectly and suppose that $\varphi_X Y = Z \neq 0$. Lemma 3.23 implies that $Z$ is in the eigenspace of $R_\xi$ with eigenvalue $\frac{\lambda_i+\lambda_j}{2}$. Since $M$ is naturally reductive, we have

$$0 \neq |Z|^2 =< \varphi_X Y, Z >= - < Y, \varphi_X Z > .$$

Thus $Y$ and $\varphi_X Z$ have the same eigenvalue: they are not orthogonal and are therefore contained in the same $\mathfrak{p}_j$. From Lemma 3.23 it follows that the eigenvalue of $Z$ is $\frac{\lambda_i+\lambda_j}{2} + \lambda_i$. Hence

$$\lambda_j = \frac{\left(\frac{\lambda_i+\lambda_j}{2} + \lambda_i\right)}{2},$$

which implies that $\lambda_i = \lambda_j$ which contradicts $i \neq j$. □

By choice of indices $V_1 \subset \mathfrak{p}_1$. Therefore $\Lambda_\mathfrak{m}(\mathfrak{m})(\mathbb{R}\xi) = V_1 \subset \mathfrak{p}_1$. Lemma 3.23 showed that $\Lambda_\mathfrak{m}(\mathfrak{p}_1)(\mathfrak{p}_1) \subset \mathbb{R}\xi \oplus \mathfrak{p}_1$. Together with Lemma 3.24 we therefore have that $\mathbb{R}\xi \oplus \mathfrak{p}_1, \mathfrak{p}_2, ..., \mathfrak{p}_r$ are $\Lambda_\mathfrak{m}$-invariant subspaces. But $\mathfrak{m}$ is $\Lambda_\mathfrak{m}$-irreducible, and therefore $\mathfrak{m} = \mathbb{R}\xi \oplus \mathfrak{p}_1$. Therefore $V = \mathfrak{p}_1$, so $R_\xi X = cX$, for any $X \in V$.

The following Lemma is asserted, though not proved, in [26]:

**Lemma 3.25.** *For $Q \in O_1 = \mathbb{R}\xi \oplus V_1$, and $W, X \in \mathfrak{m}$ we have*

$$R(Q,W)X = c(< W, X > Q- < Q, X > W).$$

*Proof.* We proceed by checking various cases of the equality. Due to the linearity of the equation, whenever we consider some $Z \in \mathbb{R}\xi$, we just assume that $Z = \xi$.

Case 1: $Q \in V_1$, $X, W \in V$. Recall that Lemma 3.18 showed that

$$< R(X,Y)W, Q >=< Q, X >< R_\xi Y, W > - < Q, Y >< R_\xi X, W >,$$

for $X, Y \in V = \mathfrak{p}_1$, $W \in \mathfrak{m}$ and $Q \in V_1$. But

$$< R(X,Y)W, Q >= - < R(X,Y)Q, W >= - < R(Q,W)X, Y >,$$

so

$$< R(Q,W)X, Y >= - < Q, X >< R_\xi Y, W > + < Q, Y >< R_\xi X, W > \quad (*)$$

for $X, Y \in V$, $W \in \mathfrak{m}$. Since $< R_\xi X, \xi >=< R(X,\xi)\xi, \xi >= 0$ we may assume that $W \in \mathfrak{p}_1$. Then we get

$$\begin{aligned} < R(Q,W)X, Y > &= - < Q, X >< R_\xi Y, W > + < Q, Y >< R_\xi X, W > \\ &= - < Q, X >< R_\xi W, Y > + < Q, Y >< R_\xi W, X > \\ &= - < Q, X >< cW, Y > + < Q, Y >< cW, X > \\ &= < - < Q, X > cW+ < cW, X > Q, Y > \\ &= < c(< - < Q, X > W+ < W, X > Q), Y >, \end{aligned}$$

for all $X, Y \in V$, $Q \in V_1$. If $Y \in \mathbb{R}\xi$, $Q \in V_1$, Lemma 3.18 then yields

$$< R(Q,W)X, Y >= 0 = - < Q, X >< cW, Y > + < Q, Y >< cX, W >,$$

since $X, Y, Z \in V$, which proves the claim of the lemma for $Q \in V_1$, $W \in V$ and $X \in V$

Case 2: $X, W \in \mathbb{R}\xi$. $Q \in O_1 = \mathbb{R}\xi \oplus V_1$, and we may assume $Q \in V_1$, since $R(\xi,\xi)\xi = 0$, and also that $X = W = \xi$. Then if $Y = \xi$ in $(*)$ we get

$$< R(Q,\xi)\xi, \xi >= 0 =< c(< \xi, \xi > Q- < Q, \xi > \xi), \xi > .$$

For $Y \in V$ we get

$$< R(Q,\xi)\xi, Y >=< cQ, Y >=< c(< \xi,\xi > Q- < Q,\xi > \xi), Y >,$$

since $V = \xi^{\perp}$, which proves this case.

Case 3: $W \in \mathbb{R}\xi$, $X \in V$. If $Y \in \mathbb{R}\xi$ in $(*)$ then we get

$$\begin{aligned} < R(Q,\xi)X,\xi > &= - < R(Q,\xi)\xi, X > \\ &= - < cQ, X > \\ &= < c(< \xi,\xi > Q- < Q,X > \xi),\xi >, \end{aligned}$$

for if $Q \in \mathbb{R}\xi$, then both sides are 0. If $Y \in V$, then we get

$$< R(Q,\xi)X, Y >=< R(X,Y)Q,\xi >= 0,$$

by Lemma 3.18, as $Q$ may be assumed to be in $V_1$. But we also have

$$< c(< \xi, X > Q- < Q, X > \xi), Y >= 0,$$

as $< \xi, X >= 0$, so the formula holds in this case as well.

Case 4: $W \in V, X \in \mathbb{R}\xi$. If $Y \in \mathbb{R}\xi$ in $(*)$ we get

$$< R(Q,W)\xi,\xi >= 0 =< c(< W,\xi > Q- < Q,\xi > W),\xi > .$$

For $Y \in V$, we first assume that $Q \in V_1$, and get

$$< R(Q,W)\xi, Y >= - < R(Q,W)Y,\xi >= 0,$$

by Lemma 3.18. But we also have $< c(< W,\xi > Q- < Q,\xi > W), Y >= 0$. If instead $Q \in \mathbb{R}\xi$, then

$$\begin{aligned} < R(\xi,W)\xi, Y > &= - < R(W,\xi)\xi, Y > \\ &= - < cW, Y > \\ &= < c(< W,\xi > \xi- < \xi,\xi > W), Y >, \end{aligned}$$

which concludes the proof of this case.

Case 5: $Q \in \mathbb{R}\xi$, $X, W \in V$. If $Y \in \mathbb{R}\xi$ in $(*)$ then we have

$$\begin{aligned} < R(\xi,W)X,\xi > &= < R(W,\xi)\xi, X > \\ &= < cW, X > \\ &= < c(< W,X > \xi- < \xi,X > W),\xi > . \end{aligned}$$

If on the other hand $Y \in V$, then

$$< R(\xi,W)X, Y >=< R(Y,X)W,\xi >= 0,$$

by Lemma 3.18. But $< c(< W,X > \xi- < \xi,X > W), Y >= 0$, so this case checks out as well. The remaining cases are obvious, and the general claim of the Lemma follows from linearity. □

**Definition 3.26.** For $X, Y, Z \in \mathfrak{m}$, $R_0$ shall denote the (1,3) tensor that satisfies

$$R_0(X,Y)Z =< Y,Z > X- < X,Z > Y,$$

and we define $\mathfrak{n}$ to be the subspace $\mathfrak{n} \subset \mathfrak{m}$ given by

$$\mathfrak{n} = \{X \in \mathfrak{m} \,|\, R(X,\cdot)(\cdot) - cR_0(X,\cdot)(\cdot) = 0\},$$

where $c \in \mathbb{R}$ is the constant of Lemma 3.19.

From Lemma 3.25 it follows that $O_1 \subset \mathfrak{n}$.

**Lemma 3.27.** [26] *$\mathfrak{n}$ is invariant under the action of $\mathfrak{h}$. In particular*

$$[[X,Y]_{\mathfrak{h}}, Z] \in \mathfrak{n},$$

*for $Z \in \mathfrak{n}$ and $X, Y \in \mathfrak{m}$.*

*Proof.* See [26]. □

We are now ready to prove the main theorem of this section.

**Theorem 3.28.** [26] *If a simply connected irreducible (as a Riemannian manifold) naturally reductive homogeneous space $M$ admits a totally geodesic hypersurface, then $M$ has constant sectional curvature.*

*Proof.* It is sufficient to show that $\mathfrak{n} = \mathfrak{m}$. We have already noted that $O_1 \subset \mathfrak{n}$, so it only remains to show that $V \subset \mathfrak{n}$. There are two cases to take care of, depending on the value of $c \in \mathbb{R}$ of Lemma 3.19.

Case 1: $c \neq 0$. Recall that Lemma 3.14 showed that

$$\begin{aligned} R_p(X,Y)Z &= -[[X,Y]_{\mathfrak{h}}, Z] + \frac{1}{4}[X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} \\ &\quad -\frac{1}{4}[Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - \frac{1}{2}[[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}}. \end{aligned}$$

or in terms of $\varphi$:

$$R(X,Y)Z = -[[X,Y]_{\mathfrak{h}}, Z] + \varphi_X\varphi_Y Z - \varphi_Y\varphi_X Z - 2\varphi_{(\varphi_X Y)} Z,$$

for $X, Y, Z \in \mathfrak{m}$. Now let $X \in V$. Then from the above equation it follows that

$$\begin{aligned} R(X,\xi)\xi &= -[[X,\xi]_{\mathfrak{h}}, \xi] - \varphi_X\varphi_\xi\xi - \varphi_\xi\varphi_X\xi - 2\varphi_{(\varphi_X\xi)}\xi \\ &= -[[X,\xi]_{\mathfrak{h}}, \xi] - \varphi_{\varphi_X\xi}\xi. \end{aligned}$$

Now $\xi \in O_1$ so $\xi \in \mathfrak{n}$, and therefore, by Lemma 3.27, so is $[[X,\xi]_{\mathfrak{h}}, \xi]$. $\varphi_{(\varphi_X\xi)}\xi = -\varphi_\xi(\varphi_X\xi)$, and $\varphi_\xi(\varphi_X\xi) \in V_1$, by definition of $V_1$. Since $V_1 \subset O_1 \subset \mathfrak{n}$, $-\varphi_{(\varphi_X\xi)}\xi \in \mathfrak{n}$, which means that $R(X,\xi)\xi \in \mathfrak{n}$. On the other hand, since $X \in V = \mathfrak{p}_1$, we have $R(X,\xi)\xi = R_\xi X = cX$, so $cX \in \mathfrak{n}$, which shows that $\mathfrak{m} \subset \mathfrak{n}$, hence $\mathfrak{m} = \mathfrak{n}$, which concludes this case.

Case 2: $c = 0$. Set $V_0 = \mathbb{R}\xi$, and define a sequence of subspaces of $\mathfrak{m}$ by setting $V_{i+1} = \{\varphi_X Z \,|\, Z \in V_i, X \in \mathfrak{m}\}$. We remark that this is consistent with our earlier definition of $V_1$. We shall now prove that each $V_i$ is contained in $\mathfrak{n}$. This has already been proved for $i = 0, 1$, and we assume that it holds for some $i \in \mathbb{N}$ and show that this implies that $V_{i+1} \subset \mathfrak{n}$. $V_{i+1}$ is spanned by elements $\varphi_X Z$, where $Z \in V_i$, and we consider three options for $X \in \mathfrak{m}$.

Case 2.1: $X \in V_j$, $0 \leq j \leq i-1$. This is trivial, since $\varphi_X Z = -\varphi_Z X$, and by the choice of $X$, $\varphi_Z X \in V_{j+1}$. But $j + 1 \leq i$, so $V_{j+1} \subset \mathfrak{n}$ by assumption.

Case 2.2: $X \in V_i$. Using Corollary 3.22 and then reorganizing the terms into three groups and cancelling we get

$$\begin{aligned} 0 &= \varphi_X(R(Z,U)V) - R(\varphi_X Z, U)V - R(Z, \varphi_X U)V - R(Z,U)\varphi_X V \\ &\quad +\varphi_Z(R(U,X)V) - R(\varphi_Z U, X)V - R(U, \varphi_Z X)V - R(U,X)\varphi_Z V \\ &\quad +\varphi_U(R(X,Z)V) - R(\varphi_U X, Z)V - R(X, \varphi_U Z)V - R(X,Z)\varphi_U V \\ &= \{-R(\varphi_X Z, U)V - R(U, \varphi_Z X)V\} \\ &\quad +\{\varphi_X(R(Z,U)V) - R(Z, \varphi_X U)V - R(Z,U)\varphi_X V + \varphi_U(R(X,Z)V) \\ &\quad -R(\varphi_U X, Z)V - R(X,Z)\varphi_U V\} \end{aligned}$$

$$\begin{aligned}
&\{+\varphi_Z(R(U,X)V) - R(U,X)\varphi_Z V - R(X,\varphi_U Z)V\} \\
= &\{-R(\varphi_X Z,U)V - R(U,\varphi_Z X)V\} \\
= &-2R(\varphi_X Z,U)V,
\end{aligned}$$

for $U, V \in \mathfrak{m}$. The second set contains only vanishing terms, since $Z \in V_i \subset \mathfrak{n}$, so $0 = R(Z,\cdot)(\cdot) - cR_0(Z,\cdot)(\cdot) = R(Z,\cdot)(\cdot)$, as $c = 0$. Similarly the terms of the third set also vanish, since we have assumed that $X \in V_i$. We are thus left with $-2R(\varphi_X Z, U)V = 0$, so $\varphi_X Z \in \mathfrak{n}$.

Case 2.3: $X \in (\sum_{k=0}^{i} V_k)^\perp$. Since $Z \in V_i$, $Z = \varphi_U Y$, for some $Y \in V_{i-1}$, and $U \in \mathfrak{m}$. Since $M$ is naturally reductive we, for any $W \in \mathfrak{m}$, get that

$$< \varphi_X Y, W >= - < \varphi_Y X, W >=< X, \varphi_Y W >= - < X, \varphi_W Y > .$$

But $\varphi_W Y \in V_i$, since $Y \in V_{i-1}$, so by choice of $X$, $< X, \varphi_W Y >= 0$, so $\varphi_X Y = 0$. This in turn implies that

$$\begin{aligned}
R(X,U)Y &= -[[X,U]_{\mathfrak{h}},Y] + \varphi_X\varphi_U Y - \varphi_U\varphi_X Y - 2(\varphi_{\varphi_X U} Y) \\
&= -[[X,U]_{\mathfrak{h}},Y] + \varphi_X Z - 2(\varphi_{\varphi_X U} Y).
\end{aligned}$$

But we also have

$$\begin{aligned}
R(X,U)Y &= -R(U,Y)X - R(Y,X)U \\
&= R(Y,U)X - R(Y,X)U \\
&= 0,
\end{aligned}$$

since by assumption $Y \in V_{i-1} \subset \mathfrak{n}$. Thus

$$0 = -[[X,U]_{\mathfrak{h}},Y] + \varphi_X Z - 2(\varphi_{\varphi_X U} Y),$$

and therefore

$$\varphi_X Z = [[X,U]_{\mathfrak{h}},Y] + 2(\varphi_{\varphi_X U} Y).$$

Since both terms on the right hand side are contained in $\mathfrak{n}$, we have $\varphi_X Z \in \mathfrak{n}$. This concludes the proof of this case, and we have proved that $V_i \subset \mathfrak{n}$ for all $i \in \mathbb{N}$. We now set $O_i = V_0 + V_1 + ... + V_i$, which implies that $O_0 \subseteq O_1 \subseteq \dots$, so there exists some $i$ such that $O_i = O_{i+1}$. Then by construction $O_i$ is $\Lambda_{\mathfrak{m}}$-invariant, and since $V_0 = \mathbb{R}\xi \subset O_i$, $O_i$ is non-empty. Then we must have $O_i = \mathfrak{m}$, and therefore $\mathfrak{n} = \mathfrak{m}$. We conclude that $R = 0$ and the theorem has been proved. □

**Theorem 3.29.** [7] *Suppose $M = G/K$ is naturally reductive with respect to a subgroup $G' \leq G$. Let $H$ be a subgroup of $G$ which contains $K$. Then the submanifold $H/K$ of $M$ with the induced Riemannian structure is naturally reductive and totally geodesic in $M$.*

*Proof.* [7] Suppose that $M = G/K$ is naturally reductive with respect to a subgroup of $G^{'}$ of $G$. This case of the Theorem can now be obtained as follows: let $K^{'} = G^{'} \cap K$, and $H^{'} = H \cap G^{'}$. We then have $K^{'} \subset H^{'}$, and since $G^{'}$ is transitive we also have $H/K = H^{'}/K^{'}$. We may therefore assume that $M$ is naturally reductive with respect to $G$ itself, and a decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{q}$. Letting $K_0$ denote the largest subgroup of $K$ which is normal in $H$, Proposition 1.46 shows that $H/K_0$ is a transitive and effective group of isometries on $N = H/K$. We set $\mathfrak{p} = \mathfrak{q} \cap \mathfrak{h}$, so that $\mathfrak{h} = \mathfrak{k} + \mathfrak{p}$, and therefore in particular $[X,Y]_{\mathfrak{q}} \in \mathfrak{p}$ for all $X, Y \in \mathfrak{h}$. Let $\overline{X}$ be the image of $X \in \mathfrak{h}$ in $\mathfrak{h}/\mathfrak{k}_0$, under the natural projection. Since $\mathfrak{p}$ is complementary to $\mathfrak{k}$, $X \mapsto \overline{X}$ is injective on $\mathfrak{p}$, and $d\pi(\mathfrak{p}) = \overline{\mathfrak{p}} = T_pN$. The induced metric on $\overline{\mathfrak{m}}$ is then given by $\overline{g}(\overline{X},\overline{Y})_{\overline{\mathfrak{p}}} = g(X,Y)$, where $g$ is the metric on $\mathfrak{q}$, and furthermore $[\overline{X},\overline{Y}]_{\overline{\mathfrak{p}}} = \overline{[X,Y]}_{\mathfrak{q}} \in \mathfrak{p}$. It follows that $N = H/K$ is naturally reductive with respect to the decomposition $\mathfrak{h}/\mathfrak{k}_0 = \overline{\mathfrak{k}} + \overline{\mathfrak{p}}$, for we have

$$\begin{aligned}
\overline{g}([\overline{X},\overline{Y}]_{\overline{\mathfrak{p}}},\overline{Z}) &= \overline{g}(\overline{[X,Y]}_{\mathfrak{q}},\overline{Z}) \\
&= g([X,Y]_{\mathfrak{q}},Z) \\
&= g(X,[Y,Z]_{\mathfrak{q}})
\end{aligned}$$

$$= \quad \overline{g}(\overline{X}, [\overline{Y}, \overline{Z}]_{\overline{\mathfrak{p}}}),$$

since $M$ was naturally reductive with respect to $G$ and $\mathfrak{g} = \mathfrak{q} + \mathfrak{k}$. We can of course identify $\mathfrak{p} \equiv \overline{\mathfrak{p}}$ and the induced metric is then just the restriction to $\mathfrak{p}$. From Theorem 1.60 we have that the geodesics starting at $p \in N$ are just $e^{tX} \cdot p$, for $X \in \mathfrak{p}$, and since $M$ is naturally reductive as well, this is a geodesic in $M$. Therefore $N$ is totally geodesic in $M$ at $p$, and homogeneity implies that $N$ is then totally geodesic in $M$. □

The following corollary is due to my own efforts.

**Corollary 3.30.** *Let $G/L$ be an irreducible Riemannian manifold which is naturally reductive with respect to some transitive subgroup of $G$. Suppose there exists a subgroup $H$ of $G$ of codimension $1$, such that $L \subset H$. Then $G/L$ has constant curvature.*

*Proof.* If $G/L = M$ is naturally reductive with respect to some subgroup of $G$, then Tsukada's theorem applies. If $H$ is of codimension 1, then $H/L \subset G/L$ is of codimension 1, i.e. $H/L$ is a hypersurface, which by Theorem 3.29 is totally geodesic. Tsukada's Theorem then implies that $G/L$ has constant sectional curvature. □

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