CURVATURES OF LIE GROUPS

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Abstract

The main purpose of this work is to write a computer program to compute the curvatures of Riemannian Lie groups. In chapter 1 we introduce the necessary notions and state the basis results on the curvatures of Lie groups. In chapter 2 and 3 we calculate the sectional and Ricci curvatures of the 3- and 4-dimensional Lie groups with standard metrics.

Throughout this work it has been my firm intention to give reference to the stated results and credit to the work of others. All theorems, propositions, lemmas and examples left unmarked are assumed to be too well known for a reference to be given.
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Chapter 1

Introduction

1.1 Riemannian manifolds

Let \( M \) be a differentiable manifold and the tangent bundle

\[ TM = \{ (p, v) \mid p \in M, \ v \in T_pM \} \]

of \( M \) be equipped with its differentiable structure induced by that of \( M \). We define a vector field as a smooth section \( X : M \to TM \) of \( TM \) and denote the set of all these by \( C^\infty(TM) \).

**Definition 1.1.** A real vector space \( V \) together with the bilinear map \([\cdot, \cdot] : V \times V \to V\) called the Lie bracket is said to be a Lie algebra when for all \( X, Y, Z \in V \)

\[ [X, X] = 0, \]

\[ [X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0. \]

The latter is commonly called the Jacobi identity.

When \([X, Y] = 0 \) for all \( X, Y \in g \) the Lie algebra is said to be abelian.

The set \( C^\infty(TM) \) of smooth vector fields on \( M \) is a vector space and forms a Lie algebra i.e. we have a bilinear skew-symmetric form

\[ [\cdot, \cdot] : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM) \]

which satisfies the Jacobi identity \([X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \) for all \( X, Y, Z \in C^\infty(TM) \).

From now on we will assume that our differentiable manifold \( M \) is equipped with a Riemannian metric

\[ g : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(M). \]

A connection on the tangent bundle is a map

\[ \nabla : C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM), \]
\[ \nabla : (X, Y) \mapsto \nabla_X Y, \]

such that

\[ \nabla_{fX + gY} Z = f \nabla_X Z + g \nabla_Y Z, \]
\[ \nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z, \]
\[ \nabla_X fY = f \nabla_X Y + X(f)Y, \]

for all \( X, Y, Z \in C^\infty(TM) \) and \( f, g \in C^\infty(M) \). A Riemannian manifold \((M, g)\) has a unique connection which is torsion free and metric, that is

\[ [X, Y] = \nabla_X Y - \nabla_Y X \]

and

\[ g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = X(g(Y, Z)), \]

for all \( X, Y, Z \in C^\infty(TM) \). This unique connection is called the Levi-Civita connection and is of great importance in Riemannian geometry.

The Riemann curvature tensor

\[ R : C^\infty(TM) \otimes C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(TM) \]

is defined by

\[ R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \]

There are several symmetries concerning the Riemann curvature tensor, this list with five fundamental ones can be found in [3]:

\[
\begin{align*}
R(X, Y) Z &= -R(Y, X) Z, \\
g(R(X, Y) Z, W) &= -g(R(X, Y) W, Z), \\
g(R(X, Y) Z, W) + g(R(Z, X) Y, W) + g(R(Y, Z) X, W) &= 0, \\
g(R(X, Y) Z, W) &= g(R(Z, W) X, Y), \\
6 \cdot R(X, Y) Z &= R(X, Y + Z) (Y + Z) - R(X, Y - Z) (Y - Z) \\
\end{align*}
\]

We can define the curvature function

\[ \kappa : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(M) \]

by

\[ \kappa(X, Y) = g(R(X, Y) Y, X) \]

When the curvature function is constantly zero we say that the manifold is flat.

The real number

\[ K_p(X, Y) = \frac{\kappa(X, Y)}{|X|^2|Y|^2 - g(X, Y)^2} \]

is called the sectional curvature of the 2-plane generated by \( X, Y \) in \((M, g)\) at the point \( p \).
The **Ricci operator**

\[ r : C^\infty(TM) \to C^\infty(TM) \]

is defined by

\[ r (X) = \sum_{k=1}^{n} R (X, E_k) E_k, \]

where \( \{E_1, \ldots, E_m\} \) is an orthonormal basis.

The **Ricci curvature**

\[ Ric : C^\infty(TM) \otimes C^\infty(TM) \to C^\infty(M) \]

is defined by

\[ Ric(X, Y) = g (r (X), Y). \]

We represent the **Ricci quadratic form** by a matrix \( Ric \) where

\[ Ric_{ij} = g (r (E_i), E_j). \]

Using the symmetries for the Riemann curvature tensor we see that

\[
\begin{align*}
g (R (X, Y) Y, Z) &= g (R (Y, Z) X, Y) \\
&= -g (R (Y, Z) Y, X) \\
&= g (R (Z, Y) Y, X).
\end{align*}
\]

This tells us that \( g (r (E_i), E_j) = g (r (E_j), E_i) \), so \( Ric \) is symmetric. Note that

\[ Ric(X, Y) = g (r (X), Y) = \sum_{k=1}^{n} g (R (X, E_k) E_k, Y). \]

In the three dimensional case, with the orthonormal basis \( \{E_1, E_2, E_3\} \), we have

\[
Ric (E_1, E_1) = \sum_{k=1}^{3} g (R (E_1, E_k) E_k, E_1)
\]

\[
= \sum_{k=1}^{3} \kappa (E_1, E_k)
\]

\[
= \kappa (E_1, E_1) + \kappa (E_1, E_2) + \kappa (E_1, E_3).
\]

Repeating this process for all the basis elements we get this system:

\[
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
\kappa (E_1, E_2) \\
\kappa (E_2, E_3) \\
\kappa (E_1, E_3)
\end{bmatrix}
= \begin{bmatrix}
Ric (E_1, E_1) \\
Ric (E_2, E_2) \\
Ric (E_3, E_3)
\end{bmatrix},
\]

where the matrix has non-zero determinant, and we can thus solve the system for the curvature function values. So in the 3-dimensional case the Ricci curvature gives us as much information about the manifold as the sectional curvature.
1.2 Lie groups and Lie algebras

Definition 1.2. A Lie group is a smooth manifold $G$ with a group structure such that the map $\rho : G \times G \to G$ with

$$\rho : (p, q) \mapsto p \cdot q^{-1}$$

is smooth.

We say that a vector field $X \in C^\infty (TG)$ is left invariant if $dL_x X = X$ for all $x \in G$, where $L_x$ is the left translation satisfying $L_x (y) = xy$. When all the left translations $L_x$ are isometries, we call $g$ a left invariant metric. Geometrically a Lie algebra $g$ of a Lie group $G$ is the set of all left invariant vector fields on the Lie group,

$$g = \{ X \in C^\infty (TG) \mid dL_p (X) = X, \ p \in G \} .$$

When $X, Y \in g$,

$$dL_p ([X, Y]) = [dL_p (X), dL_p (Y)] = [X, Y] ,$$

so $[X, Y] \in g$. Thus $g$ is a Lie subalgebra of $C^\infty (TG)$. The Lie algebra $g$ of a Lie group $G$ is isomorphic to the tangent space $T_e G$ at the identity element of $G$, thus $g$ is finite dimensional since $T_e G$ is finite dimensional. So when we want to perform calculations in $g$ we can use the elements of $T_e G$, or the other way around. If a Lie subalgebra $\mathfrak{h}$ of $g$ satisfies $[\mathfrak{h}, g] \subseteq \mathfrak{h}$, then we call $\mathfrak{h}$ an ideal in $g$.

When $g$ has no non-zero proper ideals we say it is simple. We recursively define $g^0 = g$, $g^1 = [g, g]$, $g^{j+1} = [g^j, g^j]$. The sequence

$$g = g^0 \supseteq g^1 \supseteq g^2 \supseteq \cdots$$

is called the derived series for $g$. If this series eventually reaches 0 we say that $g$ is solvable. If $g$ has no non-zero solvable ideals it is called semisimple. We also
recursively define $g_0 = g$, $g_1 = [g, g]$, $g_{j+1} = [g, g_j]$, and we call the sequence

$$g = g_0 \supseteq g_1 \supseteq g_2 \supseteq \cdots$$

the **lower central series** for $g$. If this series eventually reaches 0 we say that $g$ is **nilpotent**.

We give a few examples of Lie Groups:

**Example 1.3.** The **real general linear group** is defined by

$$\text{GL}_n (\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \det A \neq 0 \}.$$  

The **real special linear group** is given by

$$\text{SL}_n (\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \det A = 1 \}.$$  

The **special orthogonal group** is defined by

$$\text{SO} (n) = \{ A \in \mathbb{R}^{n \times n} \mid A^T \cdot A = e, \det A = 1 \}.$$  

Let $\{E_1, \ldots, E_n\}$ be an orthonormal basis for the vector space $T_e G$, then by the bilinearity of the Lie bracket it will be completely determined by the elements $[E_i, E_j]$. We can then look at the scalars $C^k_{ij}$ such that

$$[E_i, E_j] = \sum_{k=1}^{n} C^k_{ij} E_k.$$  

These are called the **structure constants** of the Lie algebra. Since we have a Riemannian metric $g (\cdot, \cdot)$ we can also write these as

$$C^k_{ij} = g ([E_i, E_j], E_k).$$  

For a linear transformation $L$ of a Euclidean space its **adjoint** $L^*$ is defined by the equation $g (Lx, y) = g (x, L^*y)$. Whenever $L^* = -L$ we call $L$ **skew-adjoint**. When we are working with left invariant vector fields the function $g (Y, Z)$ will be constant, so $X (g (Y, Z)) = 0$. Remarkably, one can by using only these identities and combining a few permutations of variables obtain the formula

$$2 \cdot g (\nabla_X Y, Z) = g ([X, Y], Z) + g ([Z, X], Y) + g ([Z, Y], X) \quad (1.1)$$

for the Levi-Civita connection. Another identity relevant to this is

$$\nabla_X Y = \sum_{k=1}^{n} g (\nabla_X Y, E_k) E_k. \quad (1.2)$$

This and the linearity of the metric means that we can express the connection as a linear combination of metrics of basis vectors,

$$\nabla_X Y = \sum_{k=1}^{n} (g ([X, Y], E_k) + g ([E_k, X], Y) + g ([E_k, Y], X)) E_k. \quad (1.3)$$
Example 1.4. The Lie algebra of $\text{GL}_n(\mathbb{R})$ is denoted $\mathfrak{gl}_n(\mathbb{R})$ and is isomorphic to $T_e \text{GL}_n(\mathbb{R}) = \mathbb{R}^{n \times n}$.

The Lie algebra of $\text{SL}_n(\mathbb{R})$ is denoted $\mathfrak{sl}_n(\mathbb{R})$ and is isomorphic to $T_e \text{SL}_n(\mathbb{R}) = \{ X \in \mathbb{R}^{n \times n} | \text{trace}(X) = 0 \}$.

The Lie algebra of $\text{SO}(n)$ is denoted $\mathfrak{so}(n)$ and is isomorphic to $T_e \text{SO}(n) = T_e \text{O}(n) = \{ X \in \mathbb{R}^{n \times n} | X^T + X = 0 \}$.

1.3 Curvatures of Lie groups

The adjoint homomorphism $\text{ad}(Z)$ is defined by $\text{ad}(Z)(Y) = [Z, Y]$. Let $i_p : G \to G$ be defined by $i_p(q) = pp^{-1}$. So $i_p(e) = e$ and we define the adjoint representation $Ad(p)$ of $G$ on $\mathfrak{g}$ by $Ad(p) = d(i_p)_e : \mathfrak{g} \to \mathfrak{g}$.

Theorem 1.5 ([8]). If the linear transform $\text{ad}(Z)$ is skew-adjoint, then

$$\kappa(Z, X) \geq 0,$$

with equality if and only if $Z$ is orthogonal to $[X, \mathfrak{g}]$.

Proof. [8] Let us assume that we have the orthonormal basis $\{E_1, \ldots, E_n\}$. Since $C_{ij}^k = g([E_i, E_j], E_k)$ we can insert the basis elements in (1.1) and write

$$2 \cdot g(\nabla_{E_i}E_j, E_k) = g([E_i, E_j], E_k) + g([E_k, E_i], E_j) + g([E_k, E_j], E_i)$$

$$= C_{ij}^k + C_{jk}^i + C_{ki}^j.$$

Applying this to (1.2) gives us

$$\nabla_{E_i}E_j = \frac{1}{2} \sum_{k=1}^{n} (C_{ij}^k + C_{ki}^j + C_{kj}^i) E_k.$$

This we just insert into $\kappa(E_1, E_2)$, and perform the following calculation:

$$\kappa(E_1, E_2) = g(R(E_1, E_2)E_2, E_1).$$

$$\nabla_{E_2}E_2 = \frac{1}{2} \sum_{k=1}^{n} (C_{22}^k + C_{k2}^2 + C_{k2}^2) E_k = \sum_{k=1}^{n} C_{k2}^2 E_k.$$

$$\nabla_{E_1}E_2 = \frac{1}{2} \sum_{k=1}^{n} (C_{12}^k + C_{k1}^2 + C_{k2}^1) E_k.$$

$$\nabla_{[E_1, E_2]}E_2 = \nabla \sum_{k=1}^{n} C_{12}^k E_k E_2 = \sum_{k=1}^{n} C_{12}^k \nabla_{E_k}E_2.$$
\[ \nabla_{E_k} E_2 = \frac{1}{2} \sum_{l=1}^{n} (C_{k2}^l + C_{lk}^2 + C_{l2}^k) E_l. \]

\[ \nabla_{E_l} E_k = \frac{1}{2} \sum_{l=1}^{n} (C_{lk}^1 + C_{lk}^1 + C_{lk}^1) E_l. \]

\[ \nabla_{E_2} E_k = \frac{1}{2} \sum_{l=1}^{n} (C_{l2}^k + C_{l2}^k + C_{l2}^k) E_l. \]

\[ g \left( \nabla_{E_1} \nabla_{E_2}, E_1 \right) = \frac{1}{2} \sum_{k=1}^{n} \left( C_{k1}^2 \sum_{l=1}^{n} (C_{1k}^l + C_{lk}^1 + C_{lk}^1) \cdot g(E_l, E_1) \right) \]

\[ = \frac{1}{2} \sum_{k=1}^{n} C_{k1}^2 \left( C_{1k}^1 + C_{1k}^1 + C_{lk}^1 \right) \]

\[ = \sum_{k=1}^{n} C_{k1}^2 C_{1k}^1. \]

\[ g \left( \nabla_{E_2} \nabla_{E_1}, E_1 \right) = \frac{1}{4} \sum_{k=1}^{n} \left( (C_{k2}^1 + C_{k2}^1) \sum_{l=1}^{n} (C_{2k}^l + C_{l2}^k + C_{l2}^k) g(E_l, E_1) \right) \]

\[ = \frac{1}{4} \sum_{k=1}^{n} (C_{k1}^1 + C_{k1}^1) \left( C_{2k}^1 + C_{l2}^k + C_{l2}^k \right). \]

\[ g \left( \nabla_{[E_1, E_2]} E_2, E_1 \right) = \frac{1}{2} \sum_{k=1}^{n} \left( C_{k2}^1 \sum_{l=1}^{n} (C_{lk}^1 + C_{lk}^1) g(E_l, E_1) \right) \]

\[ = \frac{1}{2} \sum_{k=1}^{n} C_{k2}^1 \left( C_{lk}^1 + C_{lk}^1 \right). \]

\[ \kappa(E_1, E_2) = \sum_{k=1}^{n} \left( C_{k2}^2 C_{1k}^1 - \frac{1}{4} (C_{k2}^1 + C_{k2}^1 + C_{k2}^1) (C_{1k}^1 + C_{l2}^1 + C_{l2}^1) \right) \]

\[ - \frac{1}{2} C_{k1}^2 \left( C_{lk}^1 + C_{lk}^1 + C_{l2}^1 \right) \]

\[ = \sum_{k=1}^{n} \left( -C_{k2}^2 C_{k1}^1 - \frac{1}{4} (C_{k2}^1 + C_{k2}^1 - C_{2k}^1) (C_{1k}^1 + C_{l2}^1 + C_{l2}^1) \right) \]

\[ + \frac{1}{2} C_{k1}^2 \left( -C_{l2}^1 - C_{l2}^1 - C_{l2}^1 \right) \]
And we obviously get the inequality. The sum equals 0 only when every \( C_{ik}^j = 0 \).

We end up with the formula

\[
\kappa(E_1, E_2) = \sum_{k=1}^{n} \left( \frac{1}{2} C_{12}^k (-C_{12}^k + C_{2k}^1 + C_{k1}^2) \right. \\
\left. - \frac{1}{4} (C_{12}^k - C_{2k}^1 + C_{k1}^2) (C_{12}^k + C_{2k}^1 - C_{k1}^2) - C_{k1}^1 C_{k2}^2 \right). 
\]

Now we assume \( Z \) and \( X \) are orthonormal, and that our basis is such that \( E_1 = Z, E_2 = X \). Saying \( ad(E_1) \) is skew-adjoint means \( C_{ij}^k = -C_{kj}^i \) in the language of structure constants. And applying this rule to the above formula we can do these calculations:

\[
\kappa(E_1, E_2) = \sum_{k=1}^{n} \left( \frac{1}{2} C_{12}^k (-C_{12}^k + C_{2k}^1 + C_{k1}^2) \right. \\
\left. - \frac{1}{4} (C_{12}^k - C_{2k}^1 + C_{k1}^2) (C_{12}^k + C_{2k}^1 - C_{k1}^2) + C_{ik}^j C_{jk}^2 \right) 
\]

And we obviously get the inequality. The sum equals 0 only when every \( C_{2k}^1 = 0 \).

As \( C_{2k}^1 = g([E_2, E_k], E_1) = g([X, E_k], Z) \), this equaling 0 means \([X, E_k] \) and \( Z \) are orthogonal. If this is true for every \( E_k \) then \( Z \) is orthogonal to \([X, g]\).

\[\square\]
Corollary 1.6 ([8]). If the linear transformation $\text{ad}(Z)$ is skew-adjoint, then

\[ \text{Ric}(Z, Z) \geq 0, \]

with equality if and only if $Z$ is orthogonal to $[g, g]$.

Theorem 1.7 ([8]). If $Z$ is orthogonal to $[g, g]$, then

\[ \text{Ric}(Z, Z) \leq 0, \]

with equality if and only if $\text{ad}(Z)$ is skew-adjoint.

Example 1.8. We are going to calculate the Riemann curvature and the Ricci curvature for a particular Lie group. We will define the metric by a matrix $g$ such that for two basis elements $E_i$, $E_j$ in the Lie algebra of the Lie group we have $g(E_i, E_j) = g_{ij}$. Here we are going to use

\[ g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]

For the groups algebra we will use the basis \{X, Y, Z\} and the defining equations

\[ [X, Y] = c \cdot Z, \quad [Z, X] = b \cdot Y, \quad [Y, Z] = a \cdot X. \]

We translate this to structure constants and use the technique in the proof above. We apply the identity $C_{ij}^k = g([E_i, E_j], E_k)$ and get

\[
\begin{align*}
C_{XY}^Z &= c \\
C_{YZ}^X &= b \\
C_{ZX}^Y &= a
\end{align*}
\]

This notation is not standard, but is an efficient shorthand for this calculation.

\[
\nabla_X Y = \frac{1}{2} \sum_{k \in \{X, Y, Z\}} (C_{XY}^k + C_{kX}^Y + C_{kY}^X) k
\]

\[
= \frac{1}{2} \left( 0 \cdot X + 0 \cdot Y + (c + b - a) \cdot Z \right) = \frac{c + b - a}{2} \cdot Z.
\]

When we do this for all the other combinations of basis vectors we also get

\[
\begin{align*}
\nabla_X Z &= \frac{a - b - c}{2} \cdot Y \\
\nabla_Y X &= \frac{b - c - a}{2} \cdot Z \\
\nabla_Y Z &= \frac{a + c - b}{2} \cdot X
\end{align*}
\]
\[ \nabla_Z X = \frac{b + a - c}{2} \cdot Y \]
\[ \nabla_Z Y = \frac{c - a - b}{2} \cdot X. \]

With this information we can calculate the Riemann curvature tensors:

\[ R(X, Y) Y = \nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y \]
\[ = \nabla_X 0 - \frac{c + b - a}{2} \nabla_Y Z - c \nabla_Z Y \]
\[ = -\frac{c + b - a}{2} \cdot \frac{a + c - b}{2} \cdot X - c \cdot \frac{c - a - b}{2} \cdot X \]
\[ = \left( -\frac{c + b - a}{2} \cdot \frac{a + c - b}{2} - c \cdot \frac{c - a - b}{2} \right) \cdot X, \]
\[ R(X, Z) Z = \left( -\frac{a - b - c}{2} \cdot \frac{c - a - b}{2} + b \cdot \frac{a + c - b}{2} \right) \cdot X, \]
\[ R(Y, X) X = \left( -\frac{b - c - a}{2} \cdot \frac{a - b - c}{2} + c \cdot \frac{b + a - c}{2} \right) \cdot Y, \]
\[ R(Y, Z) Z = \left( -\frac{a + c - b}{2} \cdot \frac{b + a - c}{2} - a \cdot \frac{a - b - c}{2} \right) \cdot Y, \]
\[ R(Z, X) X = \left( -\frac{b + a - c}{2} \cdot \frac{c + b - a}{2} - b \cdot \frac{b - c - a}{2} \right) \cdot Z, \]
\[ R(Z, Y) Y = \left( -\frac{c - a - b}{2} \cdot \frac{b - c - a}{2} + a \cdot \frac{c + b - a}{2} \right) \cdot Z. \]

We will save some space by making a matrix \( K \) where \( K_{ij} = \kappa(E_i, E_j) \). As the metric is orthonormal, we get the Riemannian curvature directly from these. We know that \( K \) is symmetric and that its diagonal consists of zeros, so we choose the three ones in the upper triangle of the \( K \)-matrix and simplify by multiplying together everything and completing the squares:

\[ \kappa(X, Y) = -\frac{c + b - a}{2} \cdot \frac{a + c - b}{2} - c \cdot \frac{c - a - b}{2} \]
\[ = \frac{a^2 + ac - ab - ac - c^2 + bc - ab - bc + b^2 - 2c^2 + 2ac + 2bc}{4} \]
\[ = \frac{a^2 - 2ab - 3c^2 + b^2 + 2ac + 2bc}{4} \]
\[
\kappa(X, Z) = \frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4}
\]
\[
\kappa(Y, Z) = \frac{b^2 - 2bc - 3a^2 + c^2 + 2ab + 2ac}{4}.
\]

And thus
\[
K = \begin{bmatrix}
0 & \frac{a^2 - 2ab - 3c^2 + b^2 + 2ac + 2bc}{4} & \frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4} \\
\frac{a^2 - 2ab - 3c^2 + b^2 + 2ac + 2bc}{4} & 0 & \frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4} \\
\frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4} & \frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4} & 0
\end{bmatrix}.
\]

To calculate the Ricci curvatures, and the Ricci quadratic form, we can use the relation between curvatures at the end of the first section.

\[
Ric(X, X) = \kappa(X, Y) + \kappa(X, Z)
\]
\[
= \frac{a^2 - 2ab - 3c^2 + b^2 + 2ac + 2bc}{4} + \frac{-3b^2 + c^2 - 2ac + a^2 + 2ab + 2bc}{4} \\
= \frac{2a^2 - 2c^2 - 2b^2 + 4bc}{4} \\
= \frac{a^2 - (b^2 - 2bc + c^2)}{2} \\
= \frac{a^2 - (b - c)^2}{2}
\]
\[
Ric(Y, Y) = \frac{b^2 - (a - c)^2}{2}
\]
\[
Ric(Z, Z) = \frac{c^2 - (a - b)^2}{2}.
\]

So we get
\[
Ric = \begin{bmatrix}
\frac{a^2 - (b - c)^2}{2} & 0 & 0 \\
0 & \frac{b^2 - (a - c)^2}{2} & 0 \\
0 & 0 & \frac{c^2 - (a - b)^2}{2}
\end{bmatrix}.
\]

Lemma 1.9 ([8]). Let \( g \) be a left invariant metric on a connected Lie group \( G \). This metric will also be right invariant if and only if \( \text{ad}(X) \) is skew-adjoint for every \( X \in g \).
Proof. [8] Let us mention that a left invariant metric also is right invariant if and only if for each \( p \in G \), \( \text{Ad}(p) \) is an isometry with respect to the induced metric on \( g \):
\[
g(\text{Ad}(p)(X), \text{Ad}(p)(Y)) = g(X,Y).
\]

First assume that \( g \) is both left and right invariant so that \( \text{Ad}(p) \) is an isometry. For a \( p \in G \) close to \( e \) we have \( p = \exp(X) \) for some unique \( X \) close to 0 in \( g \). We will use this identity (found for example in [10], p54):
\[
\text{Ad}(\exp(X)) = e^{ad(X)}.
\]

We have
\[
g(\text{Ad}(p)(X), \text{Ad}(p)(Y)) = g(X, \text{Ad}(p)^*(\text{Ad}(p)(Y))) = g(X,Y),
\]
and thus \( \text{Ad}(p)^* = \text{Ad}(p)^{-1} \).

We get
\[
\text{Ad}(p)^{-1} = \text{Ad}(p)^* \\
\text{Ad}(\exp(X))^{-1} = \text{Ad}(\exp(X))^* \\
e^{-ad(X)} = e^{ad(X)^*} \\
-ad(X) = ad(X)^*,
\]
and so \( ad(X) \) is skew-adjoint.

Next we assume \( ad(X) \) is skew-adjoint for all \( X \). Now \( \text{Ad}(\exp(X)) = e^{ad(X)} \) acts as isometries on \( g \), so for any \( p \in \exp(g) \), \( \text{Ad}(p) \) is an isometry on \( g \). There is an open neighbourhood \( U \) in \( \exp(g) \) containing the identity element \( e \). Since \( G \) is connected, any element in \( G \) can be written as a product of elements in \( U \). Any product of isometries is an isometry, so we see that \( \text{Ad}(p) \) is an isometry for any \( p \) in \( G \). So \( g \) is both left and right invariant.

Lemma 1.10 ([8]). A connected Lie group \( G \) admits a bi-invariant metric if and only if \( G \) is isomorphic to the cartesian product of a compact group and an abelian group.

Theorem 1.11 ([8]). Every compact connected Lie group admits a left invariant metric such that \( K_p \geq 0 \) for all \( p \).

Milnor gives in [8] a couple of interesting theorems which we will have a look at. We will use the classical Bonnet-Myers theorem in one of the proofs, so let us mention it briefly.

Theorem 1.12 (Bonnet-Myers). Let \( M \) be a complete Riemannian manifold. If there exist an \( r > 0 \) so that \( \text{Ric}(Z,Z) \geq r \cdot g(Z,Z) \) at all \( Z \), then \( M \) is compact.

Theorem 1.13 (Iwasawa, [8]). If \( G \) is a connected Lie group and \( H \) a maximal compact subgroup of \( G \), then \( G \) is homeomorphic to \( H \times \mathbb{R}^m \). Furthermore, any compact subgroup of \( G \) is contained in a maximal compact subgroup, which is a connected Lie group.
These two following theorems inform us that there are non-abelian Lie groups which are flat.

**Theorem 1.14** ([8]). If a Lie group with left invariant metric is flat, then its Lie algebra splits into an orthogonal sum $\mathfrak{h} \oplus \mathfrak{i}$. With $\mathfrak{h}$ an abelian Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{i}$ an abelian ideal, and $\text{ad}(Z)$ is skew-adjoint for every $Z \in \mathfrak{h}$.

**Proof.** [8] Let $G$ be a simply connected Lie group with a flat left-invariant metric. A flat, simply connected, complete Riemannian manifold is homeomorphic to Euclidean space, $\mathbb{R}^n$, where $n$ is the dimension of $G$. Let $K$ be any compact subgroup of $G$. According to Iwasawa above $K \subset H$ where $H$ is a maximal compact subgroup of $G$. So

$$\mathbb{R}^n \cong G \cong H \times \mathbb{R}^m,$$

where $n - m$ is the dimension of $H$. If $H$ is non-trivial we can take a volume form $\omega$ for $H$. Let $\iota : H \rightarrow H \times \mathbb{R}^m$ be the inclusion and $\pi : H \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be the projection. Then $\pi^* \omega$ is a closed form on $H \times \mathbb{R}^m \cong \mathbb{R}^n$, and hence it is also exact: $\pi^* \omega = d\alpha$. Then $\omega = \iota^* \pi^* \omega = \iota^* d\alpha = dt^* \alpha$, meaning $\omega$ is exact on $H$. But that is impossible since $H$ is compact. Thus we have a contradiction, and $H$ must be trivial. Since $K \subset H$, $K$ is also trivial. And so any compact subgroup of $G$ must be trivial.

Let $g$ have a metric, and let us look at $X \mapsto \nabla_X$ which defines a mapping $g \rightarrow o(n)$.

We are assuming the curvature tensor is identical to zero, so that

$$\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X.$$

We have $\nabla_{[X,Y]} = \nabla_X \nabla_Y - \nabla_Y \nabla_X = [\nabla_X, \nabla_Y]$ so $X \mapsto \nabla_X$ is a Lie algebra homomorphism. The kernel of a homomorphism is an ideal. In this case we call the kernel $\mathfrak{i}$. Since

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

we must for any $X,Y \in \mathfrak{g}$ have $[X,Y] = \underbrace{\nabla_X Y}_{0} - \underbrace{\nabla_Y X}_{0} = 0$ and we understand that $\mathfrak{i}$ is abelian.

Let us define $\mathfrak{h}$ as the orthogonal complement of $\mathfrak{i}$. For a $Z \in \mathfrak{h}$ and $X \in \mathfrak{i}$ we have

$$[Z,X] = \nabla_Z X - \underbrace{\nabla_Z X}_{0} = \nabla_Z X.$$ 

Due to the metric property of the Levi-Civita connection and the metric being left invariant we have

$$Z(g(X,Y)) = g(\nabla_Z X,Y) + g(X,\nabla_Z Y) = 0,$$

or $g(\nabla_Z X,Y) = -g(X,\nabla_Z Y)$ and $\nabla_Z$ is skew-adjoint. $\nabla_Z X = [Z,X] \in \mathfrak{i}$, so $\nabla_Z$ maps $\mathfrak{i}$ into itself. When $W \in \mathfrak{h}$, we get $g(\nabla_W X,Y) = -g(W,\nabla_Z X) = 0$, meaning
\( \nabla_Z W \) is orthogonal to \( X \) and \( \nabla_X \) must therefore also map \( \mathfrak{h} \) into itself. And thus \( \mathfrak{h} \) must be a Lie subalgebra of \( \mathfrak{g} \). \( \mathfrak{h} \) maps isomorphically to a Lie subalgebra of \( \mathfrak{o}(n) \). \( \mathfrak{o}(n) \) is the Lie algebra of the Lie group \( \mathbf{O}(n) \), which is compact, so it possesses some bi-invariant metric. And thus so do \( \mathfrak{h} \). This means that \( ad(Z) \) is skew-adjoint for any \( Z \in \mathfrak{h} \). Let us look at \( \mathfrak{h} \) with this bi-invariant metric, which we call \( \mathfrak{h} \) Say we have the ideal \( \mathfrak{h}_1 \subset \mathfrak{h} \), and some \( W \in \mathfrak{h} \) which is orthogonal to \( \mathfrak{h}_1 \). Since \( ad(E) \) is skew-adjoint we also get

\[
\text{Ric}(Z, Z) = -\frac{\lambda}{4} \cdot g(Z, Z),
\]

where \( Z \in \mathfrak{h}_1 \), the \( E_k \)'s are basis elements for \( \mathfrak{h}_1 \) and \( n \) is the dimension of \( \mathfrak{h}_1 \). Since \( ad(E) \) is skew-adjoint we also get

\[
\text{Ric}(Z, Z) = -\frac{1}{4} \cdot \sum_{k=1}^{n} g(Z, [E_k, Z]),
\]

Let \( C \) be the Casimir-operator

\[
C: Z \mapsto \sum_{k=1}^{n} [E_k, [E_k, Z]].
\]

Say \( \lambda \) is an eigenvalue to \( C \). Then \( \ker(C - \lambda I) \) is a non-trivial ideal to \( \mathfrak{h}_1 \), but since \( \mathfrak{h}_1 \) is simple

\[
C - \lambda I = 0
\]

so we have

\[
\text{Ric}(Z, Z) = -\frac{\lambda}{4} \cdot g(Z, Z),
\]

where \( \lambda < 0 \) According to Bonnet-Myers' theorem the Lie group \( B_i \) corresponding to \( \mathfrak{h}_i \) would be compact. The inclusion \( \mathfrak{h}_i \subset \mathfrak{h} \subset \mathfrak{g} \) would then induce a non-trivial homomorphism \( B_i \to G \). So \( G \) would contain a non-trivial compact subgroup. That is impossible. Therefore every \( \mathfrak{h}_i \) must be abelian, and thus \( \mathfrak{h} \) is abelian.

So \( \mathfrak{g} = \mathfrak{i} \oplus \mathfrak{h} \), where \( \mathfrak{i} \) is a abelian ideal, \( \mathfrak{h} \) is an abelian Lie subalgebra and \( ad(Z) \) is skew-adjoint. \( \square \)
Theorem 1.15 ([8]). Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ which splits into an orthogonal sum $\mathfrak{h} \oplus \mathfrak{i}$, with $\mathfrak{h}$ an abelian Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{i}$ an abelian ideal, and $\text{ad}(Z)$ is skew-adjoint for every $Z \in \mathfrak{h}$. Then the Lie group is flat.

Proof. [8] We look at equation (1.1) to make some conclusions about the Levi-Civita connection. Let $A \in \mathfrak{g}$, $X, Y \in \mathfrak{i}$ and $Z, W \in \mathfrak{h}$. We will look at five cases, being the different possibilities of combinations of these in the formula for the Levi-Civita connection. First:

$$2 \cdot g(\nabla_Z W, A) = g([Z, W], A) + g([A, Z], W) + g([A, W], Z)$$

$$= g([A, Z], W) - g([A, Z], W)$$

$$= 0.$$

Second:

$$2 \cdot g(\nabla_Z X, W) = g([Z, X], W) + g([W, Z], X) + g([W, X], Z) = 0.$$

Third:

$$2 \cdot g(\nabla_Z X, Y) = g([Z, X], Y) + g([Y, Z], X) + g([Y, X], Z)$$

$$= g([Z, X], Y) - g([Z, Y], X)$$

$$= g([Z, X], Y) + g([Z, X], Y)$$

$$= 2 \cdot g([Z, X], Y).$$

So $\nabla_Z = \text{ad}(Z)$. Fourth:

$$2 \cdot g(\nabla_X Y, A) = g([X, Y], A) + g([A, X], Y) + g([A, Y], X) = 0.$$

Fifth:

$$2 \cdot g(\nabla_X Z, A) = g([X, Z], A) + g([A, X], Z) + g([A, Z], X)$$

$$= -g([Z, A], X) = g([Z, X], A) = 0.$$

So $\nabla_X = 0$. We apply these identities for three different cases of the Riemann curvature tensor. First:

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{\nabla_X Y} = 0.$$

Second:

$$R(X, Z) = \nabla_X \nabla_Z - \nabla_Z \nabla_X - \nabla_{\nabla_X Z} = 0.$$
Third:

\[ R(Z, W)A = (\nabla_Z \nabla_W - \nabla_W \nabla_Z - \nabla_{[Z, W]}A)A = \nabla_Z \text{ad}(W)(A) - \nabla_W \text{ad}(Z)(A) - \text{ad}([Z, W])(A) \]

\[ = \text{ad}(Z)\text{ad}(W)(A) - \text{ad}(W)\text{ad}(Z)(A) - \text{ad}([Z, W])(A) \]


since this is the Jacobi identity. And so, the Riemann curvature tensor will always be zero.

We can look at a good example.

**Example 1.16.** The Euclidean motion group on two dimensional space, \( E(2) \), whose Lie algebra is defined by the structure equations

\[ [E_1, E_3] = -E_2, \quad [E_2, E_3] = E_1 \]

in the basis \( \{E_1, E_2, E_3\} \) is not abelian. But when equipped with a metric defined by a matrix of the form

\[
\begin{pmatrix}
  s & 0 & 0 \\
  0 & s & 0 \\
  0 & 0 & t
\end{pmatrix}
\]

it will be flat.

Heintze has shown in [11] that for a Lie group with left invariant metric to have strictly negative sectional curvature there was the condition for \( g = g^1 + \mathbb{R}X \) for some \( X \) where the eigenvalues of the restriction \( \text{ad} \ (X) |_{g^1} : g^1 \rightarrow g^1 \) have positive real part.

**Theorem 1.17** ([12]). A connected Lie group with left invariant metric with all sectional curvatures \( K_p \leq 0 \) is solvable.

The proof of this theorem requires many advanced topics not discussed in this paper. It is proved in [12] by Azencott and Wilson, where it is called Corollary 7.8.

**Theorem 1.18** ([14]). Let \( N \) be a transitive connected non-abelian nilpotent Lie group. Then for \( p \in N \) there exist two-dimensional subspaces \( R, S, T \subseteq T_p N \) such that the sectional curvatures satisfy

\[ K_p (S) < K_p (R) = 0 < K_p (T) \]

This theorem is difficult to prove, but we can prove this slightly weaker version presented by Milnor:

**Theorem 1.19** ([8]). Let \( N \) be a non-abelian nilpotent Lie group. Then there exists a direction of strictly negative and a direction of strictly positive Ricci curvature.
Proof. [8] let $k$ be the smallest number so that $g_{k+1} = 0$. Choose a unit vector $X \in g_k$, which then must be central, and contained in $g_1$. So by Corollary 1.6 we get that $Ric(X, X) > 0$. Say $\mathfrak{z}$ is the center of $\mathfrak{g}$. If $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{z}$ then

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}] + \mathfrak{z}] = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = [\mathfrak{g}, \mathfrak{g}_1] = \mathfrak{g}_2,$$

and we could not have nilpotency. So there must be a unit vector $Z \not\in \mathfrak{z}$ orthogonal to $\mathfrak{g}_1$. $ad(Z)$ is non-zero and nilpotent, so it can not be skew-adjoint. Hence we get that $Ric(Z, Z) < 0$ by Theorem 1.7.

Milnor goes on to give a generalization

**Theorem 1.20 ([8]).** Let $G$ be a Lie group. If there are three linearly independant vectors $X, Y, Z \in \mathfrak{g}$ so that $[X, Y] = Z$, then there exists a left invariant metric so that $Ric(X, X) < 0$ and $Ric(Z, Z) > 0$. 

17
Chapter 2

3-dimensional Lie Groups

Now we will look at the Lie groups of dimension 3 equipped with the metric \( g(\cdot, \cdot) \) where \( g(E_i, E_j) = g_{ij} \) and

\[
g = \begin{bmatrix}
  r^2 & 0 & 0 \\
  0 & s^2 & 0 \\
  0 & 0 & t^2
\end{bmatrix}.
\]

For these we are using the basis \( \{E_1, E_2, E_3\} \). A classification of the Lie algebras of dimension three and four (which are not products of lower dimensional algebras) is found in [9], where Patera et. al. lists the nine classes of three dimensional and twelve classes of four dimensional Lie algebras. Here is the list of 3-dimensional algebras along with their defining Lie bracket equations.

<table>
<thead>
<tr>
<th>Name</th>
<th>Structure equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{3.1} )</td>
<td>([E_2, E_3] = E_1 )</td>
</tr>
<tr>
<td>( A_{3.2} )</td>
<td>([E_1, E_3] = E_1, [E_2, E_3] = E_1 + E_2 )</td>
</tr>
<tr>
<td>( A_{3.3} )</td>
<td>([E_1, E_3] = E_1, [E_2, E_3] = E_2 )</td>
</tr>
<tr>
<td>( A_{3.4} )</td>
<td>([E_1, E_3] = E_1, [E_2, E_3] = -E_2 )</td>
</tr>
<tr>
<td>( A_{3.5}^a )</td>
<td>([E_1, E_3] = E_1, [E_2, E_3] = aE_2, (0 &lt;</td>
</tr>
<tr>
<td>( A_{3.6} )</td>
<td>([E_1, E_3] = -E_2, [E_2, E_3] = E_1 )</td>
</tr>
<tr>
<td>( A_{3.7}^a )</td>
<td>([E_1, E_3] = aE_1 - E_2, [E_2, E_3] = E_1 + aE_2, (a &gt; 0) )</td>
</tr>
<tr>
<td>( A_{3.8} )</td>
<td>([E_1, E_3] = -2E_2, [E_1, E_2] = E_1, [E_2, E_3] = E_3 )</td>
</tr>
<tr>
<td>( A_{3.9} )</td>
<td>([E_1, E_2] = E_3, [E_2, E_3] = E_1, [E_3, E_1] = E_2 )</td>
</tr>
</tbody>
</table>

We are going to use the Maple program to calculate the values of the curvature functions and the Ricci curvatures of the groups corresponding to these algebras. The last algebra, \( A_{3.9} \), corresponds to that of \( SU(2) \) whose curvatures we calculated in example 1.8 by hand. This will act as confirmation that the program works.

**Example 2.1** (\( \mathbb{R}^3 \)). But let us start by commenting on \( \mathbb{R}^3 \). The Lie algebra of \( \mathbb{R}^3 \) is defined by all brackets equaling zero. And so it is simply flat in all curvatures for any left invariant metric.

The 3-dimensional Lie groups are listed in the same order as in [9] and the table above.
Example 2.2 \((A_{3,1} = \text{Nil})\). This is the Lie group consisting of matrices of the form
\[
\begin{bmatrix}
1 & y & x \\
0 & 1 & z \\
0 & 0 & 1
\end{bmatrix},
\]
so we can define the Lie algebra of the group using the basis
\[
E_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
E_1 = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix},
E_1 = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]
The non-zero Lie brackets for this Lie algebra is
\[
[E_1, E_2] = E_3.
\]
For this algebra we get
\[
K = \begin{bmatrix}
0 & -3/4 \frac{t^2}{r^2 s^2} & 1/4 \frac{t^2}{r^2 s^2} \\
-3/4 \frac{t^2}{r^2 s^2} & 0 & 1/4 \frac{t^2}{r^2 s^2} \\
1/4 \frac{t^2}{r^2 s^2} & 1/4 \frac{t^2}{r^2 s^2} & 0
\end{bmatrix}
\]
and
\[
Ric = \begin{bmatrix}
-1/2 \frac{t^2}{r^2 s^2} & 0 & 0 \\
0 & -1/2 \frac{t^2}{r^2 s^2} & 0 \\
0 & 0 & 1/2 \frac{t^2}{r^2 s^2}
\end{bmatrix}.
\]
Eigenvalues to Ric:
\[
\left\{ 1/2 \frac{t^2}{r^2 s^2}, -1/2 \frac{t^2}{r^2 s^2}, -1/2 \frac{t^2}{r^2 s^2} \right\}
\]
Eigenvectors to Ric:
\[
\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}
\]
Example 2.3 \((A_{3,2})\). Structure equations:
\[
[E_1, E_3] = E_1, \ [E_2, E_3] = E_1 + E_2
\]
\[
K = \begin{bmatrix}
0 & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} \\
1/4 \frac{r^2 - 4s^2}{s^4 t^2} & 0 & -1/4 \frac{4s^2 + 3r^2}{s^4 t^2} \\
1/4 \frac{r^2 - 4s^2}{s^4 t^2} & -1/4 \frac{4s^2 + 3r^2}{s^4 t^2} & 0
\end{bmatrix}
\]
\[
Ric = \begin{bmatrix}
-\frac{r}{s t^2} & 1/2 \frac{4s^2 + r^2}{s^4 t^2} & 0 \\
-\frac{r}{s t^2} & 0 & 0 \\
0 & 0 & -1/2 \frac{4s^2 + r^2}{s^4 t^2}
\end{bmatrix}.
\]
Eigenvalues to Ric:
\[
\left\{ \frac{1}{2} - \frac{4s^2 + \sqrt{r^4 + 4r^2s^2}}{s^2t^2}, \frac{1}{2} - \frac{-4s^2 - \sqrt{r^4 + 4r^2s^2}}{s^2t^2}, \frac{-1}{2} \frac{4s^2 + r^2}{s^2t^2} \right\}
\]

Eigenvectors to Ric:
\[
\left\{ \left( -\frac{2sr}{\sqrt{r^4 + 4r^2s^2} - r^2}, 1, 0 \right), \left( -\frac{2sr}{-\sqrt{r^4 + 4r^2s^2} - r^2}, 1, 0 \right), (0, 0, 1) \right\}
\]

**Example 2.4** ($A_{3,3} = \mathbb{H}^3$). The Lie algebra of $\mathbb{H}^3$ is defined by the Lie brackets
\[
[E_1, E_2] = E_2, \ [E_1, E_3] = E_3.
\]

For this Lie group we have
\[
K = \begin{bmatrix}
0 & -r^{-2} & -r^{-2} \\
-r^{-2} & 0 & -r^{-2} \\
-r^{-2} & -r^{-2} & 0 \\
\end{bmatrix}
\]
and
\[
Ric = \begin{bmatrix}
-2r^{-2} & 0 & 0 \\
0 & -2r^{-2} & 0 \\
0 & 0 & -2r^{-2} \\
\end{bmatrix}.
\]

Eigenvalues to Ric:
\[
\{-2r^{-2}, -2r^{-2}, -2r^{-2}\}
\]

Eigenvectors to Ric:
\[
\{(0, 0, 1), (0, 1, 0), (1, 0, 0)\}
\]

**Example 2.5** ($A_{3,4}$). Structure equations:
\[
[E_1, E_3] = E_1, \ [E_2, E_3] = -E_2
\]
\[
K = \begin{bmatrix}
0 & t^{-2} & -t^{-2} \\
t^{-2} & 0 & -t^{-2} \\
-t^{-2} & -t^{-2} & 0 \\
\end{bmatrix}
\]
\[
Ric = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2t^{-2} \\
\end{bmatrix}.
\]

Eigenvalues to Ric:
\[
\{0, 0, -2t^{-2}\}
\]

Eigenvectors to Ric:
\[
\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}
\]
Example 2.6 \((A_{3,5}^3 = S_{\alpha})\). This Lie group consists of matrices of the form
\[
\begin{bmatrix}
  e^{\alpha x} & 0 & y \\
  0 & e^{-x} & z \\
  0 & 0 & 1 
\end{bmatrix},
\]
so the Lie algebra of this group is defined by
\[
E_1 = \begin{bmatrix}
  \alpha & 0 & 0 \\
  0 & -1 & 0 \\
  0 & 0 & 0 
\end{bmatrix},
E_2 = \begin{bmatrix}
  0 & 0 & 1 \\
  0 & 0 & 0 \\
  0 & 0 & 0 
\end{bmatrix},
E_3 = \begin{bmatrix}
  0 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 0 
\end{bmatrix}.
\]
Here the non-zero Lie brackets are
\[
[E_1, E_2] = \alpha E_2, \quad [E_1, E_3] = -E_3,
\]
and we have
\[
K = \begin{bmatrix}
  0 & -\frac{\alpha^2}{r^2} & -r^{-2} \\
  -\frac{\alpha^2}{r^2} & 0 & \frac{\alpha}{r^2} \\
  -r^{-2} & \frac{\alpha}{r^2} & 0 
\end{bmatrix}
\]
and
\[
Ric = \begin{bmatrix}
  -\frac{1+\alpha^2}{r^2} & 0 & 0 \\
  0 & \frac{-\alpha(-1+\alpha)}{r^2} & 0 \\
  0 & 0 & \frac{-1+\alpha}{r^2} 
\end{bmatrix}.
\]
Eigenvalues to Ric:
\[
\left\{-\frac{\alpha}{r^2} \left(-1+\alpha \right), \frac{-1+\alpha}{r^2}, \frac{-\alpha^2+1}{r^2}\right\}
\]
Eigenvectors to Ric:
\[
\{(0,1,0), (0,0,1), (1,0,0)\}
\]
Example 2.7 \((A_{3,6}^3)\). Structure equations:
\[
[E_1, E_3] = -E_2, \quad [E_2, E_3] = E_1
\]
\[
K = \begin{bmatrix}
  0 & 1/4 \left(-\frac{s^2+r^2}{s^2t^2r^2}\right)^2 & 1/4 \left(-\frac{3s^4+2r^2s^2r^4-r^4}{s^2t^2r^2}\right) \\
  1/4 \left(-\frac{s^2+r^2}{s^2t^2r^2}\right)^2 & 0 & -1/4 \left(-\frac{2r^2s^2+3r^4-s^4}{s^2t^2r^2}\right) \\
  1/4 \left(-\frac{s^2+r^2}{s^2t^2r^2}\right)^2 & -1/4 \left(-\frac{2r^2s^2+3r^4-s^4}{s^2t^2r^2}\right) & 0 
\end{bmatrix}
\]
\[
Ric = \begin{bmatrix}
  1/2 \left(-\frac{s^4+r^4}{s^2t^2r^4}\right) & 0 & 0 \\
  0 & -1/2 \left(-\frac{s^4+r^4}{s^2t^2r^4}\right) & 0 \\
  0 & 0 & -1/2 \left(-\frac{2r^2s^2+r^4+s^4}{s^2t^2r^4}\right) 
\end{bmatrix}.
\]
Eigenvalues to Ric:
\[ \left\{ \frac{1}{2} - \frac{s^4 + r^4}{s^2 t^2 r^2}, -1/2 - \frac{2 r^2 s^2 + r^4 + s^4}{s^2 t^2 r^2}, -1/2 - \frac{s^4 + r^4}{s^2 t^2 r^2} \right\} \]

Eigenvectors to Ric:
\[ \{ (1, 0, 0), (0, 0, 1), (0, 1, 0) \} \]

**Example 2.8** \((A_{3,7})\). Structure equations:
\[ [E_1, E_3] = \alpha E_1 - E_2, \ [E_2, E_3] = E_1 + \alpha E_2, \ (\alpha > 0) \]

\[
K = \begin{bmatrix}
0 & 1/4 -2 r^2 s^2 + r^4 + s^4 - 4 \alpha^2 s^2 r^2 & 1/4 2 r^2 s^2 - 4 \alpha^2 s^2 r^2 - 3 s^4 + r^4 \\
1/4 -2 r^2 s^2 + r^4 + s^4 - 4 \alpha^2 s^2 r^2 & 0 & -1/4 4 \alpha^2 s^2 r^2 - 2 r^2 s^2 + 3 r^4 - s^4 \\
1/4 2 r^2 s^2 - 4 \alpha^2 s^2 r^2 - 3 s^4 + r^4 & -1/4 4 \alpha^2 s^2 r^2 - 2 r^2 s^2 + 3 r^4 - s^4 & 0
\end{bmatrix}
\]

\[
Ric = \begin{bmatrix}
1/2 -4 \alpha^2 s^2 r^2 - s^4 + r^4 & \frac{(-s^2 + r^2) \alpha}{r t^2 s} & 0 \\
\frac{(-s^2 + r^2) \alpha}{r t^2 s} & -1/2 4 \alpha^2 s^2 r^2 + r^4 - s^4 & 0 \\
0 & 0 & -1/2 4 \alpha^2 s^2 r^2 - 2 r^2 s^2 + r^4 + s^4
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\left\{ \frac{1}{2} - 4 \alpha^2 s^2 r^2 + \sqrt{r^8 + 4 r^6 s^2 \alpha^2 + 4 s^6 \alpha^2 r^2 - 8 \alpha^2 s^4 r^4 - 2 r^4 s^4 + s^8}, \right.
\left. \frac{1}{2} - 4 \alpha^2 s^2 r^2 - \sqrt{r^8 + 4 r^6 s^2 \alpha^2 + 4 s^6 \alpha^2 r^2 - 8 \alpha^2 s^4 r^4 - 2 r^4 s^4 + s^8}, \right.
\left. \frac{1}{2} 4 \alpha^2 s^2 r^2 - 2 r^2 s^2 + r^4 + s^4 \right\}
\]

Eigenvectors to Ric:
\[
\left\{ \left( \frac{-2 \sqrt{r^8 + 4 r^6 s^2 \alpha^2 + 4 s^6 \alpha^2 r^2 - 8 \alpha^2 s^4 r^4 - 2 r^4 s^4 + s^8 + s^4 - r^4}}{r s \alpha}, 1, 0 \right), \left( \frac{-2 \sqrt{r^8 + 4 r^6 s^2 \alpha^2 + 4 s^6 \alpha^2 r^2 - 8 \alpha^2 s^4 r^4 - 2 r^4 s^4 + s^8 + s^4 - r^4}}{r s \alpha}, 0, 1 \right) \right\}
\]

**Example 2.9** \((A_{3,8})\). Structure equations:
\[ [E_1, E_3] = -2 E_2, \ [E_1, E_2] = E_1, \ [E_2, E_3] = E_3 \]
Here we have
\[
K = \begin{bmatrix}
0 & -\frac{r^2 t^2 - s^4}{s^2 t^2 r^2} & -\frac{3 s^4 + r^2 t^2}{s^2 t^2 r^2} \\
-\frac{r^2 t^2 - s^4}{s^2 t^2 r^2} & 0 & -\frac{r^2 t^2 - s^4}{s^2 t^2 r^2} \\
-\frac{3 s^4 + r^2 t^2}{s^2 t^2 r^2} & -\frac{r^2 t^2 - s^4}{s^2 t^2 r^2} & 0
\end{bmatrix}
\]
\[
Ric = \begin{bmatrix}
-2 \frac{s^2}{r^2 t^2} & 0 & -2 \frac{1}{r^2 t^2} \\
0 & -2 \frac{r^2 t^2 - s^4}{r^2 t^2} & 0 \\
-2 \frac{1}{r^2 t^2} & 0 & -2 \frac{s^2}{r^2 t^2}
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\begin{cases}
-2 \frac{r^2 t^2 - s^4}{s^2 t^2 r^2} , \\
2 \frac{-s^2 + rt}{r^2 t^2} , \\
-2 \frac{s^2 + rt}{r^2 t^2}
\end{cases}
\]

Eigenvectors to Ric:
\[
\{(0,1,0), (-1,0,1), (1,0,1)\}
\]

**Example 2.10** \((A_{3,9} = SU(2))\). The Lie algebra of \(SU(2)\) we define by the Lie brackets
\[
[E_1, E_2] = 2E_3, \ [E_3, E_1] = 2E_2, \ [E_2, E_3] = 2E_1.
\]

Here we have
\[
K =
\begin{bmatrix}
0 & \frac{2 r^2 t^2 + 2 s^2 t^2 - 4 t^4 + 2 r^2 s^2 + s^4 + r^4}{s^2 t^2 r^2} & \frac{2 r^2 s^2 + 2 s^2 t^2 - 3 s^4 - 2 r^2 t^2 + t^4 + r^4}{s^2 t^2 r^2} \\
\frac{2 r^2 t^2 + 2 s^2 t^2 - 4 t^4 - 2 r^2 s^2 + s^4 + r^4}{s^2 t^2 r^2} & 0 & \frac{-2 r^2 s^2 - 2 r^2 t^2 + 3 r^4 + 2 s^2 t^2 - t^4 - s^4}{s^2 t^2 r^2} \\
\frac{2 r^2 t^2 + 2 s^2 t^2 - 3 s^4 - 2 r^2 t^2 + t^4 + r^4}{s^2 t^2 r^2} & \frac{-2 r^2 s^2 - 2 r^2 t^2 + 3 r^4 + 2 s^2 t^2 - t^4 - s^4}{s^2 t^2 r^2} & 0
\end{bmatrix}
\]

and
\[
Ric = \begin{bmatrix}
\frac{2 s^2 r^2 - s^4 - t^4 + r^4}{r^2 s^2 t^2} & 0 & 0 \\
0 & -2 \frac{2 r^2 t^2 + r^4 + t^4 - s^4}{r^2 s^2 t^2} & 0 \\
0 & 0 & -2 \frac{2 r^2 s^2 + r^4 - t^4 + s^4}{r^2 s^2 t^2}
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\begin{cases}
-2 \frac{2 r^2 t^2 + r^4 + t^4 - s^4}{s^2 t^2 r^2} , \\
2 \frac{2 s^2 t^2 - s^4 - t^4 + r^4}{s^2 t^2 r^2} , \\
-2 \frac{2 r^2 s^2 + r^4 - t^4 + s^4}{s^2 t^2 r^2}
\end{cases}
\]

Eigenvectors to Ric:
\[
\{(0,1,0), (1,0,0), (0,0,1)\}
\]

If one takes example 1.8 and set \(\frac{2a}{s^2} = a, \ \frac{2b}{t^2} = b\) and \(\frac{2c}{s^2} = c\) we will get these matrices. So we know that at least in this case the program calculates correctly.
Chapter 3

4-dimensional Lie Groups

Here we will look at the curvatures of the four dimensional Lie groups with the metric defined by

\[
g = \begin{bmatrix}
q^2 & 0 & 0 & 0 \\
0 & r^2 & 0 & 0 \\
0 & 0 & s^2 & 0 \\
0 & 0 & 0 & t^2
\end{bmatrix}.
\]

As in previous chapter, the list is taken from [9].

<table>
<thead>
<tr>
<th>Name</th>
<th>Structure equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A_{4.1})</td>
<td>(E_2, E_4 = E_1, [E_3, E_4] = E_2)</td>
</tr>
<tr>
<td>(A^4_{4.2})</td>
<td>(E_1, E_4 = aE_1, [E_2, E_4] = E_2, [E_3, E_4] = E_2 + E_3, (a \neq 0))</td>
</tr>
<tr>
<td>(A_{4.3})</td>
<td>(E_1, E_4 = E_1, [E_3, E_4] = E_2)</td>
</tr>
<tr>
<td>(A_{4.4})</td>
<td>(E_1, E_4 = E_1, [E_2, E_4] = E_1 + E_2, [E_3, E_4] = E_2 + E_3)</td>
</tr>
<tr>
<td>(A_{4.5})</td>
<td>(E_1, E_4 = E_1, [E_2, E_4] = aE_2, [E_3, E_4] = bE_3, (ab \neq 0, -1 \leq a \leq b \leq 1))</td>
</tr>
<tr>
<td>(A_{4.6})</td>
<td>(E_1, E_4 = aE_1, [E_2, E_4] = bE_2 - E_3, [E_3, E_4] = E_2 + bE_3, (a \neq 0, b \geq 0))</td>
</tr>
<tr>
<td>(A_{4.7})</td>
<td>(E_2, E_3 = E_1, [E_1, E_4] = 2E_1, [E_2, E_4] = E_2, [E_3, E_4] = E_2 + E_3)</td>
</tr>
<tr>
<td>(A_{4.8})</td>
<td>(E_2, E_3 = E_1, [E_2, E_4] = E_2, [E_3, E_4] = -E_3)</td>
</tr>
<tr>
<td>(A_{4.9})</td>
<td>(E_2, E_3 = E_1, [E_1, E_4] = (1 + b)E_1, [E_2, E_4] = E_2, [E_3, E_4] = bE_3, (-1 &lt; b \leq 1))</td>
</tr>
<tr>
<td>(A_{4.10})</td>
<td>(E_2, E_3 = E_1, [E_2, E_4] = -E_3, [E_3, E_4] = E_2)</td>
</tr>
<tr>
<td>(A_{4.11})</td>
<td>(E_2, E_3 = E_1, [E_1, E_4] = -E_3, [E_2, E_4] = aE_2 - E_3, [E_3, E_4] = E_2 + aE_3, (a &gt; 0))</td>
</tr>
<tr>
<td>(A_{4.12})</td>
<td>(E_1, E_3 = E_1, [E_2, E_3] = E_2, [E_1, E_4] = -E_2, [E_2, E_4] = E_1)</td>
</tr>
</tbody>
</table>

Example 3.1 (\(A_{4.1}\)). Structure equations:

\([E_2, E_4] = E_1, [E_3, E_4] = E_2\)
Eigenvectors to Ric:
\[
K = \begin{bmatrix}
0 & \frac{1}{4} \frac{q^2}{r^4 t^2} & 0 & \frac{1}{4} \frac{q^2}{r^4 t^2} \\
\frac{1}{4} \frac{q^2}{r^4 t^2} & 0 & \frac{1}{4} \frac{r^2}{s^4 t^2} & -\frac{1}{4} \frac{3q^2 s^2 - r^4}{t r^4 s^4} \\
0 & \frac{1}{4} \frac{r^2}{s^4 t^2} & 0 & -\frac{3}{4} \frac{r^2}{s^4 t^2} \\
\frac{1}{4} \frac{q^2}{r^4 t^2} & -\frac{1}{4} \frac{3q^2 s^2 - r^4}{t r^4 s^4} & 0 & \frac{1}{4} \frac{r^2}{s^4 t^2}
\end{bmatrix}
\]

\[
Ric = \begin{bmatrix}
\frac{1}{2} \frac{q^2}{r^4 t^2} & 0 & 0 & 0 \\
0 & -\frac{1}{2} \frac{r^4 + q^2 s^2}{t r^2 s^2} & 0 & 0 \\
0 & 0 & -\frac{1}{2} \frac{r^2}{s^4 t^2} & 0 \\
0 & 0 & 0 & -\frac{1}{2} \frac{r^4 + q^2 s^2}{t r^2 s^2}
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\left\{-\frac{1}{2} \frac{r^2}{s^2 t^2}, -\frac{1}{2} \frac{r^4 + q^2 s^2}{t r^2 s^2}, -\frac{1}{2} \frac{r^2}{t^2 s^2}, 1/2 \frac{q^2}{r t^2} \right\}
\]

Eigenvectors to Ric:
\[
\{(0, 0, 1, 0), (0, 0, 0, 1), (0, 1, 0, 0), (1, 0, 0, 0)\}
\]

**Example 3.2** ($A_{4,2}^a$). Structure equations:
\[
\]

\[
K = \begin{bmatrix}
0 & -\frac{a}{t^2} & -\frac{a}{t^2} & -\frac{q^2}{t^2} \\
-\frac{a}{t^2} & 0 & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} \\
-\frac{a}{t^2} & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} & 0 & -1/4 \frac{4s^2 + 3r^2}{s^4 t^2} \\
-\frac{a}{t^2} & 1/4 \frac{r^2 - 4s^2}{s^4 t^2} & -1/4 \frac{4s^2 + 3r^2}{s^4 t^2} & 0
\end{bmatrix}
\]

\[
Ric = \begin{bmatrix}
-\frac{a (a+2)}{t^2} & 0 & 0 & 0 \\
0 & -1/2 \frac{r^2 + 4s^2 + 2as^2}{s^4 t^2} & -1/2 \frac{r(a+2)}{st^2} & 0 \\
0 & -1/2 \frac{r(a+2)}{st^2} & -1/2 \frac{4s^2 + r^2 + 2as^2}{s^4 t^2} & 0 \\
0 & 0 & 0 & -1/2 \frac{4s^2 + r^2 + 2a^2 s^2}{s^4 t^2}
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\left\{-\frac{a(a+2)}{t^2}, -2 s^2 - as^2 + 1/2 \frac{r^2 a^2 s^2 + 4 r^2 s^2 a + 4 r^2 s^2 + r^4}{s^2 t^2}, -2 s^2 - 1/2 \frac{r^2 a^2 s^2 + 4 r^2 s^2 a + 4 r^2 s^2 + r^4}{s^2 t^2}, -1/2 \frac{4 s^2 + r^2 + 2 a^2 s^2}{s^2 t^2} \right\}
\]

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Eigenvectors to Ric:
\[
\begin{align*}
(1, 0, 0, 0), & \quad \left(0, -\frac{r (a + 2) s}{\sqrt{r^2 a^2 s^2 + 4 r^2 s^2 a + 4 r^2 s^2 + r^4 - r^2}}, 1, 0\right), \\
(0, -\frac{r (a + 2) s}{-\sqrt{r^2 a^2 s^2 + 4 r^2 s^2 a + 4 r^2 s^2 + r^4 - r^2}}, 1, 0), & \quad (0, 0, 0, 1)
\end{align*}
\]

Example 3.3 \((A_{4,3})\). Structure equations:
\[
[E_1, E_4] = E_1, \ [E_3, E_4] = E_2
\]
\[
K = \begin{bmatrix}
0 & 0 & 0 & -t^{-2} \\
0 & 0 & 1/4 \frac{r^2}{s^2 t^2} & 1/4 \frac{r^2}{s^2 t^2} \\
0 & 1/4 \frac{r^2}{s^2 t^2} & 0 & -3/4 \frac{r^2}{s^2 t^2} \\
-t^{-2} & 1/4 \frac{r^2}{s^2 t^2} & -3/4 \frac{r^2}{s^2 t^2} & 0
\end{bmatrix}
\]
\[
Ric = \begin{bmatrix}
-t^{-2} & 0 & 0 & 0 \\
0 & 1/2 \frac{r^2}{s^2 t^2} & -1/2 \frac{r^2}{s^2 t^2} & 0 \\
0 & -1/2 \frac{r^2}{s^2 t^2} & -1/2 \frac{r^2}{s^2 t^2} & 0 \\
0 & 0 & 0 & -1/2 \frac{r^2 + 2 s^2}{s^2 t^2}
\end{bmatrix}
\]

Eigenvalues to Ric:
\[
\left\{-t^{-2}, 1/2 \frac{\sqrt{s^2 + r^2} r}{s^2 t^2}, -1/2 \frac{\sqrt{s^2 + r^2} r}{s^2 t^2}, -1/2 \frac{r^2 + 2 s^2}{s^2 t^2}\right\}
\]

Eigenvectors to Ric:
\[
\begin{align*}
(1, 0, 0, 0), & \quad \left(0, -\frac{sr}{\sqrt{s^2 + r^2} - r^2}, 1, 0\right), \\
(0, -\frac{sr}{-\sqrt{s^2 + r^2} - r^2}, 1, 0), & \quad (0, 0, 0, 1)
\end{align*}
\]

Example 3.4 \((A_{4,4})\). Structure equations:
\[
[E_1, E_4] = E_1, \ [E_2, E_4] = E_1 + E_2, \ [E_3, E_4] = E_2 + E_3
\]
\[
K = \begin{bmatrix}
0 & 1/4 \frac{a^2 - 4 r^2}{t^2} & -t^{-2} & 1/4 \frac{a^2 - 4 r^2}{t^2} \\
1/4 \frac{a^2 - 4 r^2}{t^2} & 0 & 1/4 \frac{a^2 - 4 r^2}{s^2 t^2} & -1/4 \frac{4 a^2 + 3 s^2 - r^4}{t^2 s^2 t^2} \\
-t^{-2} & 1/4 \frac{a^2 - 4 r^2}{s^2 t^2} & 0 & -1/4 \frac{4 a^2 + 3 s^2 - r^4}{s^2 t^2} \\
1/4 \frac{a^2 - 4 r^2}{t^2} & -1/4 \frac{4 a^2 + 3 s^2 - r^4}{t^2 s^2 t^2} & -1/4 \frac{4 a^2 + 3 s^2 - r^4}{s^2 t^2} & 0
\end{bmatrix}
\]

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Here the eigenvalues and eigenvectors will not fit on the page, and does not give us any useful information. So we only look at the case when \( r = s = t = q = 1 \).

Eigenvalues to Ric:
\[
\{-4, -3, -3 + 1/2 \sqrt{19}, -3 - 1/2 \sqrt{19}\}
\]

Eigenvectors to Ric:
\[
\{(0, 0, 0, 1), (-1, -1/3, 1, 0), (18(-1 + \sqrt{19})^{-2}, -6(-1 + \sqrt{19})^{-1}, 1, 0), (18(-1 - \sqrt{19})^{-2}, -6(-1 - \sqrt{19})^{-1}, 1, 0)\}
\]

Example 3.5 (\(A_{13}^{ab}\)). Structure equations:
\[
\]

\[
K = \begin{bmatrix} 0 & -a & -b & -t^{-2} \\ -a & 0 & -ab & -a^2 \\ -b & -ab & 0 & -b^2 \\ -t^{-2} & -a^2 & -b^2 & 0 \end{bmatrix}
\]

\[
Ric = \begin{bmatrix} -\frac{1+b+a}{t^2} & 0 & 0 & 0 \\ 0 & -\frac{a(1+b+a)}{t^2} & 0 & 0 \\ 0 & 0 & -\frac{b(1+b+a)}{t^2} & 0 \\ 0 & 0 & 0 & -\frac{b^2+a^2+1}{t^2} \end{bmatrix}
\]

Eigenvalues to Ric:
\[
\{-\frac{b(1+b+a)}{t^2}, -\frac{1+b+a}{t^2}, -\frac{b^2+a^2+1}{t^2}, -\frac{a(1+b+a)}{t^2}\}
\]

Eigenvectors to Ric:
\[
\{(0, 0, 1, 0), (1, 0, 0, 0), (0, 0, 0, 1), (0, 1, 0, 0)\}
\]

When \(a = b = 1\) this equals the Lie algebra of \(\mathbb{H}^4\).
Example 3.6 ($A_{ab}^{4,0}$). Structure equations:

\[ [E_1, E_4] = aE_1, \ [E_2, E_4] = bE_2 - E_3, \ [E_3, E_4] = E_2 + bE_3 \]

\[ K = \begin{bmatrix}
0 & -\frac{ab}{t^2} & -\frac{ab}{t^2} & -\frac{a^2}{t^2} \\
-\frac{ab}{t^2} & 0 & -1/4 \frac{2r^2s^2-r^4-s^4+4b^2s^2r^2}{s^2t^2r^2} & -1/4 \frac{2r^2s^2+4b^2s^2r^2+3s^4-r^4}{s^2t^2r^2} \\
-\frac{ab}{t^2} & -1/4 \frac{2r^2s^2-r^4-s^4+4b^2s^2r^2}{s^2t^2r^2} & 0 & -1/4 \frac{4b^2s^2r^2-2r^2s^2+3r^4-s^4}{s^2t^2r^2} \\
-\frac{a^2}{t^2} & -1/4 \frac{-2r^2s^2+4b^2s^2r^2+3s^4-r^4}{s^2t^2r^2} & -1/4 \frac{4b^2s^2r^2-2r^2s^2+3r^4-s^4}{s^2t^2r^2} & 0
\end{bmatrix} \]

\[ Ric = \begin{bmatrix}
-\frac{a(a+2b)}{t^2} & 0 & 0 & 0 \\
0 & -\frac{4b^2s^2r^2+4s^4-r^4+2abs^2}{2s^4t^2r^2} & -\frac{(-s^2+r^2)(a+2b)}{2s^4} & 0 \\
0 & 0 & -\frac{4b^2s^2r^2+4s^4-r^4+2abs^2}{2s^4t^2r^2} & 0 \\
0 & 0 & 0 & -\frac{4b^2s^2r^2-2r^2s^2+4s^4+2r^2s^2}{2s^4t^2r^2}
\end{bmatrix}. \]

Again we get eigenvalues too complicated to print out. But we will at lest look at the case where $r = s = t = q = 1$.

Eigenvalues to Ric:

\[ \{-2b^2 - a^2, -a^2 - 2ab, -2b^2 - ab, -2b^2 - ab\} \]

Eigenvectors to Ric:

\[ \{(0, 0, 0, 1), (1, 0, 0, 0), (0, 0, 1, 0), (0, 1, 0, 0)\} \]

The two last eigenvalues are not always equal. They can be written as $\lambda_3 = f(q, r, s, t) + g(q, r, s, t)$ and $\lambda_3 = f(q, r, s, t) - g(q, r, s, t)$, where $g(1, 1, 1, 1) = 0$.

Example 3.7 ($A_{4,7}$). Structure equations:

\[ [E_2, E_3] = E_1, \ [E_1, E_4] = 2E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = E_2 + E_3 \]

\[ K = \begin{bmatrix}
0 & 1/4 \frac{2q^2t^2-8s^2r^2}{s^2t^2r^2} & 1/4 \frac{q^2t^2-4s^2r^2}{s^2t^2r^2} & -4t^{-2} \\
1/4 \frac{2q^2t^2-8s^2r^2}{s^2t^2r^2} & 0 & -1/4 \frac{3q^2t^2-r^4+4r^2s^2}{s^2t^2r^2} & 1/4 \frac{r^2-4s^2}{s^2t^2} \\
1/4 \frac{2q^2t^2-8s^2r^2}{s^2t^2r^2} & -1/4 \frac{3q^2t^2-r^4+4r^2s^2}{s^2t^2r^2} & 0 & -1/4 \frac{4s^2+3r^2}{s^2t^2} \\
-4t^{-2} & 1/4 \frac{r^2-4s^2}{s^2t^2} & -1/4 \frac{4s^2+3r^2}{s^2t^2} & 0
\end{bmatrix} \]
\[
\begin{bmatrix}
1/2 -16r^2 s^2 + q^2 t^2 \\
0 \quad -1/2 -r^4 + 8r^2 s^2 + q^2 t^2 \\
0 \quad -2 \frac{r}{s t^2} \quad -1/2 \frac{8r^2 s^2 + r^4 + q^2 t^2}{t^2 s^2} \\
0 \quad 0 \quad 0 \quad -1/2 \frac{12r^2 s^2 + t^2}{s t^2}
\end{bmatrix}
\]

Eigenvectors to Ric:
\[
\left\{-4r^2 s^2 - 1/2 q^2 t^2 + 1/2 \sqrt{r^8 + 16r^6 s^2 - 4r^3 s^2 - 1/2 q^2 t^2 - 1/2 \sqrt{r^8 + 16r^6 s^2}}, \right. \\
\left. -1/2 \frac{12s^2 + r^2}{s^2 t^2}, 1/2 \frac{-16r^2 s^2 + q^2 t^2}{r^2 s^2 t^2} \right\}
\]

Eigenvectors to Ric:
\[
\left\{(0, -4 \frac{s\sqrt{s^2 + 16r^6 s^2 - r^4}}{s t^2}, 1, 0), (0, -4 \frac{s\sqrt{s^2 + 16r^6 s^2 - r^4}}{s t^2}, 1, 0), (0, 0, 0, 1), (1, 0, 0, 0)\right\}
\]

**Example 3.8** \((A_4, s)\). Structure equations:
\[
[E_2, E_3] = E_1, \quad [E_2, E_4] = E_2, \quad [E_3, E_4] = -E_3
\]

\[
K = \begin{bmatrix}
0 & \frac{q^2}{r s^2} & \frac{q^2}{r s^2} & 0 \\
\frac{q^2}{r^2 s^2} & 0 & -\frac{3q^2 t^2 - 4r^2 s^2}{r^2 s^2} & -t^{-2} \\
\frac{q^2}{r^2 s^2} & -1/4 \frac{3q^2 t^2 - 4r^2 s^2}{r^2 s^2} & 0 & -t^{-2} \\
0 & -t^{-2} & -t^{-2} & 0
\end{bmatrix}
\]

\[
Ric = \begin{bmatrix}
1/2 \frac{q^2}{r^2 s^2} & 0 & 0 & 0 \\
0 & -1/2 \frac{q^2}{r^2 s^2} & 0 & 0 \\
0 & 0 & -1/2 \frac{q^2}{r^2 s^2} & 0 \\
0 & 0 & 0 & -2 t^{-2}
\end{bmatrix}
\]

Eigenvales to Ric:
\[
\left\{-2t^{-2}, 1/2 \frac{q^2}{r^2 s^2}, -1/2 \frac{q^2}{r^2 s^2}, -1/2 \frac{q^2}{r^2 s^2} \right\}
\]

Eigenvectors to Ric:
\[
\left\{(0, 0, 0, 1), (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\right\}
\]

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Example 3.9 ($A_{4,9}^b$). Structure equations:

$$[E_2, E_3] = E_1, \ [E_1, E_4] = (1+b) E_1, \ [E_2, E_4] = E_2, \ [E_3, E_4] = bE_3.$$

$$K = \begin{bmatrix}
0 & -1/4 \frac{q^2 t^2 + 4 r^2 s^2 + 4 r^2 s^2 b}{r^2 s t^2} & -1/4 \frac{q^2 t^2 + 4 r^2 s^2 b + 4 b^2 s^2 r^2}{r^2 s t^2} & -(1+b)^2 \\
-1/4 \frac{q^2 t^2 + 4 r^2 s^2 + 4 r^2 s^2 b}{r^2 s t^2} & 0 & -1/4 \frac{q^2 t^2 + 4 r^2 s^2 b}{r^2 s t^2} & -t^2 \\
-1/4 \frac{q^2 t^2 + 4 r^2 s^2 + 4 r^2 s^2 b}{r^2 s t^2} & -1/4 \frac{3 q^2 t^2 + 4 r^2 s^2 b}{r^2 s t^2} & 0 & -\frac{b^2}{t^2} \\
-(1+b)^2 & -t^2 & -\frac{b^2}{t^2} & 0
\end{bmatrix}$$

$$Ric = \begin{bmatrix}
-1/2 \frac{4 r^2 s^2 + 8 r^2 s^2 b + 4 b^2 s^2 r^2 - q^2 t^2}{r^2 s t^2} & 0 & 0 & 0 \\
0 & -1/2 \frac{4 r^2 s^2 + 8 r^2 s^2 b + 4 b^2 s^2 r^2}{r^2 s t^2} & 0 & 0 \\
0 & 0 & -1/2 \frac{4 b^2 s^2 r^2 + q^2 t^2 + 4 r^2 s^2 b}{r^2 s t^2} & 0 \\
0 & 0 & 0 & -2 \frac{b^2 + 1+b}{t^2}
\end{bmatrix}.$$

Eigenvalues to Ric:

$$\left\{-2 \frac{b^2 + 1+b}{t^2}, -1/2 \frac{4 r^2 s^2 + q^2 t^2 + 4 r^2 s^2 b}{r^2 s t^2}, -1/2 \frac{4 b^2 s^2 r^2 + q^2 t^2 + 4 r^2 s^2 b}{r^2 s t^2}, -1/2 \frac{4 r^2 s^2 + 8 r^2 s^2 b + 4 b^2 s^2 r^2 - q^2 t^2}{r^2 s t^2}\right\}$$

Eigenvectors to Ric:

$$\{(0,0,0,1), (0,1,0,0), (0,0,1,0), (1,0,0,0)\}$$

Example 3.10 ($A_{4,10}$). Structure equations:


$$K = \begin{bmatrix}
0 & 1/4 \frac{q^2}{r^2 s^2} & 1/4 \frac{q^2}{r^2 s^2} & 0 \\
1/4 \frac{q^2}{r^2 s^2} & 0 & -1/4 \frac{3 q^2 t^2 + 2 r^2 s^2 - r^4 - s^4}{t^2 r^2 s^2} & 1/4 \frac{-3 s^4 + 2 r^2 s^2 + r^4}{t^2 r^2 s^2} \\
1/4 \frac{q^2}{r^2 s^2} & -1/4 \frac{3 q^2 t^2 + 2 r^2 s^2 - r^4 - s^4}{t^2 r^2 s^2} & 0 & -1/4 \frac{3 r^4 - 2 r^2 s^2 - s^4}{t^2 r^2 s^2} \\
0 & 1/4 \frac{-3 s^4 + 2 r^2 s^2 + r^4}{t^2 r^2 s^2} & -1/4 \frac{3 r^4 - 2 r^2 s^2 - s^4}{t^2 r^2 s^2} & 0
\end{bmatrix}$$

$$Ric = \begin{bmatrix}
1/2 \frac{q^2}{r^2 s^2} & 0 & 0 & 0 \\
0 & -1/2 \frac{s^4 - r^4 + q^2 t^2}{t^2 r^2 s^2} & 0 & 0 \\
0 & 0 & -1/2 \frac{r^4 - s^4 + q^2 t^2}{t^2 r^2 s^2} & 0 \\
0 & 0 & 0 & -1/2 \frac{-2 r^2 s^2 + r^4 + s^4}{t^2 r^2 s^2}
\end{bmatrix}.$$
Eigenvalues to Ric:

$$\left\{ \frac{1}{2} q^2 r^2 s^2 - 1/2 - 2 r^2 s^2 + r^4 + s^4 \frac{r^2 s^2 t^2}{r^2 s^2 t^2}, \frac{1}{2} r^4 - s^4 + q^2 t^2 \frac{r^2 s^2 t^2}{r^2 s^2 t^2}, -1/2 \frac{s^4 - r^4 + q^2 t^2}{r^2 s^2 t^2} \right\}$$

Eigenvalues to Ric:

$$\{(1, 0, 0, 0), (0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)\}$$

Example 3.11 ($A_{4,11}$). Structure equations:

$$[E_2, E_3] = E_1, \ [E_1, E_4] = 2aE_1, \ [E_2, E_4] = aE_2 - E_3, \ [E_3, E_4] = E_2 + aE_3$$

$$-K = \begin{bmatrix}
0 & \frac{-q^2 t^2 + 8 r^2 a^2 s^2}{4 r^2 s^2 t^2} & \frac{-q^2 t^2 + 8 r^2 a^2 s^2}{4 r^2 s^2 t^2} & 0 \\
\frac{-q^2 t^2 + 8 r^2 a^2 s^2}{4 r^2 s^2 t^2} & 0 & \frac{3 q^2 t^2 + 2 r^2 s^2 - r^4 - s^4 + 4 r^2 a^2 s^2}{4 r^2 s^2 t^2} & -\frac{2 r^2 s^2 + 4 r^2 a^2 s^2 + 3 s^4 - r^4}{4 r^2 s^2 t^2} \\
\frac{4 r^2}{t^2} & -\frac{2 r^2 s^2 + 4 r^2 a^2 s^2 + 3 s^4 - r^4}{4 r^2 s^2 t^2} & \frac{4 r^2 a^2 s^2 - 2 r^2 s^2 + 3 r^4 - s^4}{4 r^2 s^2 t^2} & 0 \\
0 & 0 & 0 & 0 
\end{bmatrix}$$

$$Ric = \begin{bmatrix}
-\frac{16 r^2 a^2 s^2}{2s^2 t^2} & 0 & 0 & 0 \\
0 & -\frac{8 r^2 a^2 s^2 + s^4 - r^4 + q^2 t^2}{2 s^2 t^2} & -\frac{2 a(s^2 + r^2)}{(r s)} & 0 \\
0 & -\frac{2 a(s^2 + r^2)}{(r s)} & -\frac{8 r^2 a^2 s^2 + r^4 - s^4 + q^2 t^2}{2 s^2 t^2 r^2} & 0 \\
0 & 0 & 0 & -\frac{12 r^2 a^2 s^2 - 2 r^2 s^2 + 4 r^4 + s^4}{2 s^2 t^2 r^2} 
\end{bmatrix}$$

Eigenvalues to Ric:

$$\left\{ -4 r^2 a^2 s^2 - 1/2 q^2 t^2 + 1/2 \sqrt{16 r^6 a^2 s^2 - 32 r^4 s^4 a^2 + 16 r^2 s^6 a^2 + r^8 - 2 r^4 s^4 + s^8} \middle/ r^2 s^2 t^2 \right\}$$

$$\left\{ -4 r^2 a^2 s^2 - 1/2 q^2 t^2 - 1/2 \sqrt{16 r^6 a^2 s^2 - 32 r^4 s^4 a^2 + 16 r^2 s^6 a^2 + r^8 - 2 r^4 s^4 + s^8} \middle/ r^2 s^2 t^2 \right\}$$

$$\left\{ -1/2 \frac{12 r^2 a^2 s^2 - 2 r^2 s^2 + r^4 + s^4}{r^2 s^2 t^2}, -1/2 \frac{16 r^2 a^2 s^2 - q^2 t^2}{r^2 s^2 t^2} \right\}$$

Eigenvalues to Ric:

$$\left\{ \left(0, -4 \frac{a(s^2 + r^2)}{(r s)} \right), \left(0, -4 \frac{a(-s^2 + r^2)}{(r s)} \right), \left(0, 0, 0, 1\right), \left(1, 0, 0, 0\right) \right\}$$
Example 3.12 ($A_{4,12}$). Structure equations:

$$[E_1, E_3] = E_1, [E_2, E_3] = E_2, [E_1, E_4] = -E_2, [E_2, E_4] = E_1$$

$$K = \begin{pmatrix}
0 & 1/4 \frac{s^2 r^2 - 2 s^2 r^2 + s^2 q^2 - 4 r^2 t^2 q^2}{s^2 r^2 + s^2 q^2} & s^{-2} & 1/4 \frac{-3 s^2 r^2 + 2 q^2 + q^4}{s^2 r^2 + s^2 q^2} \\
1/4 \frac{s^2 r^2 - 2 s^2 r^2 + s^2 q^2 - 4 r^2 t^2 q^2}{s^2 r^2 + s^2 q^2} & 0 & -s^{-2} & 1/4 \frac{-3 s^2 r^2 + 2 q^2 + q^4}{s^2 r^2 + s^2 q^2} \\
-s^{-2} & -s^{-2} & 0 & 0 \\
1/4 \frac{-3 s^2 r^2 + 2 q^2 + q^4}{s^2 r^2 + s^2 q^2} & -1/4 \frac{3 q^4 - 2 r^2 q^2 - q^4}{s^2 r^2 + s^2 q^2} & 0 & 0
\end{pmatrix}$$

$$Ric = \begin{pmatrix}
1/2 \frac{-s^2 r^2 + s^2 q^2 - 4 r^2 t^2 q^2}{s^2 r^2 + s^2 q^2} & 0 & 0 & 0 \\
0 & -1/2 \frac{s^2 q^4 - 2 s^2 r^4 + 4 r^2 t^2 q^2}{s^2 r^2 + s^2 q^2} & 0 & 0 \\
0 & 0 & -2 s^{-2} & 0 \\
0 & 0 & 0 & -1/2 \frac{3 q^4 - 2 r^2 q^2 + q^4}{s^2 r^2 + s^2 q^2}
\end{pmatrix}.$$
Appendix A

The Maple program

The program relies on the Maple package DifferentialGeometry and the subpackage LieAlgebras. For information about setting up a Lie algebra to work with, one should look at the documentation for LieAlgebraData in the LieAlgebras package.

The metric we want to work with is represented by an \( n \times n \) matrix called \( g \), where \( n \) is the dimension of the Lie algebra. The matrix must be called \( g \) and must be within the scope of the procedures.

The functionality of the program is based on the fact that we can write the Levi Civita connection of two vector fields as a linear combination of metric values for the basis vectors. Thus we must first be able to get a list of basis vectors. Maple does not have a predefined function for appending objects to lists, so this short procedure is used.

A procedure for adding objects to lists.

\[
\text{append} := \text{proc}(L1, \text{ob}) \text{options operator, arrow;} \\
\text{[seq}(L1[i], i = 1 \ldots \text{nops}(L1)), \text{ob}] \\
\text{end proc}
\]

The simplest way of getting the basis is to pick it out from the multiplication table of the internal representation of the Lie algebra. MultiplicationTable("LieDerivative") returns a table which has the actual vectors in it.

A procedure for getting a list of basis vectors for our Lie algebra.

\[
\text{getBasis} := \text{proc ()} \\
\text{local MT, B, dim, i;} \\
\text{MT} := \text{MultiplicationTable("LieDerivative");} \\
\text{B} := []; \\
\text{dim} := \text{op(MT)}[1] - 2; \\
\text{for i to dim do} \\
\text{\quad B} := \text{append}(\text{B, MT}[i+2, 1]) \\
\text{end do;} \\
\text{B} \\
\text{end proc}
\]
To understand how to use the vectors, it is important to understand how they are represented internally. The first basis vector of a Lie algebra is called $e_1$. Though it is actually a nested list with various values. It really looks like this: "vector". $A_1$=[[1],[1]]. The first sublist, "vector". $A_1$=[[1]], tells us information about this object, that it is a vector in the algebra $A_1$. The second sublist is the vector represented as a linear combination of basis vectors. [[[1],[1]]] is a list representing a sum. For a better example, we look at $a = e_1 + 5 \cdot e_2$. (Note that vector addition is called &plus and multiplication by scalar is &mult). $a$ is internally represented as ["vector", $A_1$, [[[1], [1]], [[2], 5]]]. Here we can see that in the list [[[1], [1]], [[2], 5]] the first sublist is what basis vector we have, and the coefficient of it. And the same for the second list. So there we have the information $a = 1 \cdot e_1 + 5 \cdot e_2$. Both the Lie bracket and the metric is bilinear, and the vectors are represented as linear combinations of basis vectors, so using this we only need to loop over the basis vectors used to represent the vectors and add up the results in the end. Thas is how this procedure calculates the metric value of two vectors. It extracts the indices of the basis vectors in the linear combination of the given vector, and uses that as the indice in the matrix $g$ defining the metric values. Since the metric is bilinear it adds all these up for all the basis vectors.

A procedure for calculating the metric value for two vectors

```plaintext
metr := proc (X, Y)
local ei, ej, M;
M := 0;
for ei in op(X)[2] do
    for ej in op(Y)[2] do
        M := M + ei[2] * ej[2] * g[ei[1][1], ej[1][1]]
    end do;
end do;
M
end proc
```

The most fundamental thing we need to be able to calculate to get curvatures is the Levi-Civita connection. As we saw in equations (1.1) and (1.2) it can be expressed with only Lie brackets and metric values for basis vectors in the algebra. That is what this procedure does. We give it two vectors, and it finds the basis we are working in and the metric we are using, and essentially insert equation (1.1) into (1.2) so we can get a the connection of the two given vectors as a linear combinations of basis vectors. So here we loop first over each basis vector in the basis, and in that over linear combinations of vectors that express various Lie brackets. And finally we add it all up.

The Levi-Civita connection.

```plaintext
Con := proc (X, Y)
    local B, ek, N, LB, C, K, m, i, n, j;
    B := getBasis();
```
\[ C := \text{LieBracket}(B[1], B[1]); \]
\[ \text{for } ek \text{ in } B \text{ do} \]
\[ N := 0; \]
\[ LB := \text{LieBracket}(X, Y); \]
\[ m := \text{nops}(\text{op}(LB)[2]); \]
\[ \text{for } i \text{ to } m \text{ do} \]
\[ N := N + \text{op}(LB)[2][i][2] \cdot g[\text{op}(ek)[2][1][1], \text{op}(LB)[2][i][1][1]] \]
\[ \text{end do;} \]
\[ LB := \text{LieBracket}(ek, X); \]
\[ m := \text{nops}(\text{op}(LB)[2]); \]
\[ \text{for } i \text{ to } m \text{ do} \]
\[ n := \text{nops}(\text{op}(Y)[2]); \]
\[ \text{for } j \text{ to } n \text{ do} \]
\[ N := N + \text{op}(Y)[2][j][2] \cdot g[\text{op}(LB)[2][i][1][1], \text{op}(Y)[2][j][1][1]]; \]
\[ \text{end do;} \]
\[ \text{end do;} \]
\[ LB := \text{LieBracket}(ek, Y); \]
\[ m := \text{nops}(\text{op}(LB)[2]); \]
\[ \text{for } i \text{ to } m \text{ do} \]
\[ n := \text{nops}(\text{op}(X)[2]); \]
\[ \text{for } j \text{ to } n \text{ do} \]
\[ N := N + \text{op}(X)[2][j][2] \cdot g[\text{op}(LB)[2][i][1][1], \text{op}(X)[2][j][1][1]]; \]
\[ \text{end do;} \]
\[ N := \frac{1}{2} \cdot N; \]
\[ K := \text{\&mult\,(N, ek)}; \]
\[ C := \text{\&plus\,(C, K)}; \]
\[ \text{end do;} \]
\[ C \]
\[ \text{end proc} \]

Now we can calculate the Riemann curvature tensor, since it does nothing more than take three vectors and apply them in various ways to the Levi-Civita connection.

The Riemann curvature tensor

\begin{verbatim}
RieCu := proc (X, Y, Z)
    local C, YZ, XZ, LB;
    YZ := Con(Y, Z);
    C := Con(X, YZ);
    XZ := Con(X, Z);
    C := '\&minus\,(C, Con(Y, XZ));
    LB := LieBracket(X, Y);
end proc
\end{verbatim}
C := & minus (C, Con(LB, Z));
C
end proc

Now we have all we need to calculate curvatures. First the Ricci operator:

The Ricci operator

Ric := proc (X)
local B, ek, R;
B := getBasis();
R := LieBracket(B[1], B[1]);
for ek in B do
    R := & plus (R, RieCu(X, ek, ek))
end do;
R
end proc

Then the curvature function:

The curvature function

CurFun := proc (X, Y)
local R, K;
R := RieCu(X, Y, Y);
K := metr(R, X);
K
end proc

To be able to show a lot of information at once we will generally calculate a matrix $K_{ij} = \kappa(E_i, E_j)$ for basis vectors $E_i, E_j$. This is how we do that:

A procedure for calculating a matrix where the elements are the curvature function of basis elements

CurFunMat := proc ()
local B, M, ei, ej, K;
B := getBasis();
M := Matrix(nops(B));
for ej in B do
    for ei in B do
        K := CurFun(ej, ei);
        M[op(ej)[2][1][1][1], op(ei)[2][1][1][1]] := K
    end do;
end do;
M
end proc

And a matrix representing the Ricci quadratic form, which we call $Ric$ and where $Ric_{ij} = Ric(E_i, E_j)$. 

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A procedure that calculates a matrix representing the Ricci quadratic form

\[
RicQF := \text{proc ()} \\
\quad \text{local A, B, ei, ej, m, i, R, G;}
\quad B := \text{getBasis();}
\quad A := \text{Matrix(nops(B));}
\quad \text{for ej in B do}
\quad \quad \text{for ei in B do}
\quad \quad \quad G := 0;
\quad \quad \quad R := \text{Ric(ei)};
\quad \quad \quad m := \text{nops(op(R)[2]);}
\quad \quad \quad \text{for i to m do}
\quad \quad \quad \quad \text{G := G + op(R)[2][1][1][1][2] \ast g[op(R)[2][1][1][1][1]]}
\quad \quad \quad \text{end do;}
\quad \quad \quad \text{A[op(ei)[2][1][1][1][1], op(ej)[2][1][1][1][1]] := G}
\quad \quad \text{end do;}
\quad \text{A}
\text{end proc}
\]

These procedures for the curvatures however, assume we are working with an orthonormal basis, that is that

\[
g = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

If this was the only case we could calculate the program would not be very interesting, so we need to perform some process which makes the basis we are working with orthonormal with respect to any given metric. We firstly need this little helpful procedure:

Projects a given vector to a given basis vector

\[
\text{proj} := \text{proc (v, u)} \\
\quad \text{local M, e, f, N, P;}
\quad M := \text{metr(u, v)};
\quad \text{\&mult\{M, u\}}
\text{end proc}
\]

Then we can do Gram-Schmidt for the vectors in the algebra we are using. This procedure returns a list of orthonormalized vectors.

A Gram-Schmidt procedure for our vectors.

\[
GrSc := \text{proc ()}
\]
local U, V, B, i, n, j, e, v, N;
V := getBasis(); U := V;
n := nops(V);
for i to n do
    U[i] := V[i];
    for j to i-1 do
        U[i] := &minus;'(U[i], proj(V[i], U[j]));
    end do;
    U[i] := &mult;'(1/sqrt(metr(U[i], U[i]), symbolic), U[i]);
end do;
B := U;
end proc

These vectors we can then use as an ONB. This procedure takes that list, gets the
structure equations which define the algebra we are working in, and essentially put
in the vectors of the list instead of the corresponding present vectors. It returns a
list of Lie bracket products and a list of the new basis vectors. These will use the
symbol "E", and therefore E should always be unassigned when using this program.
To use the algebra with the ONB one only need to run the result from this procedure
in LieAlgebraData, and then set up with DGsetup for that result. One also need to
set g to the identity matrix.

Uses the result from the Gram-Schmidt and makes new structure equations for a new orthonormal
basis.

makeONB := proc (onb)
    local C, e, n, i, newBasis, tab, newSEq, eq, lb, prod, bb1, bb2, bb3, fff, j, m, P, k, l, t, prodpart;
    C := onb;
    n := nops(onb);
    for i to n do
        C[i] := op(onb[i])[2][1][2]
    end do;
    newBasis := [];
    for i to n do
        newBasis := append(newBasis, cat(E, i))
    end do;
    tab := MultiplicationTable("LieBracket");
    newSEq := [];
    n := nops(tab);
    for i to n do
        lb := [cat(E, convert(Explode(convert(op(tab[i])[1][1], string))[-1], symbol)), cat(E, convert(
Explode(convert(op(tab[i])[1][2], string))[-1], symbol));
    fff := 1;
    bb1 := parse(convert(Explode(convert(op(tab[i])[1][1], string))[-1], symbol));
    bb2 := parse(convert(Explode(convert(op(tab[i])[1][2], string))[-1], symbol));
    m := nops(op(tab[i])[2]);
    if op(0, op(tab[i])[2]) = 'symbol' then
      bb3 := parse(convert(Explode(convert(op(op(tab[i])[2]), string))[-1], symbol));
      P := 1;
      for j to m-1 do
        P := P*op(op(tab[i])[2])[j]
      end do;
      prod := `&mult`((P*fff*C[bb1]*C[bb2]/C[bb3], cat(E, convert(Explode(convert(op(op(tab[i])[2]), string))[-1], symbol))
    elif op(0, op(tab[i])[2]) = '*;' then
      bb3 := parse(convert(Explode(convert(op(op(tab[i])[2])[-1], string))[-1], symbol));
      P := 1;
      k := nops(op(tab[i])[2]);
      for j to k-1 do
        P := P*op(op(tab[i])[2])[j]
      end do;
      prod := `&mult`((P*fff*C[bb1]*C[bb2]/C[bb3], cat(E, convert(Explode(convert(op(op(tab[i])[2])[-1], string))[-1], symbol))
    elif op(0, op(tab[i])[2]) = '+;' then
      k := nops(op(tab[i])[2]);
      for j to k do
        bb3 := parse(convert(Explode(convert(op(op(tab[i])[2])[-1], string))[-1], symbol));
        if op(0, op(op(tab[i])[2])[j]) = '*;' then
          P := 1;
        t := nops(op(op(tab[i])[2])[-1], symbol));
        for l to t-1 do
          P := P*op(op(tab[i])[2])[j])
        end do;
        prodpart := `&mult`(
How to use the program

As an educational example, I will show the workflow when calculating one of the curvatures in chapter 2.

\[ Base := \{X1, X2, X3\} \]

\[ a37a := [\{X1, X3\} = aX1 - X2, [X2, X3] = X1 + aX2] \]

\[ a37a := LieAlgebraData(a37aseq, Base) \]

\[ DGsetup(a37a) \]

\[ Query("Jacobi") \]

\[ g := \begin{bmatrix} r^2 & 0 & 0 \\ 0 & s^2 & 0 \\ 0 & 0 & t^2 \end{bmatrix} \]
onba37a := GrSc()
\[
\begin{bmatrix}
r^2 & 0 & 0 \\
0 & s^2 & 0 \\
0 & 0 & t^2
\end{bmatrix}
\]

\[
a37aseqonb := \text{makeONB}(\text{onba37a})
\]

\[
[[E1, E3] = \frac{ae1}{t} - \frac{se2}{rt}, [E2, E3] = \frac{re1}{st} + \frac{ae2}{t}]
\]

\[
a37aonb := \text{LieAlgebraData}(\text{a37aseqonb})
\]

\[
[[e1, e3] = \frac{ae1}{t} - \frac{se2}{rt}, [e2, e3] = \frac{re1}{st} + \frac{ae2}{t}]
\]

DGsetup(a37aonb)

'Lie algebra: L2'

Query("Jacobi")

true

\[
g := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

CFM := CurFunMat()
\[
\begin{bmatrix}
0 & -1/4 \frac{2r^2s^2 - r^4 - s^4 + 4r^2a^2s^2}{s^2t^2} & -1/4 \frac{-2r^2s^2 + 4r^2a^2s^2 + 3s^4 - r^4}{s^2t^2} \\
-1/4 \frac{2r^2s^2 - r^4 - s^4 + 3r^2a^2s^2}{s^2t^2} & 0 & -1/4 \frac{-2r^2s^2 + 4r^2a^2s^2 + 3s^4 - r^4}{s^2t^2} \\
-1/4 \frac{-2r^2s^2 + 4r^2a^2s^2 + 3s^4 - r^4}{s^2t^2} & -1/4 \frac{-2r^2s^2 + 4r^2a^2s^2 + 3s^4 - r^4}{s^2t^2} & 0
\end{bmatrix}
\]

RQF := RicQF()
\[
\begin{bmatrix}
-1/2 \frac{4r^2a^2s^2 + s^4 - r^4}{t^2s^2} & \frac{-a(s^2 + r^2)}{t^2rs} & 0 \\
-\frac{a(s^2 + r^2)}{t^2rs} & -1/2 \frac{4r^2a^2s^2 + r^4 - s^4}{t^2s^2} & 0 \\
0 & 0 & -1/2 \frac{4r^2a^2s^2 + 2r^4 + r^4 + s^4}{t^2s^2}
\end{bmatrix}
\]

\[
V, W := \text{Eigenvectors}(\text{RQF})
\]

V
\[
W = \begin{bmatrix}
\frac{1}{2} & \frac{-4r^2a^2s^2 + \sqrt{-8r^4s^4a^4 + 4r^6s^2a^2 + 4r^4s^6a^2 - 2r^4s^4 + r^8 + s^8}}{t^2a^2} \\
\frac{1}{2} & \frac{-4r^2a^2s^2 - \sqrt{-8r^4s^4a^4 + 4r^6s^2a^2 + 4r^4s^6a^2 - 2r^4s^4 + r^8 + s^8}}{t^2a^2} \\
-\frac{1}{2} & \frac{4r^2a^2s^2 - 2r^2s^2 + r^4 + s^4}{t^2a^2} \\
\frac{-2}{\sqrt{-8r^4s^4a^4 + 4r^6s^2a^2 + 4r^4s^6a^2 - 2r^4s^4 + r^8 + s^8}} & \frac{a(-s^2 + r^2)rs}{-2} \\
1 & 1 \\
0 & 0 \\
0 & 1
\end{bmatrix}
\]
Bibliography


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